# Dynamics of viscoelastic Burgers' cellular neural networks model

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# Dynamics of Viscoelastic Burgers' Cellular Neural Networks Model

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**Abstract.** In this paper we study travelling wave solutions of the viscoelastic Burgers' equation. RTD-based Cellular Neural Networks (CNN) model of this equation is presented. Using such realization new wave profiles of the travelling wave solutions are obtained.

## INTRODUCTION

Many methods used in image processing and pattern recognition can be easily implemented in RTD-based Cellular Neural Networks (CNN), however, the mathematical analysis of the phenomena as wave propagation, spatial chaos properties, and its dynamical behavior are still not fully studied. In this paper we shall provide some new wave profiles in viscoelastic Burgers' CNN model. This investigation is motivated by the paper of Hsu and Yang [5], in which the resonant tunneling diode (RTD), a class of quantum effect devices, is presented for studying the wave propagation in CNNs. RTD-based CNN is an excellent candidate for both analog and digital nanoelectronics applications because of its structural simplicity, relative easy of fabrication, inherent high speed and design flexibility.

The simplest model that couples the nonlinear convective behavior of fluids with the dissipative viscous behavior is well-known Burgers' equation:

$$u_t + uu_x = \varepsilon u_{xx}. \tag{1}$$

It is introduced by Burgers [1] as a model for turbulence. Equation (1) and its inviscid counterpart

$$u_t + uu_x = 0, \tag{2}$$

are essential for their role in modelling a wide array of physical systems such as traffic flow, shallow water waves, and gas dynamics [1]. The equations also provide fundamental pedagogical examples for many important topics in nonlinear PDEs such as travelling waves, shock formation, similarity solutions, singular perturbation, and numerical methods for parabolic and hyperbolic equations [7].

## VISCOELASTIC BURGERS EQUATION

In this paper we consider how the addition of viscoelasticity affects travelling wave solutions of Burgers' equation. The equations we consider are:

$$u_t + uu_x = v_x, \tag{3}$$

$$v_t + uv_x - vu_x = \alpha u_x - \beta v. \tag{4}$$

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The constitutive law (4) resembles a one-dimensional version of the upper convected Maxwell model [7]. The relaxation time is $\lambda = \beta^{-1}$, and $\alpha = \mu\lambda^{-1}$ could be interpreted as the elastic modulus of the material if there were no relaxation of stress ($\beta = 0$). In the other limit of instantaneous relaxation of stress ($\lambda \to 0$), (4) reduces to $v = \mu u_x$, and the system (3)-(4) is equivalent to Burgers' equation (1) with fluid viscosity $\mu = \varepsilon$.

One of the simplest constitutive laws for viscoelastic materials is the Maxwell model. Consider a linear spring and dashpot in series, with spring constant $k$ and damping coefficient $\mu$. The stress, $v$, in the element is

$$\lambda\dot{v} + v = \mu\dot{\varepsilon}, \tag{5}$$

where $\varepsilon$ is the strain in the element, and $\lambda = k/\mu$ is the relaxation time. The linear Maxwell model for a continuum is

$$\lambda v_t + v = 2\mu D, \tag{6}$$

where $2\mu D$ is the viscous stress. However, this is not a valid constitutive law because it is not frame invariant [7]. That is, the stress depends on the reference frame. Frame invariance is achieved by choosing an appropriate time derivative, akin to the material derivative for the velocity field. One frame invariant time derivative is the upper convected derivative, defined by

$$\bar{S} = S_t + u.\nabla S - \nabla u S - S\nabla u^T. \tag{7}$$

Replacing the partial time derivative in (6) with the upper convected derivative gives the upper convected Maxwell (UCM) equation

$$\lambda\bar{v} + v = 2\mu D. \tag{8}$$

The $ij$ component in (8) satisfies

$$\lambda(\frac{\partial v_{ij}}{\partial t} + u_k\frac{\partial v_{ij}}{\partial x_k} - \frac{\partial u_i}{\partial x_k}v_{kj} - v_{ik}\frac{\partial u_j}{\partial x_k}) + \tag{9}$$
$$+v_{ij} = \mu(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}),$$

where summation is over the repeated index $k$. Although there are many other frame invariant derivatives, in this paper we consider a one-dimensional reduction, in which case they yield identical reductions.

A one-dimensional version of the UCM equation is

$$\lambda(v_t + uv_x - vu_x) + v = \mu u_x. \tag{10}$$

Equation (10) is equivalent to (4). This is seen by dividing through by the relaxation time $\lambda$ to get

$$v_t + uv_x - vu_x = \alpha u_x - \beta v, \tag{11}$$

where

$$\alpha = \mu\lambda^{-1}, \tag{12}$$

$$\beta = \lambda^{-1}. \tag{13}$$

The parameter $\alpha$ could be interpreted as the elastic modulus of the material if there were no relaxation of stress ($\beta = 0$). It is somewhat arbitrary whether the constitutive law is expressed in terms of the relaxation time ($\lambda$) and viscosity ($\mu$) or elastic modulus ($\alpha$) and decay rate ($\beta$).

