On the Solvability of a Two-Point Third-Order Boundary Value Problem Under Barrier Strips

G. Mihaylova, P. Kelevedjiev

***Abstract*—We study the solvability of a two-point boundary value problem for nonlinear third-order ordinary differential equation. To prove the established existence results, we use a well-known global solvability theorem obtained by a topological method which means that the problem for the solvability of the considered boundary value problem is replaced by the problem for the existence of a fixed point of a suitable introduced operator. The application of the global theorem requires a priori bounds. These are obtained using the barrier strips technique. A result guaranteeing positive or non-negative, concave solutions is given also. The existence results are illustrated by an example.**

***Keywords*—Third-order differential equations, boundary value problem, existence, barrier strips**

I. INTRODUCTION

We consider the differential equation

$$x''' = f(t, x, x', x''), t \in (0,1), \quad (1)$$

with boundary conditions (BCs)

$$x(0) = A, \; x(1) = B, \; x'(1) - x'(0) = C \quad (2)$$

where $A, B, C \in \mathbb{R}$, and $f(t, x, p, q)$ is a scalar function, defined for $(t, x, p, q) \in [0,1] \times D_x \times D_p \times D_q$, here the sets $D_x, D_p, D_q \subseteq \mathbb{R}$ can be bounded.

A number of authors study the solvability of boundary value problems (BVPs) for equations of the form

$$x''' = f(t, x), t \in (0,1).$$

Such works are A. Cabada [3], with BCs

$$x^{(i)}(0) - x^{(i)}(1) = \gamma_i, \; \gamma_i \in \mathbb{R}, i = 0,1,2,$$

and Zh. Liu et al. [20] where the BCs are

$$x(0) = x'(0), \; \alpha x'(1) + \beta x''(1) = \gamma, \; \gamma > 0, \; \alpha, \beta \geq 0.$$

BVPs for equation (1) with various BCs have been studied in R. P. Agarwal et al. [1], J. R. Graef et al. [4], A. Granas et al. [5], M. Grossinho et al. [6], P. Kelevedjiev and Z. Todorov [8], Y. Li and Y. Li [9], D. M. Zhang and Y. U. Lu [10] and Y. Zhang and M. Pei [11].

To prove the existence of solutions to (1), (2), we apply the basic existence theorem from R. P. Agarwal et al. [2]. It is a variant of [5], Chapter V, Theorem 1.1, and its formulation requires to consider the BVP

$$x^{(n)} + \sum_{k=0}^{n-1} s_k(t) x^{(k)} = \\ = f(t, x, x', \ldots, x^{(n-1)}), t \in (0,1), \quad (3)$$

$$V_i(x) = A_i, \qquad i = \overline{1,n}, \quad (4)$$

where $s_k(t)$, $k = \overline{0, n-1}$, are continuous on $[0,1]$, $f: [0,1] \times D_0 \times D_1 \times \ldots \times D_{n-1} \to \mathbb{R}$,

$$V_i(x) = \sum_{j=0}^{n-1} [a_{ij} x^{(j)}(0) + b_{ij} x^{(j)}(1)], i = \overline{1,n},$$

where a_{ij} and b_{ij} are constants such that $\sum_{j=0}^{n-1} ({a_{ij}}^2 + {b_{ij}}^2) > 0, i = \overline{1,n}$, and $A_i \in \mathbb{R}, i = \overline{1,n}$.

For $\lambda \in [0,1]$ consider also the family of BVPs

$$x^{(n)} + \sum_{k=0}^{n-1} s_k(t) x^{(k)} = \\ = g(t, x, x', \ldots, x^{(n-1)}, \lambda), t \in (0,1), \quad (3)_\lambda$$

with boundary conditions (4). Here, the function g is defined on $[0,1] \times D_0 \times D_1 \times \ldots \times D_{n-1} \times [0,1]$, and $s_k(t)$, $k = \overline{0, n-1}$, V_i, $A_i, i = \overline{1,n}$, are as above.

Let BC denote the set of functions satisfying (4), and BC_0 denote the set of functions satisfying the homogenous BCs (4), that is,

$$V_i(x) = 0, i = \overline{1,n}.$$

Finally, let $C_{BC}^n[0,1] = C^n[0,1] \cap BC$.

Theorem 1.1. *Assume that*:

(i) *Problem* $(3)_0$, (4) *has an unique solution* $x_0 \in C^n[0,1]$.

(ii) *Problems* (3), (4) *and* $(3)_1$, (4) *are equivalent.*

(iii) *The map* $\Lambda_h: C_{BC_0}^n \to C[0,1]$, *defined by*

$$\Lambda_h x = x^{(n)} + \sum_{k=0}^{n-1} s_k(t) x^{(k)},$$

is one-to-one.

