

Boundary-Value Problem for Almost Linear Systems of Ordinary Differential Equations

N. D. Sirakova

Technical University of Sofia, Bulgaria

Corresponding author: evklid@abv.bg

Abstract. A boundary-value problem (BVP) for almost system of ordinary differential equations (ODE) is considered. An asymptotic expansion of the solution is constructed in the conditionally stable case. The boundary function method, pseudoinverse matrices and orthogonal projectors are utilized.

1. INTRODUCTION

We consider the boundary value problems

$$\varepsilon \frac{dx}{dt} = Ax + \varepsilon F(t, x, \varepsilon, f(t, \varepsilon)) + \varphi(t), \quad t \in [a, b], \quad 0 < \varepsilon \ll 1, \quad (1)$$

with boundary conditions

$$l(x(\cdot)) = h, \quad h \in R^m, \quad (2)$$

where ε is a small positive parameter.

Suppose that the following conditions are satisfied:

(C1) The $n \times n$ matrix A with constant elements has p eigenvalue with negative real part, and remaining $(n-p)$ eigenvalues have positive real part, i.e. $\lambda_i \in \sigma(A)$, $R(\lambda_i) < 0$, $i = \overline{1, p}$ and $R(\lambda_i) > 0$, $i = \overline{p+1, n}$, $\lambda_i \neq \lambda_j$ for $i \neq j$;

(C2) The vector-function $\varphi(t)$ is an n - dimensional of the class $C^\infty([a, b])$;

(C3) The function $F(x, t, \varepsilon, f(t, \varepsilon))$ is an n - dimensional vector-function, having arbitrary order continuous partial derivatives with respect to all arguments in the domain $G = [a, b] \times D_x \times [0, \bar{\varepsilon}] \times D_f$ - where $D_x \in R^n$ is some neighborhood of the solution $x_0(t)$ of the degenerate system $Ax_0(t) + \varphi(t) = 0$, $D_f \in R^p$ is bounded and closed domain, $0 < \bar{\varepsilon} < \varepsilon$. The function $f(t, \varepsilon)$ is smooth of arbitrary order with respect to all argument in the domain $G = [a, b] \times (0, \bar{\varepsilon}]$ and its values belongs to D_f ;

(C4) The vector functional $l = \text{col}(l^1, l^2, \dots, l^m)$, $l \in (C[a, b] \rightarrow R^n, R^m)$.

We assume that the function $f(t, \varepsilon)$ of (1) contain singular elements (for example, $f(t, \varepsilon) = f(\exp(-t/\varepsilon), \sin(t/\varepsilon))$). Thus the system (1) is with double singularity. On one side the small parameter ε appear before the derivative, and on the other hand bring in a singularity of the function f .

The constructing of the asymptotic solution of the problem (1), (2) is based on the boundary functions method (see, for example, [13]). The initial research for a Cauchy problem with double singularity is made in [10] in the case $\lambda_i \in \sigma(A)$, $R(\lambda_i) < 0, \forall i$. The case of a two-point boundary problem is considered in [6], [9].

Primary research of the problem (1), (2) with conditions (C1)-(C4) are considered in [3]. This paper is continuation of the article [3] and [7]. The proof of the asymptoticaly of the formal asymptotic order obtained in [7], is done in work [8]. Here we will consider a conditionally stable case of almost linear differential equations with general boundary conditions, using generalized reverse matrices and projectors [1], [11].

The problem (1), (2) with condition $R(\lambda_i) < 0, \forall i, \lambda_i \in \sigma(A)$ and (C1)-(C4) in diverse cases are research in [4], [5], [7], [12].

2.BASIC RESULTS

Instead of (1), (2) we consider BVP with two parameters $\varepsilon \in (0, \bar{\varepsilon}]$ and $\mu \in (0, \bar{\varepsilon}]$, $0 < \bar{\varepsilon} \ll 1$

$$\begin{aligned} \varepsilon \frac{dz}{dt} &= Az + \varepsilon F(z, t, \varepsilon, f(t, \mu)) + \varphi(t), \quad t \in [a, b], \quad 0 < \varepsilon \ll 1, \\ l(z) &= h. \end{aligned} \quad (3)$$

The solution of the BVP (3) we look of the form

$$z(t, \varepsilon, \mu) = \sum_{k=0}^{\infty} [z_k(t, \mu) + \Pi_k(\tau, \mu) + Q_k(v, \mu)] \varepsilon^k, \quad \tau = \frac{t-a}{\varepsilon}, \quad v = \frac{t-b}{\varepsilon}. \quad (4)$$

In (4) the functions $z_k(t, \mu)$ are the elements of the regular series, $\Pi_k(\tau, \mu)$ and $Q_k(v, \mu)$ are boundary functions in a right neighborhood of the point $t = a$ and left neighborhood of the point $t = b$ respectively and they are elements of the singular series.

