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Exact Solutions of Protter Problems for the Wave Equation in $\mathbf{R}^4$ with Third-Type Boundary Condition

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Abstract. We study four-dimensional boundary value problems for the nonhomogenous wave equation, known as Protter problems. Here the boundary conditions are posed on a characteristic surface of the domain and on non-characteristic one. It is known that the Protter problems are ill-posed in the frame of classical solvability and they have generalized solutions with strong singularities. The behavior of these singular solutions is not typical for hyperbolic equations: their singularities are isolated at one boundary point and the order of singularity do not depend on the regularity of the right-hand side of the equation. In this paper we consider a case with a third-type condition on the non-characteristic surface and we find an explicit integral representation of the generalized solution. Further, we study the adjoint homogeneous problem. The reason for the ill-posedness of the Protter problems is that their adjoint homogeneous problems have infinitely many nontrivial classical solutions. In the case of the first-type and the second-type boundary value problems these solutions are well known. In this work we extend these results for the third-type boundary value problem: we find the exact formulas of the classical solutions of the adjoint homogeneous problem.

INTRODUCTION

For $\alpha \in \mathbb{R}$ we study the following boundary value problem:

$$u_{x_1x_1} + u_{x_2x_2} + u_{x_3x_3} - u_{tt} = f(x,t) \quad \text{in} \quad \Omega, \tag{1}$$

$$u|_{\Sigma_1} = 0, \qquad [u_t + \alpha u]|_{\Sigma_0} = 0, \tag{2}$$

where Ω is the region bounded by the surfaces

$$\Sigma_0 := \{(x,t):\ t=0,\ |x|<1\},\quad \Sigma_1 := \{(x,t):\ 0<t<1/2,\ |x|=1-t\},\quad \Sigma_2 := \{(x,t):\ 0<t<1/2,\ |x|=t\}.$$

Here for the points in $\mathbb{R}^4$ we use the usual notation $(x,t) := (x_1,x_2,x_3,t)$ and, correspondingly, $|x| := \sqrt{x_1^2+x_2^2+x_3^2}$. Note that Σ_1 and Σ_2 are characteristic surfaces for equation (1).

This problem is one of the so called Protter problems, which M. Protter formulated while he investigated some problems from supersonic fluid dynamics ([1, 2]). Actually, his problems are multidimensional generalization of the two-dimensional Darboux problems, where the boundary conditions are prescribed on a characteristic segment and on non-characteristic one.

It is well known that in the general case the Protter problems are not classically solvable. Instead of this, they have generalized solutions $u(x,t)$ in properly defined space of functions with strong singularities at the origin $O(0,0,0,0)$. The behavior of the singular solutions $u(x,t)$ is not typical for hyperbolic equations: their singularities are isolated at the point O and the order of singularity does not depend on the regularity of the right-hand side function $f(x,t)$. The adjoint homogeneous problems of the Protter problems have infinitely many nontrivial classical solutions $v_k(x,t)$, $k = 1,2,\dots$ ([3, 4, 5]). Actually, a necessary condition for existence of a bounded (or even classical) solution $u(x,t)$ is the orthogonality of $f(x,t)$ to all the functions $v_k(x,t)$. More detailed information on Protter problems for the multidimensional (4-D or 3-D) wave equation can be found in [6, 7, 8, 9, 10].

Note that for $\alpha = 0$ the boundary condition on Σ_0 turns into a second-type condition. At all, according to the type of the boundary condition on Σ_0, we will use the following terminology: in the case when $\alpha \neq 0$ we have a *third-type boundary value problem*, otherwise we have a *second-type boundary value problem*. If instead of (2) we set the Dirichlet boundary conditions $u|_{\Sigma_1} = 0$, $u|_{\Sigma_0} = 0$, we obtain a *first-type boundary value problem*.

The *first and the second-type boundary value problems* are much more well studied. For these problems explicit representation formulas for the generalized solutions $u(x,t)$, as well as for the nontrivial clasical solutions $v_k(x,t)$ of the corresponding adjoint homogeneous problems, are known.

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The *third-type boundary value problem*, or rather some its analogues and generalizations, are studied in [6, 11, 12, 13, 14]. At certain conditions for the right hand-side $f(x,t)$ existence and uniqueness results are derived and a priori estimates for the singular solutions are obtained. Nevertheless, these works do not give any explicit formulas for the solutions of the third-type boundary value problem or its adjoint homogeneous problem. In this parer we give such explicit formulas.