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(iv) *Each solution* $x \in C^n[0,1]$ *to family* $(3)_\lambda$, (4) *satisfies the bounds*

$$m_i \le x^{(i)}(t) \le M_i \; for \; t \in [0,1], i = \overline{0,n},$$

where the constants $-\infty < m_i, M_i < \infty, i = \overline{1,n},$ *are independent of* λ *and* x.

(v) *There is a sufficiently small σ>0, such that*

$$[m_i - \sigma, M_i + \sigma] \subseteq D_i, i = \overline{0, n-1},$$

and the function $g(t, p_0, p_1, \dots, p_{n-1}, \lambda)$ *is continuous on* $[0,1] \times J \times [0,1]$, *where*

$$J = [m_0 - \sigma, M_0 + \sigma] \times [m_1 - \sigma, M_1 + \sigma] \times \dots \times \\ \times [m_{n-1} - \sigma, M_{n-1} + \sigma],$$

and $m_i, M_i, \; i = \overline{0, n-1}$ *are as in* (iv).

Then boundary value problem (3), (4) *has at least one solution in* $C^n[0,1]$**.**

Let us notice that the considered BVP (1), (2) is a particular case of boundary value problem (3), (4) and we can use Theorem 1.1 with $n = 3$.

In our consideration, the equation $(3)_\lambda$ has the form

$$x''' = \lambda f(t, x, x', x''), t \in [0,1]. \qquad (1)_\lambda$$

In fact, we apply Theorem 1.1 on the family of BVPs for $(1)_\lambda$, (2), which is of the form $(3)_\lambda$, (4). Moreover, we need results, which assure the a priori bounds from (iv) for the eventual $C^3[0,1]$-solutions to the family boundary value problems $(1)_\lambda$, (2). These auxiliary results rely on the following assumptions:

(H1) There are constants F_i, L_i, $i = 1,2$, such that

$$F_2 < F_1 \le C \le L_1 < L_2, [F_2, L_2] \subseteq D_q,$$
$$f(t,x,p,q) \ge 0$$
$$\text{for } (t,x,p,q) \in [0,1] \times D_x \times D_p \times [L_1, L_2],$$
$$f(t,x,p,q) \le 0$$
$$\text{for } (t,x,p,q) \in [0,1] \times D_x \times D_p \times [F_2, F_1].$$

(H2) There are constants F_i', L_i', $i = 1,2$, such that

$$F_2' < F_1' \le C \le L_1' < L_2', [F_2', L_2'] \subseteq D_q,$$
$$f(t,x,p,q) \le 0$$
$$\text{for } (t,x,p,q) \in [0,1] \times D_x \times D_p \times [L_1', L_2'],$$
$$f(t,x,p,q) \ge 0$$
$$\text{for } (t,x,p,q) \in [0,1] \times D_x \times D_p \times [F_2', F_1'].$$

(H3) There are constants $m_i \le M_i, \; i = \overline{0,2}$, such that

$$[m_0 - \sigma, M_0 + \sigma] \subseteq D_x, \; [m_1 - \sigma, M_1 + \sigma] \subseteq D_p,$$
$$[m_2 - \sigma, M_2 + \sigma] \subseteq D_q,$$

and $f(t,x,p,q)$ is continuous on $[0,1] \times J$, where

$$J = [m_0 - \sigma, M_0 + \sigma] \times [m_1 - \sigma, M_1 + \sigma] \times \\ \times [m_2 - \sigma, M_2 + \sigma]$$

and $\sigma > 0$ is sufficiently small.

Let us recall, the conditions **(H1)** and **(H2)** are of barrier strips type, see P. Kelevedjiev [7]. The barrier strips technique has been used also in [1, 8, 10, 11].

II. EXISTENCE RESULTS

The proofs of the following two lemmas can be found in R. P. Agarwal et al. [1].

