Pade Series Approximation of Static Nondifferentiable Nonlinearities

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***Abstract* – This paper considers the problem of smooth function approximation of certain nondifferentiable nonlinear characteristics by applying Pade series. Unified approach for this approximation is presented, which is based on using the ideal relay characteristic to describe the nonlinear effect of the approximated nonlinearity. The ideal relay serves as a switching device to carry out changes in the nonlinear element behavior. The main idea of the presented approach is to transform the problem of smooth function approximation of certain nonlinear characteristics to the problem of smooth function approximation of the ideal relay characteristic. The jumping in the relay behavior is replaced by parameterized hyperbolic tangent function, which on its own turn is described in terms of an exponential function. Furthermore, Pade series is used to approximate the exponential function in the hyperbolic tangent expression. Some popular nonlinear characteristics are considered and the approximation mechanism is explained by considering different ranges of operation for the input argument. The approximation power is increased by using parametrization of the hyperbolic tangent function.**

***Index Terms*—hyperbolic tangent function, ideal relay, nondifferentiable nonlinearities, Pade series, smooth function approximation**

I. Introduction

Most physical processes in practice contain elements with nonlinear characteristics. A specific feature of such processes is the inability to apply linear methods for system analysis and design. In order to overcome such problems is to apply system linearization around the operating point. The linearization techniques allow using linear techniques for nonlinear control problems. One of the difficulties of using linear techniques is the range of system operation, i.e. the operating signals should be small and the system behavior is reliable only in small neighborhood around the equilibrium position. Therefore, linearization requires and introduces some limitations on the system behavior.

Another problem with linearization is the requirement for differentiability of the nonlinear characteristics under consideration. The condition for differentiability of nonlinear characteristics is additional limitation over the system description. In this sense, linearization is reliable only when the nonlinear element characteristics are continuous and have unique derivatives around the point of operation. Nonlinear systems, whose characteristics are not differentiable around the point of operation, are not linearizable. Such characteristics include relays, all kinds of linearizable. Such characteristics include relays, all kinds of hysteresis characteristics, backlash, saturation, dead zone and others. For some of these characteristics, the nonlinear part is not essential for the system operation. Such characteristics can be replaced by certain linear characteristics. For some other characteristics however, the nonlinear part is essential for system operation and they cannot be replaced by linear models. Such characteristics include abrupt changes in system behavior or discontinuous jumps in the nonlinearity description. One possible approach for such system element description is to replace the nondifferentiable part with a certain analytic approximation and to use the differentiable approximation for solving the system problems.

One possible application of this approach is when the nonlinear system is presented in terms of Volterra series. Two main difficulties in using Volterra series representations are the series convergence and the Volterra kernels computation. These difficulties are consequence of using nondifferentiable nonlinearities in system description. One possible way to avoid such difficulties is to use their analytical approximation. Another application of the analytic approximation approach is the sliding mode operation in robotics. In sliding mode operation, the system trajectory switches abruptly from one regime of operation to another. The switching behavior is governed by using the relay element in the control algorithm implementation. The relay type switching in the system behavior leads to chattering in the performance of the switching mechanism. The fast relay switching leading to chattering causes also fast oscillation of the state variables. When analyzing the sliding mode operation, such kind of performance is not desirable. The usual approach to reduce the effect of such behavior is to use analytic approximation of the relay switching operation. In many other applications, the ideal relay characteristic having jump at the zero point, is replaced by the characteristic of saturation, which has continuous kind of behavior. One more application of the analytic approximation approach is when using the Lyapunov theory for exploring system stability and design. It is well known that the Lyapunov function needs to be differentiable. The analytic approximation approach allows differentiation at the point of switching.

This paper considers the problem of analytic approximation of certain nondifferentiable nonlinear characterristics. The nondifferentiable nonlinear characteristics, which are presented are some popular static nonlinearities. It is shown that, the nondifferentiable nonlinear characteristics can be presented by using the ideal relay characteristic. The ideal relay characteristic exhibits jump behavior from one level to another. This jump behavior serves to model the switching performance between positive and negative levels of operation. Therefore, in order to approximate smoothly

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the nondifferentiable nonlinear characteristic, it is desirable to use analytic approximation of the ideal relay. The ideal relay analytic approximation is achieved by using Pade series representation. The factors, which increase the accuracy of such approximation, are also considered.

