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Dynamic Management of the Activation of Participants in a Social Network in Continuous Time

Petar Marinov

***Abstract*— The present work is devoted to the problem of dynamic control of the activation of participants in a social network in continuous time, which is formulated and solved as a choice of the number of activators introduced at each moment of time. Participants are in two states – active or inactive. Each participant makes a decision to be in one of these two states, taking into account the opinion of other participants with whom he contacts, the share of the latter determining his threshold. It is assumed that the change in the initial share of participants is described by a first-degree dynamic equation of the distribution function of the thresholds of the participants. Introducing control (adding activators) leads to a controllable dynamic system that has an equilibrium state. Initially, the reachability set and the monotonicity of the control trajectories are studied. Then, the case of continuous control is investigated. The problem of activating all participants in the social network as well as the case of positional control are also investigated.**

***Keywords*— Social networks, dynamic model, management**

I. Introduction

This work is devoted to models for controlling participants in social networks in continuous time [2]. A model of the dynamics of collective behavior, in which at the initial moment of discrete time all participants are inactive, then at each of the subsequent moments of time the participants act simultaneously and independently is given in [1]. The problem of dynamically controlling the activation of participants in a social network in continuous time is formulated and solved by choosing the number of activators introduced at each moment of time. The participant model itself is a generalization of the Granovetter model [8] for the case of continuous time [3],[4]. It is assumed that if x0 ϵ [0; 1] is the fraction of participants acting at the initial (zero) moment of time, then its evolution in time x(t), t ≥ 0 is described by the equation

$$\dot{x} = F(x) - x\,, \tag{1}$$

where F(·) is a known continuous function that has the properties of a distribution function and is essentially a function of the distribution of the thresholds of the participants, so that F(0) = 0, F(1) = 1. Introducing control (representing the addition of activators) u(t) ϵ [0; 1] by analogy with the way it was done in [1],[5],[6] leads to a controllable dynamical system

$$\dot{x} = u(t) + (1 - u(t))\,F(x) - x\,. \tag{2}$$

The sequence of presentation of the material in this section is as follows: First, the reachability set and the property monotonicity of trajectories of control are studied. Then, the case of constant control is studied. Third, the tasks of activating all participants in the social network are studied. Finally, the case of positional control is studied.

II. Reachability set and monotonicity

Let us consider the functions $G_1(x,t)$ and $G_2(x,t)$, such that $R \times [t_0, +\infty) \to R$, which we assume to be continuously differentiable with respect to x and continuous with respect to t. In addition, we assume that the functions G_1 and G_2 yield solutions to the Cauchy problems for the system of differential equations

$$\left|\begin{array}{l}\dot{x}_1 = G_1(x,t)\\ \dot{x}_2 = G_2(x,t)\end{array}\right. \tag{3-0}$$

with initial conditions $(t_0, x_0), x_0 \in R$ to be infinitely extendible in t. Let us denote these solutions of the Cauchy problems as $x_1(t,(t_0,x_0))$ and $x_2(t,(t_0,x_0))$ respectively. In such a case, if for every x∈R for every $t \geq t_0$_0 it is fulfilled $G_1(x,t) > G_2(x,t)$, then for every $t > t_0$ it is fulfilled $x_1(t,(t_0,x_0)) > x_2(t,(t_0,x_0))$ are satisfied.

$$\left.\frac{d}{dt}x_1(t,(t_0,x_0))\right|_{t=t_0} > \left.\frac{d}{dt}x_2(t,(t_0,x_0))\right|_{t=t_0}.$$

Therefore, one can find a number ε > 0 such that

$$\forall t \in (t_0, t_0 + \varepsilon] \to x_1(t,(t_0,x_0)) > x_2(t,(t_0,x_0))\,,$$

i.e. the graph of the solution of the first equation lies above the graph of the second on the entire half-interval (t0, t0 + ε]. Such an arrangement of the graphs is valid for all t > t0.