Lemma 2.1. (Lemma 2, [1]) *Let* $x \in C^3[a,b]$ *be a solution to* $(1)_\lambda$. *Suppose **(H1)** holds with [0,1] replaced by* $[a,b]$ *and* $x''(b) = C$. *Then*

$$F_1 \le x''(t) \le L_1 \text{ for } t \in [a,b].$$

Lemma 2.2. (Lemma 3, [1]) *Let* $x \in C^3[a,b]$ *be a solution to* $(1)_\lambda$. *Suppose **(H2)** holds with [0,1] replaced by* $[a,b]$ *and* $x''(a) = C$. *Then*

$$F_1' \le x''(t) \le L_1' \; for \; t \in [a,b].$$

Lemma 2.3. *Let* $x \in C^3[0,1]$ *be a solution of family* $(1)_\lambda$, *(2). Assume that **(H1)** and **(H2)** hold. Then*

$$min\{F_1, F_1'\} \le x''(t) \le max\{L_1, L_1'\}, t \in [0,1].$$

Proof. We know $x'(t)$ is continuous and differentiable in [0,1]. Therefore we can apply Lagrange theorem, according to which there is a $\mu \in (0,1)$, such that

$$x'(1) - x'(0) = x''(\mu)(1 - 0), \\ x''(\mu) = x'(1) - x'(0) = C \qquad (5)$$

It is not difficult to see that the conditions of Lemma 2.1 are satisfied on the interval [0, μ]. According to this lemma we have

$$F_1 \le x''(t) \le L_1 \; for \; t \in [0, \mu].$$

On the other hand Lemma 2.2 yields

$$F_1' \le x''(t) \le L_1' \; for \; t \in [\mu, 1].$$

As a result, we get the statement in the whole interval [0,1].

Lemma 2.4. *Let* $x(t) \in C^3[a,b]$ *be a solution of family* $(1)_\lambda$, *(2). Assume that **(H1)** and **(H2)** hold. Then*

$$|x'(t)| \le |B - A| + max\{|F_1|, |F_1'|, |L_1|, |L_1'|\}, t \in [0,1],$$

$$|x(t)| \le |A| + |B - A| + \max\{|F_1|, |F_1'|, |L_1|, |L_1'|\},$$
$t \in [0,1]$,

Proof. According to the mean value theorem there is a $\nu \in (0,1)$, such that

$$x(1) - x(0) = x'(\nu)(1 - 0),$$

i.e.,

$$x'(\nu) = B - A.$$

Now for each $t \in (\nu, 1)$, there is a $\xi \in (\nu, t)$, with the property

$$x'(t) - x'(\nu) = x''(\xi)(t - \nu),$$

$$x'(t) = x'(\nu) + x''(\xi)(t - \nu),$$

$$|x'(t)| = |x'(\nu) + x''(\xi)(t - \nu)| \le \\ \le |x'(\nu)| + |x''(\xi)(t - \nu)| = \\ = |B - A| + |x''(\xi)||t - \nu| \le |B - A| + |x''(\xi)|.$$

But, $|x''(\xi)| \leq max\ \{|F_1|, |F_1'|, |L_1|, |L_1'|\}$ by Lemma 2.3. Thus,

$$|x'(t)| \leq |B - A| + \max\{|F_1|, |F_1'|, |L_1|, |L_1'|\}$$

for $t \in [\nu, 1]$. Similarly, for each $t \in [0, \nu)$, there is an $\eta \in (t, \nu)$, such that

$$x'(\nu) - x'(t) = x''(\eta)(\nu - t),$$

from where, as above, obtain

$$|x'(t)| \leq |B - A| + \max\{|F_1|, |F_1'|, |L_1|, |L_1'|\}$$

for $t \in [0, \nu]$ and so

$$|x'(t)| \leq |B - A| + \max\{|F_1|, |F_1'|, |L_1|, |L_1'|\} \quad (6)$$

for $t \in [0,1]$.

Next, again from the mean value theorem, for all $t \in (0,1]$, there is a $\theta \in (0, t)$ for which

$$x(t) - x(0) = x'(\theta)(t - 0),$$

$$x(t) = x(0) + x'(\theta)t,$$

from where, using (6), establish the bound for $|x(t)|$.

Theorem 2.5. *Let **(H₁)** and **(H₂)** hold. Let in addition **(H₃)** hold for*

$$M_0 = |A| + |B-A| + max\{|F_1|, |F_1'|, |L_1|, |L_1'|\},\ m_0 = -M_0,$$

$$M_1 = |B-A| + max\{|F_1|, |F_1'|, |L_1|, |L_1'|\},\ m_1 = -M_1,$$

$$M_2 = max\{|L_1|, |L_1'|\},\ m_2 = min\{|F_1|, |F_1'|\}.$$

Then BVP (1),(2) has at least one solution in $C^3[0,1]$.