**Remark 1** *We find that the solutions develop into travelling waves, however, with jump discontinuities in the wave profile. When solving equations with discontinuities care must be taken in order to capture the correct solution. There are several questions that arise: In the case of the double-shock solution, what determines the shock profile? What determines the shape of the solution between two shocks? Why is that we see a double-shock solution? To answer all these questions we shall apply RTD-based Cellular Neural Networks approach.*

## RTD-BASED CELLULAR NEURAL NETWORKS

Cellular Neural Networks (CNNs)[2] are complex nonlinear dynamical systems, and therefore one can expect interesting phenomena like bifurcations and chaos to occur in such nets. It was shown that as the cell self-feedback coefficients are changed to a critical value, a CNN with opposite-sign template may change from stable to unstable. Namely speaking, this phenomenon arises as the loss of stability and the birth of a limit cycles.

We shall apply in this study one-dimensional original RTD-based CNN without input and threshold terms [5]. The dynamics of our RTD-based CNN model for the system of viscous Burger's equation (3), (4) will be the following:

$$\begin{aligned} \frac{du_j}{dt} &+ u_j A_1 * u_j = A_1 * v_j \\ \frac{dv_j}{dt} &+ u_j A_1 * v_j - v_j A_1 * u_j = \\ &= \alpha A_1 * u_j - \beta v_j, \end{aligned} \tag{14}$$

$1 \leq j \leq M$, where $A_1 = (1, -2, 1)$, is one-dimensional discretized Laplacian CNN template, $*$ is the convolution CNN operator [2].

We study here the structure of the travelling wave solutions of the RTD-based CNN model (14) of (3), (4) having the form:

$$\begin{aligned} u_j &= \Phi(j - ct), \\ v_j &= \Psi(j - ct) \end{aligned} \tag{15}$$

$\Phi$, $\Psi$ being continuous functions. Let us substitute (15) in (14). Therefore we consider solutions $\Phi(s; c)$, $\Psi(s; c)$, $s = j - ct$ of:

$$\begin{aligned} -c\Phi^{'}(s; c) + G_1(\Phi(s; c)) = 0, \\ -c\Psi^{'}(s; c) + G_2(\Psi(s; c)) = 0 \end{aligned} \tag{16}$$

where $G_1(\Phi), G_2(\Psi) \in \mathbf{R}^1$. We consider travelling waves that correspond to heteroclinic connections between two equilibrium points with given velocity values at infinity. The equilibrium points of the system (16) correspond to all states with $\Psi = 0$, and thus we assume the following asymptotic boundary conditions:

$$\begin{aligned} lim_{s\to-\infty}\Phi(s; c) = u_l, \\ lim_{s\to\infty}\Phi(s; c) = u_r, \end{aligned} \tag{17}$$

$$\begin{aligned} lim_{s\to-\infty}\Psi(s; c) = 0, \\ lim_{s\to\infty}\Psi(s; c) = 0, \end{aligned} \tag{18}$$

for some $c > 0$.

Below we propose the following result.

**Theorem 1** *Suppose that $u_j(t) = \Phi(j - ct)$ and $v_j = \Psi(j - ct)$ are travelling wave solutions of the CNN model (14) of the system of viscoelastic Burgers' equations (3), (4). Then there exists $c = \frac{u_l + u_r}{2} > 0$ such that*

*(i) for $\alpha > \frac{d^2}{4}$, $d = u_l - u_r$ smooth travelling wave solution of (14) exists;*

*(ii) for $\frac{d^2}{8} < \alpha < \frac{d^2}{4}$ piecewise smooth travelling wave solution with two jump discontinuities exists;*

*(iii) for $\alpha < \frac{d^2}{8}$ single shock wave solution exists.*

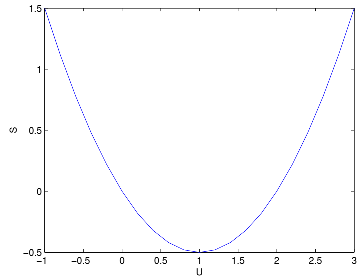


**FIGURE 1.** Heteroclinic orbit corresponding to travelling wave solution of (14)

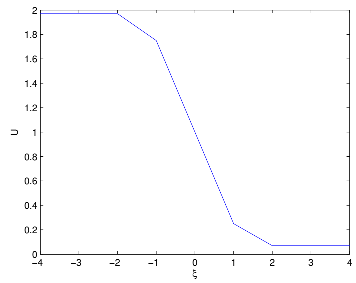


**FIGURE 2.** The wave profile of the RTD-based CNN model (14) for different values of the parameter sets: a). $u_l = 2$, $u_r = 0$, $\beta = 1$, $\alpha = 1.2$