If we doubt the statement about the arrangement of the graphs and assume that it is possible to exist tˆsuch that

$$\exists \hat{t} : x_1(\hat{t},(t_0,x_0)) = x_2(\hat{t},(t_0,x_0)) = \hat{x}.$$

where tˆis the first moment at which the graph x2 reaches the graph x1, i.e.

$$\hat{t} = \inf \quad t > t_0 + \varepsilon : x_1(t,(t_0,x_0)) = x_2(t,(t_0,x_0)) \quad < +\infty.$$

It is clearly seen that $\hat{t} \geq t_0 + \varepsilon > t_0$ and that

$$\forall t \in (t_0, \hat{t}) \to x_1(t,(t_0,x_0)) > x_2(t,(t_0,x_0))\,.$$

Therefore, for $\tau \in [0, \hat{t} - t_0)$ we will have

$$x_1(\hat{t}-\tau,(t_0,x_0)) - x_1(\hat{t},(t_0,x_0)) > x_2(\hat{t}-\tau,(t_0,x_0)) - x_2(\hat{t},(t_0,x_0))\,,$$

since the second integrands in both parts of this inequality are the same and equal to $\hat{x}$. Moreover, if I divide both parts of the inequality by (– τ) (the sign of the inequality will change to the opposite) and we move to the limit transition at $\tau \to 0$, as a result we will obtain a dependence for the derivatives of the solutions x_1 and x_2 at the point $\hat{t}$ of the form:

$$\frac{d}{dt} x_1(t,(t_0,x_0))\Big|_{t=\hat{t}} \leq \frac{d}{dt} x_2(t,(t_0,x_0))\Big|_{t=\hat{t}}$$

from which it will follow that $G_1(\hat{x},\hat{t}) \leq G_2(\hat{x},\hat{t})$. This contradicts the accepted conditions and shows that the doubts are unfounded.

In this case, it is not necessary to consider the inequality $G_1(\hat{x},\hat{t}) > G_2(\hat{x},\hat{t})$ for all x ϵ R, it is enough to limit ourselves to the union of the reachability sets of equations (3-0) for the chosen values of the initial conditions (t_0, x_0).

Let us denote by $x_t(u)$ the value of the share of active participants at time t, using control u(·). Due to the fact that $\forall\, x \in [0; 1]$ $F(x) \leq 1$ and the right-hand side of expression (1) monotonically increases under the control u for each t, then the following statement is true:

Let the function F(x) be such that F(x) < 1 for x ∈ [0; 1). If

$$\forall t \geq t_0 \to u_1(t) > u_2(t) \text{ and } x_0(u_1) = x_0(u_2) \quad (x_0 < 1)$$

then

$$\forall t > t_0 \;\; x_t(u_1) > x_t(u_2).$$

Indeed, by virtue of the conditions of the statement, for all t and x < t the inequality is satisfied

$$u_1(t) + (1-u_1(t))F(x) - x > u_2(t) + (1-u_2(t))F(x) - x,$$
$$u(t) \leq \Delta,\; t \geq t_0, \tag{3}$$

where Δ ∈ [0; 1] is some constant.

In what follows, we will assume that $t_0 = 0$, $x(t_0) = x(0) = 0$, i.e. at the initial moment the actors are not activated.

If the efficiency criterion is the fraction of actors acting at a given moment T > 0, then the corresponding final control problem has the form:

$$\begin{cases} x_T(u) \to \max\limits_{u(\cdot)}, \\ \dot{x} = u(t) + (1-u(t))F(x) - x \\ u(t) \leq \Delta,\; t \geq t_0 \end{cases} \tag{4}$$

The solution to problem (4) has the form: u(t) = Δ, t ∈ [0;T].

Let us denote by $\tau(\hat{x},u) = \min\{t \geq 0 \mid x_t(u) \geq \hat{x}\}$ the earliest moment of time at which the share of active participants reaches a given value $\hat{x}$ (if the set $\{t \geq 0 \mid x_t(u) \geq \hat{x}\}$ is empty, then we put $\tau(\hat{x},u) = +\infty$).). Within the framework of the considered model, the following task of rapid response can be formulated:

$$\begin{cases} \tau(\hat{x},u) \to \min\limits_{u(\cdot)}, \\ \dot{x} = u(t) + (1-u(t))F(x) - x \\ u(t) \leq \Delta,\; t \geq t_0 \end{cases} \tag{5}$$

The solution to problem (5) has form: u(t) = Δ, t ∈ [0; τ].