II. Analytic Approximation of the Ideal Relay Characteristic

The ideal relay characteristic can be presented by the following expression:

$$f(x) = sgn(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1 & x < 0 \end{cases} \quad (1)$$

This characteristic is nondifferentiable at the point $x = 0$, where the relay behavior changes with jump. Therefore, the relay characteristic at this point is not only nondifferentiable, but it is also discontinuous. The discontinuous jump of any nonlinear characteristic imposes heavy limitations for its analytic approximation. Therefore, in order to assure differentiability, one has to look for some popular practical approximation procedures. There exists no convergent Taylor series representation of the relay characteristic, which is due to the jump behavior at the zero point. Orthogonal series representations have also very slow rate of convergence. One possible solution of the approximation problem for the ideal relay (1) is by employing gate functions [5]. By using the mechanism of gate function approximation, we present the relay characteristic in terms of a hyperbolic tangent function as:

$$f(x) = sgn(x) \cong tanh(Ax), \quad (2)$$

where A is a parameter, determining the error of approximation [4]. Figure 1 shows the gate function approximation of the ideal relay, for different values of the parameter A. It is shown that, for small values of the parameter, $A = 1$ (- - -) or $A = 2$ (….), the gate function clearly deviates from the true characteristic. For medium values $A = 10$ (-.-.-), the error of approximation clearly reduces, while for $A = 100$ (---) or larger than hundred, the error of approximation is very small.

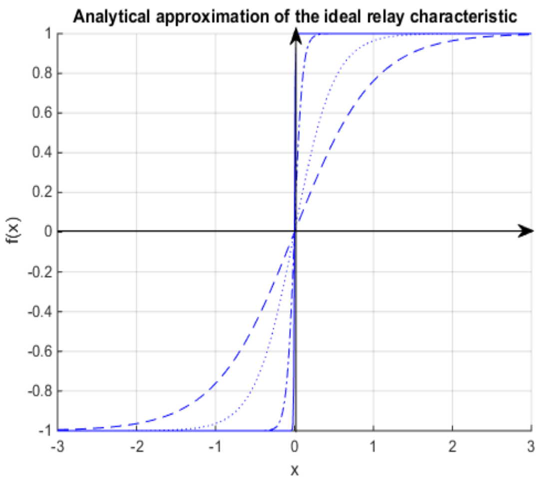

Fig. 1. Hyperbolic tangent approximation of the ideal relay

The computer representation of the hyperbolic tangent function is discussed in [2]. We consider a floating point base B with t fractional digits. It is shown that there exist four different regions, where the $tanh(x)$ is computed. In the interval from zero to x_{small}, the hyperbolic tangent function representation is $\tanh(x) = x$, which is the Taylor series expansion, truncated to the first term. The upper bound value is determined as: $x_{small} = \sqrt{3}B^{(-t-1)/2}$ and for double precision representation, $x_{small} = 1.29 \cdot 10^{-8}$. In the interval from x_{small} to x_{medium}, $tanh(x)$ is represented by accurate rational polynomial approximation, developed in [3]. The value of the upper bound is obtained as $x_{medium} = \ln(2B-1)/2$, which for double precision representation is computed as: $x_{medium} = 0.5493$. The main interest for the hyperbolic tangent computation is the third interval from x_{medium} to x_{large}, which is computed as $\tanh(x) = 1 - \frac{2}{1+\exp(2x)}$, which can be derived from using the formula $\tanh(x) = \frac{\exp(x)-\exp(-x)}{\exp(x)+\exp(-x)}$. The value of the upper bound is computed as $x_{large} = (t+2)\ln(2)/2$ [2]. The value of x_{large} for double precision arithmetic is $x_{large} = 19.06154$. For values of x larger than x_{large}, $\tanh(x) = 1$. Since in the derivations to follow, we use the scaling parameter A, the range of operation is significantly reduced, and the argument of approximation is restricted to two intervals: for $x \in [0, x_{large}]$ and for $x > x_{large}$. In the first interval, $\tanh(Ax) = 1 - \frac{2}{1+\exp(2Ax)}$ and in the second interval $\tanh(Ax) \cong 1$. In order to avoid using exponential terms in the denominator, we assume rational function approximation of the function $exp(x)$ in terms of Pade series [4].

