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On the Solvability of a Fourth-Order Initial Value Problem Under Barrier Strips Conditions

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***Abstract*—This article is devoted to the existence of global solutions of the initial value problem for fourth-order nonlinear ordinary differential equation. To prove the established existence results we use a global existence theorem obtained in our previous work by a topological method, that is, the problem for the solvability of the considered initial value problem is replaced by the problem for the existence of a fixed point of a suitable introduced operator, the existence of a fixed point follows from the topological transversality theorem. The application of the global existence theorem needs a priori bounds which follow from imposed barrier strips conditions. In addition to an existence result, a result guaranteeing the existence of at least one positive, increasing (non-negative, non-decreasing), convex solution is also obtained.**

***Keywords*—barrier strips, existence, fourth-order differential equations, initial value problem**

I. INTRODUCTION

[1]We study the existence of global solutions to the initial value problem (IVP)

$$x^{(4)} = f(t, x, x', x'', x'''), t \in [0,1], \quad (1)$$

$$x(0) = A, x'(0) = B, x''(0) = C, x'''(0) = D, \quad (2)$$

where $A, B, C, D \in \mathbb{R}$, and $f(t, x, p, q, r)$ is a scalar function defined for $(t, x, p, q, r) \in [0,1] \times D_x \times D_p \times D_q \times D_r$, here the sets $D_x, D_p, D_q, D_r \subseteq \mathbb{R}$ can be bounded.

The well-known Peano theorem guarantees a local solution to problem (1), (2) if $f(t, x, p, q, r)$ is continuous and bounded in a neighborhood of $(0, A, B, C, D)$, see for example P. Hartman [4].

The solvability of IVPs has been studied in W. Mydlarczyk [7] and N. Faried et al. [2]. The first article considers the problem

$$u'''(t) = g(u(t)), t > 0,$$

$$u(0) = u'(0) = u''(0) = 0,$$

where $g: (0, \infty) \to [0, \infty)$ is continuous. The second one is devoted to the fifth-order differential equation

$$\left(-\frac{d^2}{dt^2} + A^2\right)\left(\frac{d}{dt} + A\right)^3 u(t) + \sum_{j=1}^{5} A_j \frac{d^{5-j}}{dt^{5-j}} u(t) = f(t),$$

$t \in \mathbb{R}$, with initial conditions

$$\frac{d^s u(0)}{dt^s} = 0, s = 0,1,2,3,$$

where A is a self-adjoint positively defined operator and $A_j, j = 1,2,3,4,5$, are linear unbounded operators.

Numerical methods for fourth-order IVPs have been used in K. Hussain et al. [5], S. J. Kayode et al. [6] and I. Singh and G. Singh [8]; a Runge-Kutta type method, a block[2] method and the adomian decomposition method, respectively.

In our considerations we use a priori bounds. They are provided under the following assumptions.

(H₁) There are constants F_i, L_i, $i = 1,2$, such that

$$F_2 < F_1 \le D \le L_1 < L_2, [F_2, L_2] \subseteq D_r,$$

$$f(t, x, p, q, r) \le 0 \quad (3)$$

for $(t, x, p, q, r) \in [0,1] \times D_x \times D_p \times D_q \times [L_1, L_2]$,

$$f(t, x, p, q, r) \ge 0 \quad (4)$$

for $(t, x, p, q, r) \in [0,1] \times D_x \times D_p \times D_q \times [F_2, F_1]$.

(H₂) There are constants $-\infty < m_i, M_i < \infty$, $i = \overline{0,3}$, such that

$$[m_0 - \sigma, M_0 + \sigma] \subseteq D_x, [m_1 - \sigma, M_1 + \sigma] \subseteq D_p,$$

$$[m_2 - \sigma, M_2 + \sigma] \subseteq D_q, [m_3 - \sigma, M_3 + \sigma] \subseteq D_r,$$

where $f(t, x, p, q, r)$ is continuous on the set $[0,1] \times J$,

$$J = [m_0 - \sigma, M_0 + \sigma] \times [m_1 - \sigma, M_1 + \sigma] \times$$
$$\times [m_2 - \sigma, M_2 + \sigma] \times [m_3 - \sigma, M_3 + \sigma],$$

and $\sigma > 0$ is a sufficiently small.