Essentially, as in discrete-time models [6], within the framework of problem (4) or (5), it is most profitable for the control center to introduce the maximum permissible number of activators among the participants of the social network at the initial time point and to do nothing else (not to try, for example, to then reduce and then again increase the number of introduced activators at subsequent time points). This structure of the optimal solution is due to the fact that in models (4) and (5) the control center does not bear the costs of introducing and/or maintaining activators.

We study the properties of the reachability set

$$D = \bigcup_{u(t)\in[0;\Delta]} x_T(u), \quad D \subseteq [0;1]$$

due to the fact that the right-hand side of the dynamic system (2) becomes zero at x = 1.

From the point of view of possible applications, the case of constant controls: u(t) = v, t ≥ 0, when using which the share v ϵ [0;Δ] of the activators is the same at all times. Let us denote by $x_T(\Delta) = x_T(u(t)) \equiv \Delta$, t ∈ [0;T], and by

$$D_0 = \bigcup_{v\in[0;\Delta]} x_T(v) \subseteq [0;1]$$

denote the reachability set under constant control. Since $x_T(v)$ is a monotone continuous map of [0; Δ] in [0; 1] and $x_T(0) = 0$, the reachability set at the initial time has the form $D_0 = [0; x_T(\Delta)]$.

Let us consider models that take into account the costs of the control center. For a fixed cost λ ≥ 0 for maintaining one activator per unit time, the costs of the center for time τ ≥ 0 will take the form:

$$c_\tau(u) = \lambda \int_0^\tau u(t)\,dt \tag{6}$$

Let the following monotonic functions be given: 1) the final profit H(·) of the control center from the share of active participants and 2) the current profit h(·). Then the following problem (7) will be a generalization of problem (4)

$$\begin{cases} H(x_T(u)) + \int_0^T h(x(t))\,dt - c_T(u) \to \max\limits_u, \\ \dot{x} = u(t) + (1-u(t))F(x) - x \\ u(t) \leq \Delta,\; t \geq t_0 \end{cases} \tag{7}$$

If constraints are given on the total costs C of the control center, then the problem of type (7) can be formulated as follows

$$\begin{cases} H(x_T(u)) + \int_0^T h(x(t))\,dt \to \max\limits_u, \\ \dot{x} = u(t) + (1-u(t))F(x) - x \\ c_T(u) \leq C \end{cases} \tag{8}$$

A possible variant of problems of type (4), (5), (7), (8) would be the problem of minimizing the costs of providing a given share xˆof active participants at time T:

$$\begin{cases} c_T(u) \to \min_u, \\ x_T(u) \geq \hat{x}, \\ \dot{x} = u(t) + (1 - u(t))\,F(x) - x \end{cases} \qquad (9)$$

Problems of type (7)-(9) reduce to standard optimal control problems.

Let us consider problem (9) in the case when F(x) = x. Suppose that $x_0 = 0$ and the center costs are given by (6) with $\lambda_0 = 1$. As a result, we obtain the following optimal program control problem with fixed ends:

$$\begin{cases} \dot{x} = u(1 - x), \\ x(0) = 0,\ x(T) = \hat{x}, \\ 0 \leq u \leq \Delta, \end{cases} \qquad (10\text{-}1)$$

We are looking for a solution to this problem provided that

$$\int_0^T u(t)\,dt \to \min_{u \in [0, \Delta]} \qquad (10\text{-}2)$$

We solve a conditional extremal problem as a problem of fast response. The Pontryagin function for problem (10) will be $H = \psi.(u.(1 - x))$. According to the Pontryagin maximum principle, this function must take maximum values with respect to u. Since the function is linear with respect to u, the maximum is achieved at the ends of the interval [0, Δ] depending on the sign of the coefficient in front of u, i.e. we can write

$$u = \Delta.\operatorname{sign}(\psi.(1 - x)) \qquad (11)$$

The linearity of the Pontryagin function in control is a consequence of the linearity of both the right-hand side of the dynamical system (2) and the functional (6). Therefore, if the constraints in optimal control problems of type (7)–(9) are linear in control, then the optimal program control will have a structure described by expression (11), within which structure the control value at any time is equal to either the maximum possible or the minimum possible value.