The Pade series approximation of the exponential function is a rational function, which is developed from the Taylor series and contains as many parameters as the corresponding parameters in the Taylor series. For a Taylor series expansion with N parameters, a Pade series representation can be obtained as follows:

$$P_{L,M}(x) = \frac{a_0+a_1x+a_2x^2+\cdots+a_Lx^L}{1+b_1x+b_2x^2+\cdots+b_Mx^M}, \quad (3)$$

where $L + M + 1 = N$. In order to compute the Pade series parameters, we use the correspondence between Taylor series and Pade series coefficients, as follows [1]:

$$\sum_{i=0}^{\infty} c_i x^i = \frac{a_0+a_1x+a_2x^2+\cdots+a_Lx^L}{1+b_1x+b_2x^2+\cdots+b_Mx^M} \quad (4)$$

can be written as:

$$(c_0 + c_1x + c_2x^2 + \cdots + c_{L+M}x^{L+M} + \cdots)(1 + b_1x + \cdots + b_Mx^M) = (a_0 + a_1x + \cdots + a_Lx^L)$$

The above equality transforms to the following set of equations:

$$\begin{bmatrix} c_1 & c_2 & c_3 & \cdots & c_7 \\ c_2 & c_3 & c_4 & \cdots & c_8 \\ c_3 & c_4 & c_5 & \cdots & c_9 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_7 & c_8 & c_9 & \cdots & c_{13} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_7 \end{bmatrix} = - \begin{bmatrix} c_8 \\ c_9 \\ c_{10} \\ \vdots \\ c_{14} \end{bmatrix}, \quad (5)$$

where in (5), we have considered the case $L = M = 7$. The parameters a_i, $i = 0,1,\cdots,7$ can be obtained from the equations:

$$a_0 = c_0,$$
$$a_1 = c_0b_1 + c_1, \ldots, a_7 = c_0b_7 + c_1b_6 + \cdots + c_6b_1 + c_7 \quad (6)$$

Solving equations (5) and (6), we obtain the expression for $P_{7,7}(x)$ as follows:

$$P_{7,7}(x) = \frac{1+\beta_1 x+\beta_2 x^2+\cdots+\beta_7 x^7}{1-\beta_1 x+\beta_2 x^2-\cdots-\beta_7 x^7}, \quad (7)$$

where the parameters β_i, $i = 1,2,\cdots,7$ are computed as in [4]:
$\beta_1 = 0.5$, $\beta_2 = 0.1154$, $\beta_3 = 0.016$, $\beta_4 = 0.0015$, $\beta_5 = 8.74 \cdot 10^{-5}$, $\beta_6 = 3.24 \cdot 10^{-5}$, $\beta_7 = 5.78 \cdot 10^{-8}$.
We approximate the exponential function $exp(2Ax) \approx P_{7,7}(2Ax)$ and therefore, the Pade series approximations for the ideal relay characteristic on the interval $x \in [0, x_{large}/A]$ can be computed as: $sgn(x) \approx \tanh(Ax) \approx 1 - \frac{2}{1+P_{7,7}(2Ax)}$. For $x > x_{large}/A$, the value of the hyperbolic tangent function is one.

Remark

From the above derivations follows that, the approximation of the ideal depends to a great extend on the parameter A. The larger the parameter A, the smaller is the interval length, where the differentiable approximation is defined. The upper bound on this interval is x_{large}/A, i.e. for $A = 100$, the upper bound is $x_{large}/A = 0.19062$. However, this circumstance does not create any difficulties for the approximation problem, since the discontinuous jump in the relay characteristic appears at $x = 0$ and the main effort to overcome the abrupt change in the jump behavior is exactly in the zero point and in close neighborhood around it. In this sense, no matter how small the approximation interval is, the goal of smooth replacement of the switching jump will be satisfied.

III. Smooth Function Approximation of Nondifferentiable Nonlinearities

The smooth approximation of the ideal relay characteristic serves as a base for smooth function approximation of many other nonlinear characteristics. Consider first the dead zone nonlinear characteristic, which is presented in fig.2.

$$f(x) = \begin{cases} 0, & |x| \le a \\ x-a, & x > a \\ x+a, & x < -a \end{cases}, \quad (8)$$

where a is a parameter, showing the length of the dead zone interval. In the case presented in fig.2, the parameter $a = 1.0$. Obviously, at the points $x = \pm a$, the dead zone nonlinear characteristic is not differentiable.