**Proof:**

Without loss of generality we shall fix $\beta = 1$. The equilibrium points of the system (16) with the boundary conditions (17) and (18) are $E_1 = (u_l, 0)$ and $E_2 = (u_r, 0)$. We are looking for travelling wave solution of the RTD-base CNN model (14). It is a heteroclinic orbit connecting the two equilibrium points $E_1$ and $E_2$ ( see Fig.1).

After integrating system (16) and under the conditions (17), (18) we obtain:

$$\Psi(s;c) = \frac{\Phi(s;c)^2}{2} - c\Phi(s;c) + R, \tag{19}$$

where $c = \frac{u_l+u_r}{2}$ and $R = \frac{u_l u_r}{2}$. Substituting (19) in (16) we obtain for the wave profile $\Phi(s;c)$:

$$\Phi^{'}(s;c) = \tag{20}$$

$$\frac{-\beta(\Phi(s;c) - u_l)(\Phi(s;c) - u_r)}{(\Phi(s;c) - u_l)(\Phi(s;c) - u_r) + 2((\frac{u_l-u_r}{2})^2 - \alpha)}.$$

From (20) it is clear that we have the following possible cases: Travelling wave solution of (14) exists if and only if

$$u_l > u_r \quad \text{and} \quad \alpha > (\frac{u_l - u_r}{2})^2;$$

$$u_l < u_r \quad \text{and} \quad 2\alpha < (\frac{u_l - u_r}{2})^2;$$

Equivalently, no travelling wave solutions exist if

$$(\frac{u_l - u_r}{8})^2 \leq \alpha \leq (\frac{u_l - u_r}{4})^2.$$

We shall give the simulation of the RTD-based CNN model (14) on Figure 2.

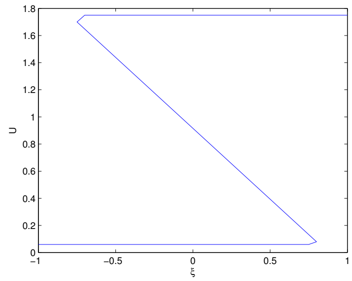


**FIGURE 3.** b). $u_l = 2, u_r = 0, \beta = 1, \alpha = 0.9$

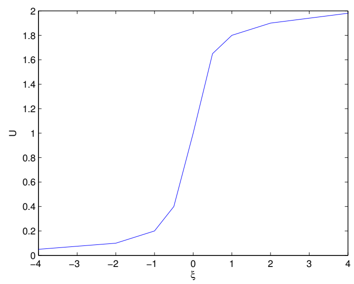


**FIGURE 4.** c). $u_l = 2, u_r = 0, \beta = 1, \alpha = 0.25$

**Remark 2** *For the parameter values given in Figure 2, a travelling wave exists when $\alpha > 1$ (a). As $\alpha$ approaches 1, the wave profile approaches the piecewise linear function. As $\alpha$ decreased further, the curve becomes multivalued and the asymptotic values are no longer satisfied (b).As $\alpha$ decreases even further, the solution returns to being single-valued but no longer yields a travelling wave solution with the given asymptotic limits (c). This transition occurs at* $\alpha = \frac{1}{2}(\frac{u_l - u_r}{2})^2$.

## CONCLUSIONS

In this paper we study the wave profiles of the travelling wave solutions of viscoelastic Burgers' equation. We apply the RTD-based Cellular Neural Networks in the one-dimensional integer lattice. A circuit implementation of the RTD-based CNN can be found in [6]. It is also pointed out that the bistable RTD-based CNN exhibits good performance for a number of interesting image processing applications because of its high-speed processing and high cell density. The study of travelling wave solutions of partial differential equations and lattice dynamical systems has drawn considerable attention in the past decades.

Recall that $\alpha = \mu/\lambda$, where $\mu$ and $\lambda$ are the viscosity and relaxation time, respectively. From Theorem 1 the following conclusions can be made. For a fixed relaxation time $\lambda$, each of the three types of wave solutions is possible, depending on the size of the viscosity. For large enough viscosity $\mu > \frac{d^2}{4}\lambda$ the wave profile of the solutions is smooth. As $\frac{d^2}{4}\lambda < \mu < \frac{d^2}{8}\lambda$ is decreased the wave solutions becomes double-shock and then when $\mu < \frac{d^2}{4}$ change to single shock wave.

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