Let us recall that the strips $[0,1] \times [L_1, L_2]$ and $[0,1] \times [F_2, F_1]$ of (H₁) are called barrier ones, in this case for the third derivative of the eventual solutions $x \in C^4[0,1]$ to a suitable family of boundary value problems (BVPs) containing IVP (1), (2).

We use the a priori bounds to apply the basic existence theorem given in R. Agarwal and P. Kelevedjiev [1]. In fact, it is a variant of A. Granas et al. [3], Chapter I, Theorem 5.1 and Chapter V, Theorem 1.2. To formulate this theorem we consider the BVP

$$x^{(4)} + a(t)x''' + b(t)x'' + c(t)x' + d(t)x =$$
$$= f(t, x, x', x'', x'''), t \in [0,1], \quad (5)$$

$$V_i(x) = r_i, \ i = 1,2,3,4, \quad (6)$$

where $a, b, c, d \in C([0,1], \mathbb{R})$, $f: [0,1] \times D_x \times D_p \times D_q \times D_r \to \mathbb{R}$,

$$V_i(x) = \sum_{j=0}^{2} [a_{ij} x_0^{(j)} + b_{ij} x_1^{(j)}], i = \overline{1,4},$$

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with $\sum_{j=0}^{2}({a_{ij}}^2+{b_{ij}}^2)>0, i=\overline{1,4}$, and $r_i\in\mathbb{R}, i=\overline{1,4}$.

Introduce also the following family of BVPs

$$x^{(4)}+a(t)x'''+b(t)x''+c(t)x'+d(t)x= \\ =g(t,x,x',x'',x''',\lambda), t\in[0,1], \qquad (5)_\lambda$$

with boundary conditions (6), where $\lambda\in[0,1]$, the function g is defined on $[0,1]\times D_x\times D_p\times D_q\times D_r\times[0,1]$, and a,b,c,d are as above.

Let BC denote the set of functions satisfying (6), and BC_0 denote the set of functions satisfying the homogenous BCs (6), that is,

$$V_i(x)=0, i=\overline{1,4}.$$

Finally, let $C^4_{BC}[0,1]=C^4[0,1]\cap BC$ and $C^4_{BC_0}[0,1]=C^4[0,1]\cap BC_0$.

Now we are ready to formulate our basic tool.

Theorem 1.1. [Theorem 4, [1]]. *Assume that*:

(i) *Problem* $(5)_0$, (6) *has a unique solution* $x_0\in C^4[0,1]$.

(ii) *Problems* (5), (6) *and* $(5)_1$, (6) *are equivalent.*

(iii) *The map* $L_h: C^4_{BC_0}\to C[0,1]$, *defined by*

$$L_hx=x^{(4)}+a(t)x'''+b(t)x''+c(t)x'+d(t)x,$$

is one-to-one.

(iv) *Each solution* $x\in C^4[0,1]$ *to family* $(5)_\lambda$, (6) *satisfies the bounds*

$$m_i\le x^{(i)}(t)\le M_i \; for\; t\in[0,1], i=\overline{0,4},$$

where the constants $-\infty<m_i,M_i<\infty, i=\overline{1,4}$ *are independent of* λ *and* x.

