Edge of Chaos in Nanoscale Memristor CNN

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***Abstract*— In this paper analytical results are derived for nanoscale memristor CNN (NM-CNN) in which neurons operate in a regime called edge of chaos. The system describing the model consists of highly nonlinear differential equations. We propose new algorithm based on the generalized local activity scheme for the determination of the edge of chaos regime in nanoscale memristor CNN model under consideration. MATLAB implementation of algorithms based on a numerical integration of the NM-CNN state equations allowing a reliable and accurate determination of the edge of chaos parameter regime is proposed. Application of the obtained results for pattern formation is presented.**

Keywords—nanoscale memristor, CNN, edge of chaos, numerical integrarion, spatial pattern formation

I. INTRODUCTION

Realized nanoscale CNN have been recently considered in a fast growing number of investigations dealing with image processing problems and pattern formation. Computer experiments conclude that the variable memristor synapses bestow more behavioral degrees of freedom to the networks, allowing them to outperform the comparative synapse types. Nanoscale CNN have so far been studied via numerical integrations [7, 9].

It is known [4, 10] that CNN (Fig.1) operating in the edge of chaos regime can exhibit computational complexity and can have applications in future computational systems.

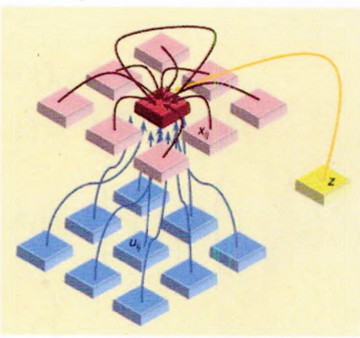

Fig. 1. Illustration of the CNN coupling structure for r = 1.

Since the cell size cannot be decreased considerably in conventional CMOS technology, nano-elements will play an important role in future CNN-UM chip realizations. Especially, memristors [2] which are considered for synaptic connections in first realizations [11], will play an important role for the realization of future CNN-UM sensor-processor systems by taking their rich dynamical behavior into account.

However, a deep mathematical treatment of CNN with memristors, briefly called memristor CNN in the following, hasn't been provided so far. Especially, the derivation of methods allowing the determination of the parameter space of a memristor CNN showing emergent complex behavior, is being essentially important in the development of CNN based computational methods [3,4,5].

In [9] niobium dioxide (NbO_2) Mott meristors are incorporated into a relaxation oscillator which leads to periodic and chaotic self-oscillations. The quasi-static memristors current-voltage plot exhibited a region of current-controlled negative differential resonance (NDR) at low currents and then a reproducible box-like hysteresis at higher currents. NbO_2 Mott memristors could be useful in some neural-inspired computations when a pseudo-random signal is introduced in order to prevent global synchronization. In [9] dynamic behavior of such memristors is obtained experimentally by building a relaxation oscillator, and the resulting plots demonstrate excellent agreement between the quasi-static and dynamic measurement. Moreover, it is shown that incorporating such memristors into the hardware of a Hopfield computing network can improve the efficiency and accuracy of converging to a solution for computationally difficult problems.

In this paper we consider NbO_2 Mott meristors and incorporate them into hysteresis CNN working in relaxation oscillator mode. In particular, for hysteresis CNN [5, 8], one can determine the domain of the cell parameters of locally active cells, and thus potentially capable of exhibiting complexity. We study the dynamics of the obtained model via local activity theory. We determine the edge of chaos domain in which complex behavior emerges. We propose numerical integration of the obtained algorithm and provide its application for spatial pattern formation.

II. THE MODEL

We consider hysteresis CNN made of first-order cells with hysteresis switches (see Fig.2). Our model will operate in relaxation oscillator mode and in this way it can have many applications. When CNN operates the relaxation oscillator mode then various patterns and nonlinear waves can be generated. Moreover, associative (static) and dynamic memories functions can be derived from the hysteresis CNN [8].

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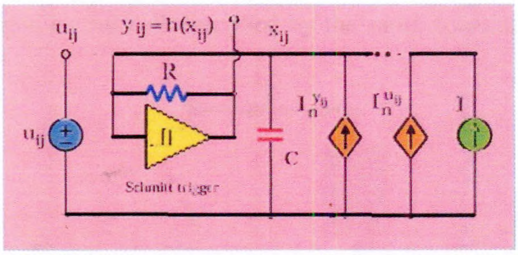

Fig.2. Circuit implementation of hysteresis CNN [8].