Proof. We will check that the family of BVPs $(1)_\lambda$, (2) and the BVP (1), (2) satisfy all hypotheses of Theorem 1.1. To verify (i), we have to show that the BVP

$$x''' = 0,$$
$$x(0) = A, x(1) = B, x'(1) - x'(0) = C,$$

has a unique solution. The general solution of the differential equation is

$$x(t) = c_1 + c_2\,t + c_3\,t^2$$

with

$$x'(t) = c_2 + 2\,c_3\,t$$

Next, to find a solution to the BVP, we use the BCs and obtain the system

$$\begin{vmatrix} c_1 & = A \\ c_1 + c_2 + c_3 & = B \\ 2c_3 & = C \end{vmatrix}$$

Its determinant is

$$\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 2 \end{vmatrix} = 2(-1)^{3+3}\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 2.1.1 = 2 \neq 0.$$

Consequently the system has a unique solution (c_1^*, c_2^*, c_3^*), and

$$x(t) = c_1^* + c_2^*\,t + c_3^*\,t^2$$

is the unique solution of the BVP, that is, (i) of Theorem 1.1 holds. Apparently (ii) also holds. Next, by standard reasoning, we check that for an arbitrary $y(t) \in C[0,1]$ the BVP

$$x''' = y(t),$$

$$x(0) = 0, x(1) = 0, x'(1) - x'(0) = 0,$$

has a unique solution in $C^3[0,1]$. So, the map $\Lambda_h: C^3_{BC_0}[0,1] \to C[0,1]$, defined by $\Lambda_h x = x'''$, is one-to-one. Thus, (iii) holds. Furthermore, for each solution $x(t) \in C^3[a,b]$ to $(1)_\lambda$, (2) we have

$$m_0 \leq x(t) \leq M_0, t \in [0,1], \text{by Lemma 2.4},$$

$$m_1 \leq x'(t) \leq M_1, t \in [0,1], \text{by Lemma 2.4},$$

$$m_2 \leq x''(t) \leq M_2, t \in [0,1], \text{by Lemmas 2.3}.$$

Because of the continuity of f on [0,1]×J, there are constants m_3 and M_3 such that

$$m_3 \leq \lambda f(t, x, p, q) \leq M_3$$

for $\lambda \in [0,1]$ and $(t, x, p, q) \in [0,1] \times J$. But, for $t \in (0,1)$we have $(x(t), x'(t), x''(t)) \in J$. Thus, the equation $(1)_\lambda$, implies

$$m_3 \leq x'''(t) \leq M_3\ for\ t \in [0,1].$$

Hence, (iv) also holds. Finally, (v) follows again from the continuity of f on the set J. Therefore, we can apply Theorem 1.1 to conclude that the assertion is true.

Under a suitable combination of the signs of A,B,C and D, **(H₁)** and **(H₂)** guarantee solutions with important properties.

Theorem 2.6. *Let $A, B > 0$ $(A \geq 0, B \geq 0)$ and $C < 0$. Suppose **(H₁)** and **(H₂)** hold with $L_1, L_1' \leq 0$ and **(H₃)** holds for $m_i, M_i, i = 0,1,2$, as in Theorem 2.5. Then BVP (1), (2) has at least one positive (non-negative), concave solution in* $C^3[0,1]$.

Proof. According to Theorem 2.5, BVP (1), (2) has a solution $x(t) \in C^3[a,b]$. For this solution we know that

$$x''(t) \leq max\{L_1, L_1'\} \leq 0, t \in [0,1],$$

from where the assertion follows immediately.

Example 2.7. *Consider the BVP*

$$x''' = P_n(x''), t \in [0,1],$$

$$x(0) = 2,\ \ x(1) = 1,\ \ x'(1) - x'(0) = -4,$$

where the polynomial $P_n(r), r \geq 2$ has simple zeros r_1 and r_2 such that $r_1 < -4 < r_2 < 0$.

Consider the case

$$P_n(r) > 0 \text{ for } r \in (r_2, r_2 + \theta)$$

and

$$P_n(r) < 0 \text{ for } \in (r_1 - \theta, r_1)$$

where $\theta > 0$ is such that

$$r_1 + \theta \leq -4 \leq r_2 - \theta,$$

$$r_2 + \theta \leq 0,$$

and

$$P_n(r) \neq 0 \;\; for\; r \in U_{i=2}^{2}((r_i - \theta, r_i + \theta)\backslash\{r_i\});$$

in the other cases, for the sign of $P_n(r)$ around the zeros, the reasoning is analogous. It is not difficult to see that **(H1)** is satisfied for

$$F_2 = r_1 - \theta, F_1 = r_1, L_1 = r_2 \text{ and } L_2 = r_2 + \theta,$$

(H2) is satisfied for

$$F_2' = r_1,\; F_1' = r_1 + \theta,\; L_1' = r_2 - \theta \text{ and } L_2' = r_2,$$

and **(H3)** is obvious. So, we can apply Theorem 2.6 to conclude that the considered problem has a positive, concave solution in $C^3[0,1]$.

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