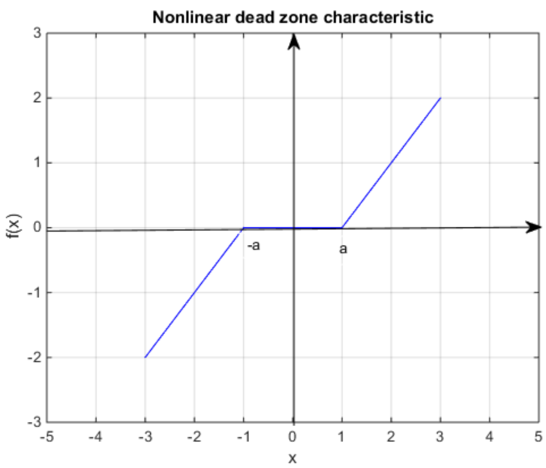

Fig. 2. Dead zone nonlinear characteristic

One possible way to approximate the dead zone nonlinearity is by using smooth approximation of the ideal relay element. The dead zone characteristic can be presented by the following expression:

$$f(x) = 0.5(x-a)[sgn(x-a)+1] + \\ +0.5(x+a)[sgn(-x-a)+1] \quad (9)$$

From expression (9) is clear that, when $|x| < a$, $sgn(x - a) = -1$ and the first term in expression (9) is zero, and similarly $sgn(-x-a) = -1$ and the second term in expression (9) is zero. When $x = a$, the first term in expression (9) is zero, since $(x - a) = 0$. The second term in expression (9) is also zero, since $sgn(-x - a) + 1 = 0$. For $x > a$, $sgn(x - a) = 1$ and the first term in (9) is equal to $(x - a)$, while the second term in expression (9) is zero, since $sgn(-x - a) + 1 = 0$. Similarly, for $x < -a$, the first term in (9) is zero, since $sgn(x - a) + 1 = 0$, while the second term in (9) is equal to $(x + a)$. Obviously, the problem of dead zone characteristic smooth approximation reduces to smooth approximation of the ideal relay characteristic, which matter was discussed in section two.

We consider now the characteristic of nonlinearity with saturation, presented in fig.3. The nonlinearity with saturation characteristic can be described by the following expression:

$$f(x) = \begin{cases} kx, & |x| \le b \\ c, & x > b \\ -c, & x < -b \end{cases}, \quad (10)$$

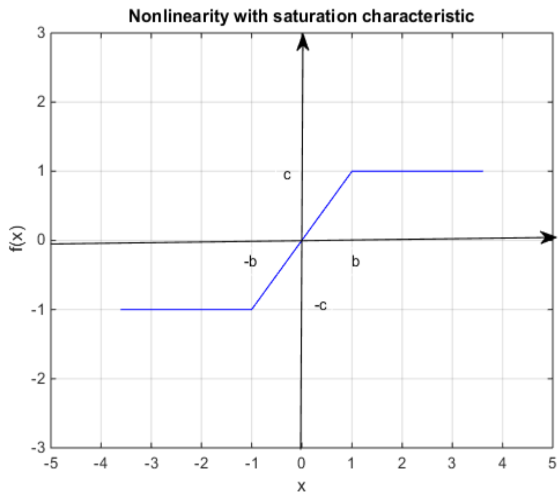

Fig. 3. Nonlinearity with saturation characteristic

where $b > 0$ and $k = \frac{c}{b}$, (in fig.3, $b = 1$, $c = 1$ and $k = 1$). By using the ideal relay characteristic, we can write:

$$f(x) = 0.5kx[sgn(x+b) - sgn(x-b)] + \\ +0.5c[sgn(x-b) - sgn(-x-b)] = 0.5kxf_1(x) + \\ +0.5cf_2(x) \quad (11)$$