CORRESPONDING 2-D PROBLEM

Let $f \in C^1(\bar{\Omega})$. In this case $f(x,t)$ can be expanded into a Fourier series in terms of spherical functions:

$$f(x,t) = \sum_{m=0}^{\infty} \sum_{p=1}^{2m+1} f_m^p(|x|,t)\, Y_m^p(x/|x|), \tag{3}$$

where $Y_m^p(x)$, $m = 0,1,2,\dots,\ p = 1,2,\dots,2m+1$ are the three-dimensional spherical functions, defined on the unit sphere $|x| = 1$. In this paper we study the solution $u(x,t)$ only in the case when the right-hand side function $f(x,t)$ is a single term from the expansion (3), i.e. for some fixed $n \in \mathbb{N}$ and $s \in \mathbb{N},\ s \le 2n+1$ the function $f(x,t)$ has the form:

$$f(x,t) = g(|x|,t)\, Y_n^s(x/|x|). \tag{4}$$

Clearly, if the Fourier expansion (3) consists of finite number of terms, then the resulting solution can be obtained as a sum of such "single-term" solutions. The case of an infinite series requires a deep study of the convergence of the formally obtained solution, which is a difficult task and we do not do it here. Such a study has been carried out for the first and the second-type problems ([9, 10]).

It is known that if $f(x,t)$ is of the form (4), then the generalized solution of (1)-(2) has the form:

$$u(x,t) = \frac{1}{|x|}\, w(|x|,t)\, Y_n^s(x/|x|)$$

and, passing to the characteristic coordinates $\xi = 1-|x|-t,\ \eta = 1-|x|+t$, the function

$$U(\xi,\eta) := w\left(\frac{2-\xi-\eta}{2}, \frac{\eta-\xi}{2}\right)$$

is a solution of the following 2-D problem:

Problem $\mathbf{P}_\alpha$. *Find a function $U(\xi,\eta)$ solving the equation*:

$$U_{\xi\eta} - \frac{n(n+1)}{(2-\xi-\eta)^2}\, U = G(\xi,\eta) \tag{5}$$

in $D := \{(\xi,\eta):\ 0 < \xi < \eta < 1\}$, *satisfying the boundary conditions*:

$$U(0,\eta) = 0, \qquad \big(U_\xi - U_\eta\big)(\xi,\xi) = \alpha U(\xi,\xi), \tag{6}$$

where

$$G(\xi,\eta) = \frac{1}{8}\,(2-\xi-\eta)\, g\left(\frac{2-\xi-\eta}{2}, \frac{\eta-\xi}{2}\right).$$

Now, we will concentrate on Problem P_α and we will formulate our results only for this problem. After inverse transformation, these results can be easily reformulated for problem (1)-(2).

It is known that for $G \in C^1(\bar{D})$ Problem P_α has an unique generalized solution $U \in C^1(\bar{D} \backslash \{(1,1)\})$, $U_{\xi\eta} \in C(D)$, with a possible singularity at the point $(\xi,\eta) = (1,1)$. Here we claim that this result still holds under the weaker condition $G \in C(\bar{D})$, and we give an explicit formula for the solution. Actually, such results for the particular case $n = 1$ have been achieved in our paper [15]. In this way, the results in this work may be considered as a continuation and generalization of the partial result from [15].

Obviously, in the case of the second-type problem, the corresponding 2-D problem is **Problem $\mathbf{P_0}$** (i.e. Problem P_α with $\alpha = 0$). In the case of the first-type problem, the corresponding 2-D problem is **Problem $\mathbf{P_d}$**, with the Dirichlet boundary conditions: $U(0,\eta) = U(\xi,\xi) = 0$ instead of (6). Problems P_0 and P_d are completely studied and we will comment below the connection between our new results on Problem P_α ($\alpha \neq 0$) with the known results on P_0 and P_d.

EXPLICIT FORMULA FOR THE GENERALIZED SOLUTION

We construct the explicit formula for the solution $U(\xi,\eta)$ of Problem P_α via Riemann-Hadamard function.