Hamilton's equations look like this:

$$\dot{x} = \frac{\partial H}{\partial \psi} = u(1 - x), \quad \dot{\psi} = -\frac{\partial H}{\partial x} = u\psi.$$

Boundary conditions are imposed only on the first equation. The solution of the first equation for u = 0 will be a constant, and for u = Δ the solution is the function

$$x(t) = 1 - (1 - x(t_0))\,e^{-\Delta(t - t_0)}$$

From this expression arises a constraint on the maximum number of activators needed to be able to move participants in the social network from position zero to position

$$\hat{x}:\ \Delta \geq \frac{1}{T}\log\frac{1}{1 - \hat{x}}.$$

There is minimal time involved

$$t_{\min} = \frac{1}{\Delta}\log\frac{1}{1 - \hat{x}},$$

for which the control must take the maximum value Δ, and at other times the control must be equal to zero. In particular, one of the solutions to problem (10) will be

$$u = \begin{cases} \Delta, & t \leq t_{\min} \\ 0, & t_{\min} < t \leq T \end{cases} \qquad (12)$$

when the maximum number of activators is introduced at once and kept constant for time $t_{\min}$.

The structure of the optimal solution to this problem (a partially constant function taking values 0 or Δ) may lead to the need to minimize the number of control switching. Such an additional restriction is essentially justified by the fact that the control center can bear some additional costs for introducing or removing activators. In the event that there is such a restriction in the considered problem, from the entire set of optimal controls, the best remain either (12), or control

$$u = \begin{cases} \Delta, & t \in [T - t_{\min}, T], \\ 0, & t < T - t_{\min}. \end{cases}$$

III. CONSTANT CONTROL

From expression (6) it follows that for constant controls u(t) = v, t ≥ 0, at a fixed cost λ ≥ 0 for maintaining one actuator per unit time, the center costs for time τ ≥ 0 will be $c_\tau(v) = \lambda.v.\tau$. For given functions F(·) (i.e., for a known dependence $x_t(v)$) problems (7)–(9) are reduced to typical scalar optimization problems.

As an example, let us consider the case when F(x) = x, T = 1, $x_0 = 0$, H(x) = x, h(x) = γ x, where γ ≥ 0 is a known constant. From equation (1) we find:

$$x_t(u) = 1 - \exp\left(-\int_0^t u(y)\,dy\right) \qquad (13)$$

With constant controls we get

$$x_t(v) = 1 - e^{-vt}.$$

Problem (7) will take the form of the following scalar optimization problem:

$$e^{-v}\left(\frac{\gamma}{v} - 1\right) - \frac{\gamma}{v} - \lambda v \to \max_{v \in [0;\Delta]} \qquad (14)$$

and problem (8) will take the form of the following scalar optimization problem

$$e^{-v}\left(\frac{\gamma}{v} - 1\right) - \frac{\gamma}{v} \to \max_{v \in [0;\Delta]} \qquad (15)$$

Problem (9) takes the form and has a solution accordingly:

$$\begin{cases} v \to \min\limits_{v\in[0;1]}, \\ 1 - e^{-v} = \hat{x}. \end{cases} \quad \text{and} \quad v = \ln\left(\frac{1}{1-\hat{x}}\right).$$

IV. Activation of all participants in the social network

Let us consider the asymptotes of the problems presented above at T = +∞. We assume that the function F(·) has one inflection point, F(0) = 0 (shown in [7]), the equation F(x) = x has a unique solution on the interval (0; 1) – the point q > 0, and

$$\forall x \in (0;q)\ F(x) < x,\ \forall x \in (q;1)\ F(x) > x.$$

We will also assume that the goal of minimal cost management is to activate all participants.

From the introduced assumptions about the properties of the function F(·) it follows that if for some time τ x(τ) > q is satisfied, then even for u(t) ≡ 0 for each t > 0, the trajectory xt(u) will be non-decreasing and . Essentially, this property means that all participants in the social network are such that the region of attraction of the zero equilibrium position in the absence of management (without introduced activators) is the half-interval [0; q). In other words, for the participants it is enough to ensure activation by external influence of a larger than q share of participants and then, even in the absence of management, they themselves will strive for the single equilibrium state.