(v) *There is a sufficiently small* $\sigma>0$, *such that*

$$[m_0-\sigma,M_0+\sigma]\subseteq D_x, [m_1-\sigma,M_1+\sigma]\subseteq D_p,$$

$$[m_2-\sigma,M_2+\sigma]\subseteq D_q, [m_3-\sigma,M_3+\sigma]\subseteq D_r,$$

and the function $g(t,x,p,q,r,\lambda)$ *is continuous for* $(t,x,p,q,r,\lambda)\in[0,1]\times J\times[0,1]$,

where

$$J=[m_0-\sigma,M_0+\sigma]\times[m_1-\sigma,M_1+\sigma]\times \\ \times[m_2-\sigma,M_2+\sigma]\times[m_3-\sigma,M_3+\sigma],$$

and $m_i,M_i,\; i=\overline{0,3}$ *are as in* (iv).

Then boundary value problem (5), (6) *has at least one solution in* $C^4[0,1]$.

In this paper the equation $(5)_\lambda$ has the form

$$x^{(4)}=\lambda f(t,x,x',x'',x'''), t\in[0,1]. \qquad (1)_\lambda$$

In fact, we apply Theorem 1.1 on the family of BVPs $(1)_\lambda$, (2), which is of the form $(5)_\lambda$, (6), and on the IVP (1), (2), which is of the form (5), (6).

II. A Priori Bounds

In this part we state results, which assure the a priori bounds from (iv) for the eventual $C^4[0,1]$-solutions to the family IVPs $(1)_\lambda$, (2).

Lemma 2.1. *Let* $x\in C^4[0,1]$ *be a solution to* $(1)_\lambda$, (2) *and* **(H1)** *hold. Then*

$$F_1\le x'''(t)\le L_1 \text{ for } t\in[0,1].$$

Proof. Suppose that there is a $t\in[0,1]$ for which $x'''(t)>L_1$. Then, from the continuity of $x'''(t)$ on the interval $[0,1]$ and from $x'''(0)\le L_1$ it follows that the set

$$S_-=\{t\in[0,1]: L_1<x'''(t)\le L_2\}$$

is not empty and there is a $\varphi\in S_-$ such that $x^{(4)}(\varphi)>0$. Since $x(t)$ is a $C^4[0,1]$-solution to differential equation $(1)_\lambda$, we have in particular

$$x^{(4)}(\varphi)=\lambda f\big(\varphi,x(\varphi),x'(\varphi),x''(\varphi),x'''(\varphi)\big).$$

But,

$$(\varphi,x(\varphi),x'(\varphi),x''(\varphi),x'''(\varphi))\in S_-\times\mathbb{R}^3\times(L_1,L_2].$$

This means that we can use (3) to conclude that $x^{(4)}(\varphi)\le 0$. The obtained contradiction shows that

$$x'''(t)\le L_1 \text{ for } t\in[0,1].$$

Similarly, using (4), we establish

$$F_1\le x'''(t) \text{ for } t\in[0,1].$$

Lemma 2.2. *Let* $x\in C^4[0,1]$ *be a solution to* $(1)_\lambda$, (2) *and* **(H1)** *hold. Then*

$$|x(t)|\le|A|+|B|+|C|+max\{|F_1|,|L_1|\}, t\in[0,1],$$

$$|x'(t)|\le|B|+|C|+max\{|F_1|,|L_1|\}, t\in[0,1],$$

$$|x''(t)|\le|C|+max\{|F_1|,|L_1|\}, t\in[0,1].$$

Proof. By the mean value theorem, for each $t\in(0,1]$ there exists an $\eta\in(0,t)$ such that

$$x''(t)-x''(0)=x'''(\eta)(t-0), t\in(0,1],$$

$$|x''(t)|\le|x''(0)|+|x'''(\eta)||t|, t\in(0,1],$$

$$|x''(t)|\le|C|+|x'''(\eta)|, t\in[0,1].$$

But from Lemma 2.1 we know that

$$F_1\le x'''(t)\le L_1, t\in[0,1],$$

that is,

$$|x'''(t)|\le max\{|F_1|,|L_1|\}, t\in[0,1],$$

and in particular

$$|x'''(\eta)|\le max\{|F_1|,|L_1|\}.$$

Thus,

$$|x''(t)|\le|C|+max\{|F_1|,|L_1|\}, t\in[0,1].$$

To prove the bound for $|x'(t)|$, we use again the mean value theorem. Now, for each $t\in(0,1]$ there is a $\zeta\in(0,t)$ such that