NDR has been modelled using a highly nonlinear transport relationship with temperature as the state variable of the memristor quasi-static conduction equation [6]. The memristor dynamical equation for the state variable x for NbO_2 Mott meristors is Newton's law of cooling:

$$\frac{dx}{dt} = \frac{i_m v_m}{C_{th}} - \frac{x - T_{amb}}{C_{th} R_{th}(x)} \quad (1)$$

where T_{amb} = 300 K is the ambient temperature, $C_{th} = 10^{-16} WsK^{-1}$ is the thermal capacitance, R_{th} is the temperature-dependent effective thermal resistance of the device. The basic requirements for chaotic oscillations in a constant-voltage-based electronic circuit are an element that displays local activity along with three dynamic state variables or two state variables and coupling to an oscillator [10].

In this paper we propose the following memristor CNN model:

$$\left| \begin{array}{l} \frac{dx_{ij}}{dt} = -x_{ij} + M(x_{ij}, y_{ij}, u_{ij}, t) - 2\,h(x_{ij}), \\ \quad y_{ij}(t) = G(x_{ij}, u_{ij}) u_{ij} \end{array} \right. \quad (2)$$

where x_{ij} is the state variable, $y_{ij} \triangleq i_m$, $u_{ij} \triangleq v_m$, $h(x_{ij})$ is dynamic hysteresis function defined by [8]:

$$h(x(t)) = \begin{cases} 1, for\ x(t) > -1, f(x(t_-)) = 1 \\ -1, for\ x(t) = -1 \\ -1, for\ x(t) < 1, f(x(t_-)) = -1 \\ 1, for\ x(t) = 1, \end{cases} \quad (3)$$

$$t_- = \lim_{\varepsilon \to 0} (t - \varepsilon), \varepsilon > 0,$$

$$M(x_{ij}, y_{ij}, u_{ij}, t) = \frac{y_{ij} u_{ij}}{C_{th}} - \frac{x_{ij} - T_{amb}}{C_{th} R_{th}},$$

$$G(x_{ij}, u_{ij}) = A_1(x_{ij}) \left\{ B_1(x_{ij})^2 \left(1 + \left(\frac{\sqrt{\frac{u_{ij}}{d}}}{B_1(x_{ij})} - 1 \right) e^{\frac{\sqrt{\frac{u_{ij}}{d}}}{B_1(x_{ij})}} \right) + \frac{1}{2d} \right\},$$

$A_1(x_{ij}) = \sigma_0 e^{-\frac{0.301}{2k_b x_{ij}}} A$, $k_b = 8.617 \times 10^{-5} eV$ is the Boltzmann constant, A is the lateral device area, $B_1(x_{ij}) = \frac{k_b x_{ij}}{\omega}$, ω and σ_0 are material constants and d is the thickness of NbO_2 [8]. Model (2), which we shall call nanoscale memristor CNN (NM-CNN), is a system of highly nonlinear differential and algebraic equations. Simulations of this model show chaotic oscillations (see Fig.3).

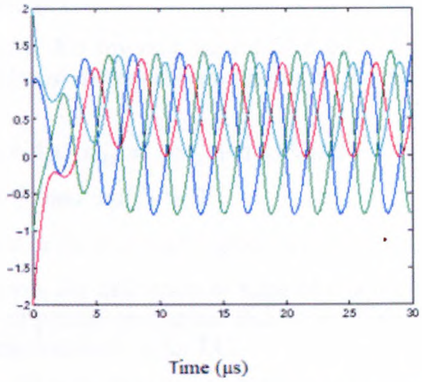

Fig.3. Simulated NM-CNN in relaxation oscillator mode.

We shall consider relaxation oscillator for system (2) given on the Figure 4 below:

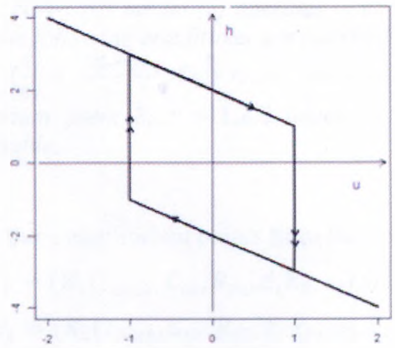

Fig.4. Relaxation oscillator.

It is known that hysteresis CNN [8] working in relaxation oscillator mode may generate some interesting patterns shown below:

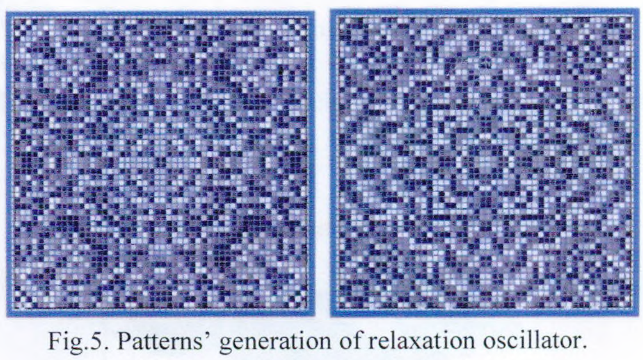
Fig.5. Patterns' generation of relaxation oscillator.