For $k = 0,1,2,\ldots$ define the functions:

$$\Psi_k(\xi,\eta;\xi_0,\eta_0) := F_3\left(n+1,n+1,-n,-n,k+1;\frac{\xi_0-\eta}{2-\xi-\eta},\frac{\eta-\xi_0}{2-\xi_0-\eta_0}\right)$$
$$= \sum_{i=0}^{n}\sum_{j=0}^{n}\frac{(n+1)_i(n+1)_j(-n)_i(-n)_j}{(k+1)_{i+j}\,i!\,j!}\,\frac{(-1)^j\,(\xi_0-\eta)^{i+j}}{(2-\xi-\eta)^i(2-\xi_0-\eta_0)^j},$$

where

$$F_3(a_1,a_2,b_1,b_2,c;x,y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(a_1)_i(a_2)_j(b_1)_i(b_2)_j}{(c)_{i+j}\,i!\,j!}\,x^iy^j$$

is the Appell series (basic relations and properties of the Appell series are given for example in [16]). Note that $\Psi_k(\xi,\eta;\xi_0,\eta_0)$ are polynomials because $n \in \mathbb{N}\cup\{0\}$.

Next, let

$$R(\xi,\eta;\xi_0,\eta_0) = {}_2F_1\left(n+1,-n,1;\frac{-(\xi_0-\xi)(\eta_0-\eta)}{(2-\xi-\eta)(2-\xi_0-\eta_0)}\right).$$

It is well known that $R(\xi,\eta;\xi_0,\eta_0)$ is the Riemann function for equation (5).

Theorem 1. *Let $G \in C(\bar{D})$. Then Problem P_α has an unique generalized solution. This solution has the following integral representation at any point $(\xi_0,\eta_0) \in D$:*

$$U(\xi_0,\eta_0) = \int_0^{\xi_0}\int_{\xi}^{\eta_0}\Phi(\xi,\eta;\xi_0,\eta_0)G(\xi,\eta)\,d\eta d\xi, \tag{7}$$

where:

$$\Phi(\xi,\eta;\xi_0,\eta_0) := \begin{cases} R(\xi,\eta;\xi_0,\eta_0), & \eta > \xi_0,\\ Q(\xi,\eta;\xi_0,\eta_0), & \eta < \xi_0,\end{cases}$$

with

$$Q(\xi,\eta;\xi_0,\eta_0) := R(\xi,\eta;\xi_0,\eta_0)+R(\xi,\eta;\eta_0,\xi_0)+2\sum_{k=1}^{\infty}\frac{\alpha^k(\xi_0-\eta)^k}{k!}\Psi_k(\xi,\eta;\xi_0,\eta_0).$$

A direct computation of the derivatives shows that the function defined by (7) indeed solves Problem P_α.

Further, the uniqueness of this solution can be justified if we multiply the both sides of equation (5) by $\Phi(\xi,\eta;\xi_0,\eta_0)$ and integrate over the region $\{(\xi,\eta):\ 0<\xi<\xi_0,\ \xi<\eta<\eta_0\}$. Using integration by parts, after some transformations, we come to the equality (7), which means that (7) gives the only possible solution of the problem.

From formula (7) it follows that $U(\xi,\eta)$ may have singularity of n-th order at the point $(\xi,\eta) = (1,1)$. In the general case this singularity really exists and it can be reduced only if $G(\xi,\eta)$ satisfies special conditions.

Remark 1. *It is easy to check that in the case $n = 1$ the function $\Phi(\xi,\eta;\xi_0,\eta_0)$ coincides with the function $\Phi(\xi,\eta;\xi_0,\eta_0)$ from Theorem 1 in* [15].

Remark 2. *In the case $n = 0$ the function $\Phi(\xi,\eta;\xi_0,\eta_0)$ becomes very simple. Then $R(\xi,\eta;\xi_0,\eta_0) \equiv 1$, and*

$$Q(\xi,\eta;\xi_0,\eta_0) = 2\,e^{\alpha(\xi_0-\eta)}.$$

Remark 3. *Obviously, for $\alpha = 0$*

$$Q(\xi,\eta;\xi_0,\eta_0) := R(\xi,\eta;\xi_0,\eta_0)+R(\xi,\eta;\eta_0,\xi_0).$$

Then the function $\Phi(\xi,\eta;\xi_0,\eta_0)$ coincides, as expected, with the known Riemann-Hadamard function for Problem P_0 (see for example [17]).