Let us denote by u_τ the solution to the following problem:

$$\int_0^\tau u(t)\,dt \to \min_{u:\,u(t)\in[0;\Delta],\ x_\tau(u)>q} \qquad (16)$$

We calculate

$$Q_\tau = \int_0^\tau u^\tau(t)\,dt$$

and find

$$\tau^* = \arg\min_{\tau\ge 0} Q_\tau.$$

The solution to problem (16) exists under the condition

$$\Delta > \Delta^* = \max_{x\in[0,q]} \frac{x - F(x)}{1 - F(x)} \qquad (17)$$

Given the introduced assumptions about the properties of the distribution function for the structure of the optimal solution to the problem under consideration, it is seen that if condition (17) is satisfied, then uτ(t) ≡ 0 for t > τ

V. Positional control

In tasks for controlling the activation of participants in a social network, a synthesis of time-optimal program control can be used, which was considered above. However, positional control can also be used in these tasks. Here we consider two possible options that correspond to real situations.

In the first option, the task consists in finding a positional control law $\tilde{u}(x) : [0;1] \to [0;1]$ that ensures maximum activation of participants (in the sense of tasks (4) or (5)) under one or another restriction on the trajectory of the system and/or on control.

Let the control by analogy with expression (3) be limited in the following way:

$$\tilde{u}(x) \le \Delta,\ x \in [0;1] \qquad (18)$$

by placing an additional constraint on the trajectory

$$\dot{x}(t) \le \delta,\ t \ge 0 \qquad (19)$$

where δ > 0 is a given constant. In essence, condition (19) means, for example, that a too rapid increase (per unit time) in the share of activated participants is noticed by the corresponding megacenter monitoring the state of the social network, which intervenes and makes further management impossible. Thus, the management center, whose goal is to manage the activation of the participants of the social network, should strive to maximize the value of the share of activated participants under conditions (18) and (19). The solution to the problem formulated in this way, due to the properties of the dynamic system (2) established above, has a simple form:

$$\tilde{u}^*(x) = \min\left\{\Delta;\ \max\left\{0;\ \frac{x + \delta - F(x)}{1 - F(x)}\right\}\right\} \qquad (20)$$

The fraction in expression (20) is obtained by equating the right-hand side of Eq. (1) to the constant δ. For small values of δ, there may not be a non-negative control satisfying Eq. (19).

For F(x) = x, T = 1, $x_0 = 0$, H(x) = x, h(x) = γ.x, where γ ≥ 0 and δ = 0.35 is chosen, the optimal value of the position control for this case is shown in Fig. 20.

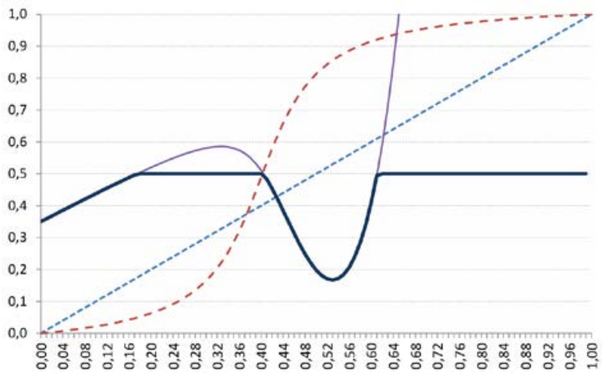

Fig. 20. Optimal positional control (solid line) The angle bisector and the function F(·) (dashed lines), The upper bound in expression (21) (thin continuous line).

The second case of positional control refers to the problem of calming the participants in the network, in which the control center is interested in reducing the share of active participants, and the possible controls on its part are the number (or part) of the calmers implemented in the network - additional participants that are always inactive.

If w ∈ [0; 1] is the share of calmers, then the dynamics of the share of active participants will satisfy the equation

$$\dot{x} = (1 - w)F(x) - x,\ x \in [0;\ 1) \tag{21}$$

Let $\tilde{w}$ (x): [0;1] → [0;1] be a positional control. Then, if the goal of the control center is to reduce the share of active participants, then constraint (19) on the magnitude of the change in the state of the social network will have the form

$$\dot{x}(t) \leq 0\ ,\ t \geq 0 \tag{22}$$

and from Eq. (21) we obtain the following condition for positional control:

$$\tilde{w}(x) \geq 1 - \frac{x}{F(x)} \tag{23}$$

The minimum limit on the magnitude of the controls at any time point in which the system (21) is controllable in the sense of Eq. (22) is equal to

$$\Delta_{\min} = \max_{x \in [0;1]} (1 - \frac{x}{F(x)})$$

Fig. 21 shows the value of the position control under condition (23) and constraint $\Delta_{\min}$ (in the example $\Delta\min = 0.385$).