$$x'(t)-x'(0)=x''(\zeta)(t-0), t\in(0,1],$$

$$|x'(t)|\le|x'(0)|+|x''(\zeta)||t|, t\in(0,1],$$

$$|x'(t)|\le|B|+|x''(\zeta)|, t\in[0,1],$$

$$|x'(t)|\le|B|+|C|+max\{|F_1|,|L_1|\}, t\in[0,1].$$

Finally, we apply the Lagrange theorem on $x(t)$. For each $t\in(0,1]$ there is a $\nu\in(0,t)$ for which

$$x(t)-x(0)=x'(\nu)(t-0), t\in(0,1],$$

$$|x(t)|\le|x(0)|+|x'(\nu)||t|, t\in(0,1],$$

$$|x(t)|\le|A|+|x'(\nu)|, t\in[0,1],$$

$$|x(t)|\le|A|+|B|+|C|+max\{|F_1|,|L_1|\}, t\in[0,1].$$

Lemma 2.3. *Let* **(H1)** *holds and* **(H2)** *holds for*

$$M_0=|A|+|B|+|C|+max\{|F_1|,|L_1|\}, m_0=-M_0,$$

$$M_1=|B|+|C|+max\{|F_1|,|L_1|\}, m_1=-M_1,$$

$$M_2=|C|+max\{|F_1|,|L_1|\}, m_2=-M_2,$$

$$M_3=L_1, m_3=F_1.$$

Then there are constants m_4 *and* M_4 *such that* $m_4 \le x^{(4)}(t) \le M_4, t \in [0,1]$.

Proof. Because of the continuity of $f(t,x,p,q,r)$ on the set $[0,1] \times J$, there exist constants m_4 and M_4 such that

$$m_4 \le f(t,x,p,q,r) \le M_4 \text{ for } (t,x,p,q,r) \in [0,1] \times J.$$

Since from Lemmas 2.1 and 2.2 we have $(x(t), x'(t), x''(t), x'''(t)) \in J$ for $t \in [0,1]$, equation $(1)_\lambda$ implies

$$m_4 \le x^{(4)}(t) \le M_4 \text{ for } t \in [0,1].$$

Corollary 2.4. *Let* $A,B,C,D \ge 0$ *and* **(H1)** *hold with* $F_1 \ge 0$. *Then each solution* $x(t) \in C^4[0,1]$ *to* $(1)_\lambda$, (2) *satisfies the bounds*

$$A \le x(t) \le A + B + C + L_1, t \in [0,1],$$
$$B \le x'(t) \le B + C + L_1, t \in [0,1],$$
$$C \le x''(t) \le C + L_1, t \in [0,1].$$

Proof. By Lemma 2.1 we have $x'''(t) \ge F_1 \ge 0, t \in [0,1]$, which means that $x''(t)$ is non-decreasing on $[0,1]$ and so

$$x''(t) \ge C, t \in [0,1].$$

Besides, by applying Lemma 2.2, we get

$$x''(t) \le C + L_1, t \in [0,1].$$

Next, using $x''(t) \ge C \ge 0$ for $t \in [0,1]$ and Lemma 2.2, we establish

$$B \le x'(t) \le B + C + L_1, t \in [0,1].$$

The bound for $x(t)$ follows similarly.

III. Existence Results

Theorem 3.1. *Let* **(H1)** *holds. Let in addition* **(H2)** *holds for*

$$M_0 = |A| + |B| + |C| + \max\{|F_1|, |L_1|\}, m_0 = -M_0,$$
$$M_1 = |B| + |C| + \max\{|F_1|, |L_1|\}, m_1 = -M_1,$$
$$M_2 = |C| + \max\{|F_1|, |L_1|\}, m_2 = -M_2,$$
$$M_3 = L_1, m_3 = F_1.$$

Then IVP (1),(2) *has at least one solution in* $C^4[0,1]$.