where $f_1(x) = sgn(x + b) - sgn(x - b)$ and $f_2(x) = sgn(x - b) - sgn(-x - b)$. From expression (11), it is clearly seen that, if $|x| < b$, $sgn(x - b) = -1$ and $sgn(-x - b) = -1$ and therefore, $f_2(x) = 0$. Since $sgn(x + b) = 1$, the first term $f_1(x) = 2$ and therefore, $f(x) = kx$. If $x = b$, then the second term $f_2(x) = 1$ and the first term $f_1(x) = 1$. Therefore, $f(x) = 0.5kb + 0.5c = c$. If $x = -b$, then the second term $f_2(x) = -1$ and the first term is $f_1(x) = 1$ and $f(x) = -0.5kb - 0.5c = -c$, since $kb = c$. If $x > b$, the first term $f_1(x) = 0$ and the second term $f_2(x) = 2$ and therefore, $f(x) = c$. If $x < -b$, then the first term $f_1(x) = 0$ and the second term $f_2(x) = -2$ and therefore, $f(x) = -c$. Thus, the expression (11) completely describes the behavior of the nonlinearity with saturation element. It is clear from (11) that, the only nondifferentiable part of this characteristic is the $sgn(x)$ function, which can be approximated by Pade series, which was shown in section 2. Using smooth function approximation for the ideal relay characteristic, we can smoothly approximate the saturation nonlinearity.

Finally, we consider the ideal relay with dead zone characteristic, shown in fig.4. The mathematical description of the relay with dead zone can be written as follows:

$$f(x) = \begin{cases} 0, & |x| < b \\ a, & x > b \\ -a, & x < -b \end{cases}, \qquad (12)$$

where $b > 0$ and $a > 0$, (in fig.4, $b = 1.5$ and $a = 1$). By using the relay characteristic $sgn(x)$, the relay with dead zone curve can be described by the expression:

$$f(x) = 0.5a[sgn(x - b) + sgn(x + b)] \qquad (13)$$

It is clear that, if $|x| < b$, $f(x) = 0$. For $x > b$, $f(x) = a$ and for $x < -b$, $f(x) = -a$. Similarly to the previous cases, developed above, the only nondifferentiable part in (12) is the $sgn(x)$ function. By using Pade series approximation of the sign function, we can develop nonlinear characteristic, which is differentiable on the whole interval of observation.

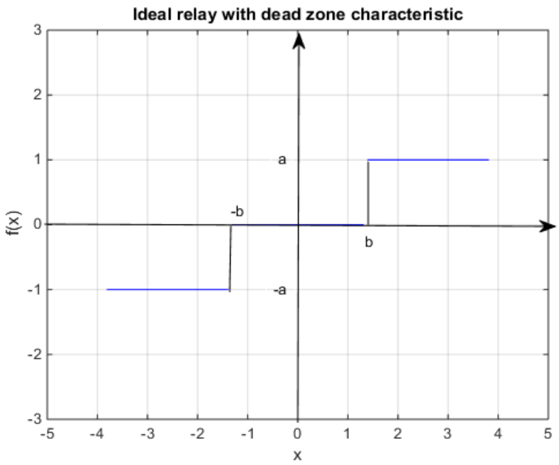

Fig. 4. Ideal relay with dead zone characteristic

By using Pade series approximation of the sign function, we can develop nonlinear characteristic, which is differentiable on the whole interval of observation. Therefore, we can develop smooth function approximation for the jump behavior of relay with dead zone and replace the discontinuous characteristic by a differentiable one.

IV. CONCLUSION

This paper considers the problem of smooth function approximation of certain nondifferentiable nonlinearities. The smooth function approximation approach is required, when time and space derivatives of certain functions is in the base of the applied nonlinear methods, like the Lyapunov based functions methods, the sliding mode operation and others. The presented approach is constructed by using smooth function approximation of the ideal relay characteristic. The ideal relay characteristic can be used to present some other nondifferentiable nonlinear characteristics like, nonlinearity with saturation, dead zone and relay with dead zone nonlinearities. Therefore, the smooth function approximation of the relay characteristic can be used to obtain smooth approximation of the other nonlinear characteristics. Explicit relations are provided, which describe the behavior of these characteristics, thus avoiding the if-then-else algorithmic representation. The proposed approximation method uses the gate functions approach and is based on Pade series representation of the computed hyperbolic tangent function. The Pade series representation is utilized for rational function approximation of the ideal relay characteristic. In this terms, we derive rational function approximations of certain nondifferentiable nonlinear characteristics.

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