Also, we may rewrite the function $Q(\xi,\eta;\xi_0,\eta_0)$ as:

$$Q(\xi,\eta;\xi_0,\eta_0) = R(\xi,\eta;\xi_0,\eta_0) - R(\xi,\eta;\eta_0,\xi_0) + 2\sum_{k=0}^{\infty} \frac{\alpha^k(\xi_0-\eta)^k}{k!}\Psi_k(\xi,\eta;\xi_0,\eta_0),$$

because $\Psi_0(\xi,\eta;\xi_0,\eta_0) = R(\xi,\eta;\eta_0,\xi_0)$, which follows from the known relation

$$F_3(a,c-a,b,c-b,c;x,y) = (1-y)^{a+b-c}\,{}_2F_1(a,b,c;x+y-xy).$$

From here we may note that

$$Q(\xi,\eta;\xi_0,\eta_0) \to R(\xi,\eta;\xi_0,\eta_0) - R(\xi,\eta;\eta_0,\xi_0) \quad as \quad \alpha \to -\infty.$$

In this case $\Phi(\xi,\eta;\xi_0,\eta_0)$ tends to the known Riemann-Hadamard function for the Dirichlet problem P_d. It turns out, that the Dirichlet problem P_d may be considered as the limit case of Problem P_α when $\alpha \to -\infty$.

NONTRIVIAL CLASSICAL SOLUTIONS OF THE ADJOINT HOMOGENEOUS PROBLEM

Consider the adjoint homogeneous problem of Problem P_α:

$$V_{\xi\eta} - \frac{n(n+1)}{(2-\xi-\eta)^2}V = 0 \quad \text{in} \quad D, \tag{8}$$

$$V(\xi,1) = 0, \qquad \big(V_\xi - V_\eta\big)(\xi,\xi) = \alpha V(\xi,\xi). \tag{9}$$

For $n \in \mathbb{N}$ this problem has nontrivial classical solutions.

For $p = 1,2,\dots,n$ define the functions:

$$\begin{aligned} V_p^n(\xi,\eta) := \sum_{i=0}^{n} \frac{(n+1)_i(-n)_i}{i!\,(2-\xi-\eta)^i} \sum_{k=\delta}^{\infty} \frac{\alpha^k(1-\eta)^{k+p+i}}{(k+p+i)!} \\ = \sum_{k=\delta}^{\infty} \frac{\alpha^k\,(1-\xi)^{p+k}(1-\eta)^{p+k}}{(2-\xi-\eta)^{p+k}}\,{}_2F_1\left(p+k+n+1, p+k-n, p+k+1; \frac{1-\eta}{2-\xi-\eta}\right), \end{aligned} \tag{10}$$

where $\delta = 1$ if $n-p$ is an odd number, and $\delta = 0$ otherwise. Let these functions be continuously extended at the point $(\xi,\eta) = (1,1)$:

$$V_p^n(1,1) := \lim_{(\xi,\eta)\to(1,1)} V_p^n(\xi,\eta) = 0.$$

Theorem 2. *Let $n \in \mathbb{N}$. For $p = 1,2,\dots,n$ the functions $V_p^n(\xi,\eta)$ are classical solutions of problem* (8)-(9)*, belonging to $C^2(D)\cap C(\bar{D})$.*

Remark 4. *The function $V_1^1(\xi,\eta)$ was announced in our paper* [15].

ASYMPTOTIC EXPANSION OF THE SINGULAR SOLUTIONS

For $p = 1,\dots,n$ define the coefficients:

$$\beta_p^n := \int_D V_p^n(\xi,\eta)G(\xi,\eta)\,d\xi d\eta,$$

as well as the functions:

$$H_p^n(\xi,\eta) := {}_2F_1\left(p+n+1, p-n, p+1; \frac{1-\eta}{2-\xi-\eta}\right).$$

The functions $H_p^n(\xi,\eta)$ are bounded in $\bar{D}$.

Analogously to the known results for Problems P_0 and P_d, it is expected for the coefficients β_p^n to be responsible for the behavior of the singularities of the generalized solution $U(\xi,\eta)$ of Problem P_α. In the next theorem we give an asymptotic formula for the singular solutions $U(\xi,\eta)$. The singular terms in this asymptotic expansion indeed are controlled by the coefficients β_p^n.