In the next section we shall apply local activity theory in order to determine edge of chaos regime in which our model (2) will exhibit complex behavior.

III. EDGE OF CHAOS

The theory of local activity which will be applied in this paper offers a constructive analytical method. In particular, for hysteresis CNN [5], one can determine the domain of the cell parameters of locally active cells, and thus potentially capable of exhibiting complexity. The physical basis of the concept of local activity is instructive. We shall associate the variables with the voltage and current of a 2-D terminal electronic circuit cell described by the same equations (see Fig.6).

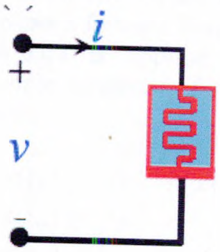

Fig.6. 2-D terminal device.

From this perspective, each cell is assumed to be operating near an equilibrium point. If there is at least one equilibrium point for which the circuit model of the cell acts like a source of small signal power, in a precise sense defined in [10], i.e. if the cell is capable of injecting a net small-signal average into the passive resistive grids, then the cell is said to be locally active. The central theme of the local activity theory is that emergence and complexity can be rigorously explained by explicit mathematical criteria given to identify a relatively small subset of the local-active parameter region, called the edge of chaos. A locally-active cell kinetic equation can exhibit complex dynamics such as limit cycles or chaos, even if the cells are uncoupled from each other (by setting all diffusion coefficients to zero). It is not surprising that coupling such cells could give rise to complex spatio-temporal phenomena, such as scroll waves, and spatio-temporal chaos.

We shall present here an algorithm for determination of edge of chaos for our NM-CNN model (2):

1. Find the equilibrium points – the corresponding discrete system can have one, two, or m- real equilibrium points. They can be found numerically, or by explicit mathematical formulas. In general, the equilibrium points are functions of the cell parameters;
2. Calculate the cell coefficients of the Jacobian matrix about each equilibrium point;
3. Calculate the trace Tr and the determinant - Δ of the Jacobian matrix about each equilibrium point;
4. Determine stable and locally active region at each equilibrium point;
5. Determine the region called edge of chaos (EC).

We apply this algorithm to our NM-CNN model (2). It is known [10] that the equilibrium points should satisfy the system:

$$\left| \begin{aligned} 0 &= -x_{ij} + M(x_{ij}, y_{ij}, u_{ij}, t) - 2\,h(x_{ij}), \\ 0 &= G(x_{ij}, u_{ij})u_{ij} \end{aligned} \right. \quad (4)$$

This system may have three real roots as functions of the cell parameters. Then we calculate the cell coefficients $a_{11}(E_r), a_{12}(E_r), a_{21}(E_r), a_{22}(E_r)$, $r = 1,2,3$ of the Jacobian matrix at each equilibrium point.

We shall apply in our study the following definition of Local Activity [4]: Discrete system of equations are called locally active if, and only if, its associated cells are locally active at some cell equilibrium point. Otherwise, they are said to be locally passive.

We define stable and locally active region for the NM-CNN model (2).

Definition 1. *We say that the cell is both stable and locally active region at the equilibrium point E_r for NM-CNN model (2) if*

$a_{22} > 0$ *or* $4a_{11}a_{22} < (a_{12} + a_{21})^2$ *and*

$Tr(E_k) < 0$ *and* $\Delta(E_k) > 0$.

This region in the parameter space is called $SLAR(E_r)$.

Until now the definition of edge of chaos (EC) is known only via empirical examples. Below we give more precise mathematical definition for EC.

Definition 2. *NM-CNN model (2) operates in edge of chaos regime if and only if at least one equilibrium point which is both locally active and stable exists.*

Based on the above algorithm the main theorem in this paper holds:

Theorem 1. NM-*CNN model (2) operates in edge of chaos if and only if the following conditions are satisfied:* $C_{th}R_{th} < 1$ and $\frac{4}{C_{th}R_{th}} < (\frac{K_r}{2d} + \frac{2K_r k_B}{\omega})^2$. *This means that there is at least one equilibrium point* $E_r, r = 1,2,3$ *which is both locally active and stable.*

Proof.

We find the three equilibrium points from the system (4):

$$E_1 = (K_1(T_{amb}, C_{th}, R_{th}, d, k_b, \omega), 0,0),$$

$$E_2 = (K_2(T_{amb}, C_{th}, R_{th}, d, k_b, \omega), 0,0),$$

$$E_3 = (K_3(T_{amb}, C_{th}, R_{th}, d, k_b, \omega), 0,0).$$

Then we find cell coefficients of the Jacobian matrix of system (4) - a_{kl} at each equilibrium point $E_r, r = 1,2,3$.