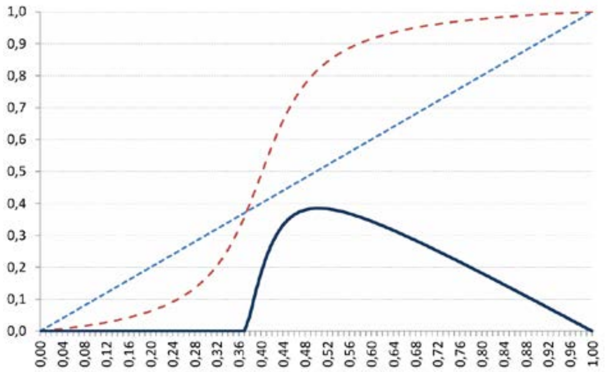

Fig. 21. Minimal positional control: positional control (bold), angle bisector and distribution function F(·) (dashed)

VI. Conclusion

This section describes the tasks of controlling the activation of participants in a social network in continuous time by introducing activators or depressants into it.

Of considerable interest for further research is the consideration of the control of a social network as a differential game, describing the situation of informational confrontation between two control centers, making decisions in continuous time about the shares (or quantities) of the introduced activators u and depressants w, respectively (the static task as a starting point in [3]). In this case, the controlled object will be described by the dynamical system.

$$\dot{x} = u(1 - w) + (1 - u - w + 2uw)F(x) - x\,.$$

Another promising direction is the consideration of the tasks of dynamic (program and/or positional) control of participants, described by a transfer equation:

$$\frac{\partial}{\partial t} p(x,t) + \frac{\partial}{\partial x}\left[u + (1-u)F(x) - x\right] p(x,t) = 0$$

i.e. a model in which the state of the participants at any point in time is described not by a scalar proportion of active participants, but by the corresponding probability distribution p(x, t).

References

[1] G. Albi, E. Cristiani, L. Pareschi, D. Peri, (2020). Mathematical Models and Methods for Crowd Dynamics Control, In: Gibelli, L. (eds) Crowd Dynamics, Vol. 2. Modeling and Simulation in Science, Engineering and Technology, pp 159–197, Birkhäuser, Cham. https://doi.org/10.1007/978-3-030-50450-2_8.

[2] N. Bellomo, C. Bianca, V. Coscia, On the modeling of crowd dynamics: An overview and research perspectives, SeMA Springer, vol. 54, pp. 25–46, 2011. https://doi.org/10.1007/BF03322586.

[3] M. Wiedermann, E.K. Smith, J. Heitzig, et al., A network-based microfoundation of Granovetter's threshold model for social tipping, Sci Rep 10, 11202, 2020. https://doi.org/10.1038/s41598-020-67102-6.

[4] E.J. Dansu, H. Seno, A Mathematical Model for the Dynamics of Information Spread under the Effect of Social Response, Interdisciplinary Information Sciences, vol. 28, no. 1, pp.75-93, July 2022. DOI:10.4036/iis.2022.R.03.

[5] П. Маринов, Д. Ганчева, Стохастичен модел на управление на социална мрежа при един център на въздействие, IX-та национална научна конференция TechCo 2025, 27 Юни 2025г, България.

[6] P. Marinov, A Model for Managing Participants in Social Networks with Discrete Variable Impact, 51nd International conference AMEE 2025, June 2025, Sozopol, Bulgaria. (in press).

[7] P.V. Marinov, "Mathematical approach to social network modelling", 32nd National Conf. with International Participation Telecom 2024, Sofia, Bulgaria, November 2024, https://ieeexplore.ieee.org/xpl/conhome/1835184/all-proceedings.

[8] M. Granovetter, Threshold Models of Collective Behavior, The American Journal of Sociology. – 1978, Vol. 83, N. 6, pp. 1420-1443.