Theorem 3. *Let $F \in C(\bar{D})$. Then the generalized solution of Problem P_α has the following asymptotic representation in D:*

$$U(\xi,\eta) = \sum_{p=1}^{n} c_p^n \beta_p^n H_p^n(\xi,\eta)(2-\xi-\eta)^{-p} + H(\xi,\eta), \tag{11}$$

where c_p^n are nonzero constants and the function $H(\xi,\eta)$ is bounded in $\bar{D}$.

It is essential to note that $H_p^n(\xi,1) = 1$ for $0 \le \xi < 1$ and hence $H_p^n(\xi,1) \to 1 \neq 0$ as $\xi \to 1$. This means that if for some fixed index $p = p_0$ in the expansion (11) the corresponding coefficient $\beta_{p_0}^n$ is different from zero, then the order of singularity of the solution $U(\xi,\eta)$ is at least of p_0-th order. Bounded solution is possible only if all the coefficients β_p^n, $p = 1,\dots,n$, are equal to zero, i.e. only if the right-hand side function $G(\xi,\eta)$ is orthogonal in $L_2(D)$ to all the functions $V_p^n(\xi,\eta)$.

Remark 5. *The result in Theorem 3 corresponds to the known results for the asymptotic behavior of the solutions of Problems P_0 and P_d.*

For $p = 1,\dots,n$ define the functions:

$$E_p^n(\xi,\eta) := \frac{(1-\xi)^p(1-\eta)^p}{(2-\xi-\eta)^p} H_p^n(\xi,\eta).$$

Now, let p be a fixed number and examine two cases for p:

- *Let $n-p$ be an even number. For $\alpha = 0$, according to* (10)*, $V_p^n(\xi,\eta) \equiv E_p^n(\xi,\eta)$, which means that $E_p^n(\xi,\eta)$ is a nontrivial classical solutions of Problem P_0. In this case the function $E_p^n(\xi,\eta)$ can be transformed into the following form:*

$$E_p^n(\xi,\eta) = k_p^n \, {}_2F_1\left(\frac{p+n+1}{2}, \frac{p-n}{2}, \frac{1}{2}; \frac{(\eta-\xi)^2}{(2-\xi-\eta)^2}\right),$$

where $k_p^n = \mathrm{const} \neq 0$. At the same time, $V_p^n(\xi,\eta) \to 0$ as $\alpha \to -\infty$.

- *Let $n-p$ be an odd number. Then $V_p^n(\xi,\eta) \to -E_p^n(\xi,\eta)$ as $\alpha \to -\infty$, as well as the function $E_p^n(\xi,\eta)$ solves equation* (8) *and satisfies the Dirichlet boundary conditions $V(\xi,1) = V(\xi,\xi) = 0$. In this case the function $E_p^n(\xi,\eta)$ can be also presented as:*

$$E_p^n(\xi,\eta) = h_p^n \frac{\eta-\xi}{2-\xi-\eta} \, {}_2F_1\left(\frac{p+n+2}{2}, \frac{p-n+1}{2}, \frac{3}{2}; \frac{(\eta-\xi)^2}{(2-\xi-\eta)^2}\right),$$

where $h_p^n = \mathrm{const} \neq 0$. At the same time, $V_p^n(\xi,\eta) \equiv 0$ for $\alpha = 0$.

In view of all this, the asymptotic formula for the solution of Problem P_0 becomes:

$$U(\xi,\eta) = \sum_{k=0}^{[(n-1)/2]} a_k^n \beta_{n-2k}^n \, {}_2F_1\left(n-k+\frac{1}{2}, -k, \frac{1}{2}; \frac{(\eta-\xi)^2}{(2-\xi-\eta)^2}\right)(2-\xi-\eta)^{2k-n} + H(\xi,\eta),$$

and the asymptotic formula for the solution of Problem P_d becomes:

$$U(\xi,\eta) = \sum_{k=0}^{[n/2]-1} b_k^n \beta_{n-2k-1}^n \frac{\eta-\xi}{2-\xi-\eta} \, {}_2F_1\left(n-k+\frac{1}{2}, -k, \frac{3}{2}; \frac{(\eta-\xi)^2}{(2-\xi-\eta)^2}\right)(2-\xi-\eta)^{2k+1-n} + H(\xi,\eta),$$

where $a_k^n, b_k^n = \mathrm{const} \neq 0$.

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