Following Definition 1 we find $SLAR(E_r)$: $C_{th}R_{th} < 1$ and $\frac{4}{C_{th}R_{th}} < (\frac{K_r}{2d} + \frac{2K_r k_B}{\omega})^2$, $r = 1,2,3$. According to Definition 2 there is at least one equilibrium point which is both locally active and stable.

The simulation on Figure 7 below presents the EC region of our NM-CNN model (2).

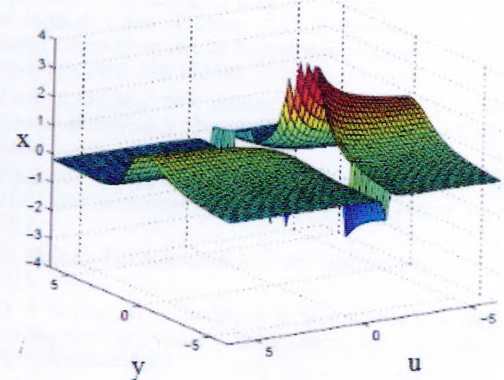

Fig.7. Edge of chaos region in the cell parameters' set.

IV. Simulations and Application

In this section we shall present numerical simulations based on the above algorithm for determination of edge of chaos. All numerical results are analyzed on software MATLAB in the case of the two typical applications under consideration. In this paper a forward Euler algorithm with a time step size $\Delta t = 0.01$ is applied to all computer simulations. The dynamic hysteresis function $h(x)$ (3) is programmed as follows:

$$h(x(t_n)) = \begin{cases} 1, \; for \;\; x(t_n) > -1, & h(x(t_{n-1})) = 1 \\ -1, \; for \;\; x(t_n) = -1, & \\ -1, \; for \; x(t_n) < 1, & h(x(t_{n-1})) = -1 \\ 1, \; for \; x(t_n) = 1, & \end{cases}$$

where $t_n = n\Delta t, n = 1,2, \ldots$

Validation of the obtained theoretical results is provided on Figure 8:

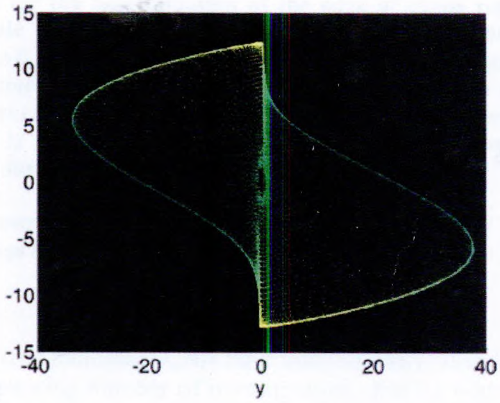

Fig.8. Validation of EC region for NM-CNN.

Through extensive numerical simulations we obtain that non uniform spatial patterns are generated in our NM-CNN model depending on initial conditions (see Fig. 9).

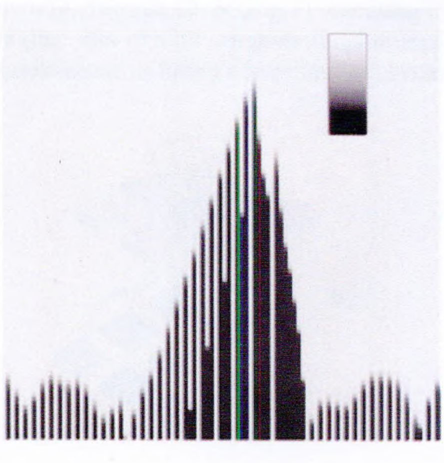

Fig.9. Spatial pattern formation in NM-CNN model.

Figure 9 shows the spatial (non-uniform) patterns developed over time. Surprisingly, the developed patterns are periodic, like Turing patterns, and they reached equilibrium at around 2×10^4 s.

V. Conclusions

In this paper we propose analytical results for nanoscale memristor CNN model. Our model incorporates NbO_2 Mott meristors [9] into hysteresis CNN working in relaxation oscillator mode. In this way we can obtain different applications of the model under consideration. Strong mathematical inequalities are obtained for determination of edge of chaos regime in which complex behavior and spatial patterns can occur.

We propose constructive procedure which is applicable to any system whose cells and couplings are described by deterministic mathematical models. The crux of the problem is to derive testable necessary and sufficient conditions which guarantee that the system has a unique steady state solution at $t \to \infty$. This is done in section 3 of the paper.

Non-uniform spatial patterns are generated depending on the initial condition of the model. This due to the fact that homogeneous non conservative medium cannot exhibit complexity unless the cells, or the coupling network is locally active.

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