Proof. We will show that the family of BVPs $(1)_\lambda$, (2) and the IVP (1), (2) satisfy all hypotheses of Theorem 3.1. By standard reasoning check that (i) is fulfilled, see the proof of Theorem 2.5 of T. Todorov [9]. Apparently (ii) also holds. To check (iii) we establish, by standard reasoning, that for an arbitrary y(t) ∈ C[0, 1] the IVP

$$x^{(4)} = y(t),$$
$$x(0) = 0, x'(0) = 0, x''(0) = 0, x'''(0) = 0,$$

has a unique solution in $C^4[0,1]$, that is, the map $\Lambda_h: C^4_{\mathrm{BC}_0}[0,1] \to C[0,1]$, defined by $\Lambda_h x = x^{(4)}$, is one-to-one. Hence, (iii) holds. Furthermore, for each solution $x(t) \in C^4[0,1]$ to $(1)_\lambda$, (2) we have

$$m_i \le x^{(i)}(t) \le M_i, t \in [0,1], i = \overline{0,4}, \text{ by Lemma 2.3.}$$

Therefore, (iv) also holds. Finally, (v) follows from the continuity of f on the set J. Thus, we can apply Theorem 3.1 to conclude that the assertion is true.

Under a suitable combination of the signs of A, B, C and D, (H1) guarantees solutions with important properties.

Theorem 3.2. *Let* $A, B > 0 (A = B = 0), C \ge 0, D \ge 0$. *Suppose* **(H1)** *holds with* $F_1 \ge 0$ *and* **(H2)** *holds for*

$$m_0 = A, M_0 = A + B + C + L_1, m_1 = B + C,$$
$$M_1 = B + C + L_1,$$
$$m_2 = C, M_2 = C + L_1, m_3 = F_1, M_3 = L_1.$$

Then IVP (1), (2) *has at least one positive, increasing (non-negative, non-decreasing), convex solution in* $C^4[0,1]$.

Proof. Following the proof of Theorem 3.1, we establish that there is a solution $x(t) \in C^4[0,1]$. Now, the bounds $m_i \le x^{(i)}(t) \le M_i, t \in [0,1], i = 0,1,2,$ follow from Corollary 2.4. Actually Corollary 2.4 implies in particular

$$x(t) \ge A > 0, x'(t) \ge B > 0 \quad (x(t) \ge 0, x'(t) \ge 0),$$
$$x''(t) \ge C \ge 0 \text{ for } t \in [0,1],$$

which yields the assertion.

IV. An Example

Consider the initial value problem

$$x^{(4)} = P_n(x'''), t \in [0,1],$$
$$x(0) = A, x'(0) = B, x''(0) = C, x'''(0) = D,$$

where the polynomial $P_n(r), n \ge 2$, has simple zeros z_1 and z_2 such that $z_1 < D < z_2$.

Obviously, there is a $\theta > 0$ such that

$$z_1 + \theta \le D \le z_2 - \theta$$

and

$$P_n(p) \ne 0 \text{ for } r \in \bigcup_{i=1}^{2} ((z_i - \theta, z_i + \theta) \backslash \{z_i\}).$$

Let $P_n(r) > 0$ for

$$r \in (z_1 - \theta, z_1) \text{ and } P_n(r) < 0 \text{ for } p \in (z_2, z_2 + \theta);$$

analogously we can consider the other cases. Then **(H1)** holds for $F_2 = z_1 - \theta, F_1 = z_1, L_1 = z_2, L_2 = z_2 + \theta$. Since $P_n(r)$ is continuous, **(H2)** also holds. So, we can apply Theorem 3.1 which means the considered problem has at least one solution in $C^4[0,1]$.

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