After finding the functions $z_k(t, \mu)$, $\Pi_k(\tau, \mu)$ and $Q_k(v, \mu)$ the solution of the BVP (1), (2) is obtained in the form

$$x(t, \varepsilon) = \sum_{k=0}^{\infty} [z_k(t, \varepsilon) + \Pi_k(\tau, \varepsilon) + Q_k(v, \varepsilon)] \varepsilon^k.$$

We substitute the series (4) in the system (3)

$$\begin{aligned} \varepsilon \sum_{k=0}^{\infty} \left(\frac{d}{dt} z_k(t, \mu) + \varepsilon^{-1} \frac{d}{d\tau} \Pi_k(\tau, \mu) + \varepsilon^{-1} \frac{d}{dv} Q_k(v, \mu) \right) \varepsilon^k \\ = A \sum_{k=0}^{\infty} [z_k(t, \mu) + \Pi_k(\tau, \mu) + Q_k(v, \mu)] \varepsilon^k \\ + \varepsilon F(t, \sum_{k=0}^{\infty} [z_k(t, \mu) + \Pi_k(\tau, \mu) + Q_k(v, \mu)] \varepsilon^k, \varepsilon, f(t, \mu)) + \varphi(t), \end{aligned} \quad (5)$$

We represent the function $F(t, z, \varepsilon, f(t, \mu))$ in the form

$$\begin{aligned} F(t, \sum_{k=0}^{\infty} (z_k(t, \mu) + \Pi_k(\tau, \mu) + Q_k(v, \mu)) \varepsilon^k, \varepsilon, f(t, \mu)) \\ = \bar{F}(t, \varepsilon, \mu) + \Pi F(\tau, \varepsilon, \mu) + QF(v, \varepsilon, \mu), \end{aligned}$$

where

$$\begin{aligned} \bar{F}(t, \varepsilon, \mu) &= F(t, \sum_{k=0}^{\infty} z_k(t, \mu) \varepsilon^k, \varepsilon, f(t, \mu)), \\ \Pi F(\tau, \varepsilon, \mu) &= F(a + \varepsilon\tau, \sum_{k=0}^{\infty} (z_k(a + \varepsilon\tau, \mu) + \Pi_k(\tau, \mu)) \varepsilon^k, \varepsilon, f(a + \varepsilon\tau, \mu)) \\ &\quad - F(a + \varepsilon\tau, \sum_{k=0}^{\infty} z_k(a + \varepsilon\tau, \mu) \varepsilon^k, \varepsilon, f(a + \varepsilon\tau, \mu)), \\ QF(v, \varepsilon, \mu) &= F(b + \varepsilon v, \sum_{k=0}^{\infty} (z_k(b + \varepsilon v, \mu) + Q_k(v, \mu)) \varepsilon^k, \varepsilon, f(b + \varepsilon v, \mu)) \\ &\quad - F(b + \varepsilon v, \sum_{k=0}^{\infty} z_k(b + \varepsilon v, \mu) \varepsilon^k, \varepsilon, f(b + \varepsilon v, \mu)), \end{aligned}$$

We decompose the function $\bar{F}(t, \varepsilon, \mu)$ in Taylor series in a neighborhood of the point $(t, z_0(t), 0, f)$

$$\bar{F}(t, \varepsilon, \mu) = \sum_{k=0}^{\infty} \bar{F}_k(t, \mu) \varepsilon^k,$$

where

$$\bar{F}_k(t, \mu) = \begin{cases} F(t, z_0(t), 0, f(t, \mu)) & k=0 \\ F_z(t, z_0(t), 0, f(t, \mu))z_k(t, \mu) \\ + g_k(z_0(t), \dots, z_{k-1}(t, \mu), f(t, \mu)), & k=1, 2, \dots \end{cases} \quad (6)$$

We develop other features also in Taylor series - $\Pi F(\tau, \varepsilon, \mu)$ of the point $(a, z_0(a) + \Pi_0(\tau), 0, f)$ and $QF(v, \varepsilon, \mu)$ in a neighborhood of the point $(b, z_0(b) + Q_0(v), 0, f)$.

$$\Pi F(\tau, \varepsilon, \mu) = \sum_{k=0}^{\infty} \Pi F_k(\tau, \mu) \varepsilon^k,$$

where

$$\Pi F_k(\tau, \mu) = \begin{cases} F(a, z_0(a) + \Pi_0(\tau), 0, f(a, \mu)) \\ - F(a, z_0(a), 0, f(a, \mu)), & k=0 \\ F_z(a, z_0(a) + \Pi_0(\tau), 0, f(a, \mu)) \Pi_k(\tau, \mu) \\ + G_k(\tau, \Pi_0(\tau), \dots, \Pi_{k-1}(\tau, \mu), f(a, \mu)), & k=1, 2, \dots \end{cases},$$

$$QF(v, \varepsilon, \mu) = \sum_{k=0}^{\infty} QF_k(v, \mu) \varepsilon^k,$$

where

$$QF_k(v, \mu) = \begin{cases} F(b, z_0(b) + Q_0(v), 0, f(b, \mu)) \\ - F(b, z_0(b), 0, f(b, \mu)), & k=0 \\ F_z(b, z_0(b) + Q_0(v), 0, f(b, \mu)) Q_k(v, \mu) \\ + R_k(v, Q_0(v), \dots, Q_{k-1}(v, \mu), f(b, \mu)), & k=1, 2, \dots \end{cases}$$

The functions g_k, G_k and R_k contain derivative up to $(k-1)$ -th order of the function $F(t, z, \varepsilon, f(t, \mu))$ with respect to t, z and ε .

Let a linear system $\frac{dx}{dt} = Ax$ have a fundamental solution the matrix $X(t) = \exp(At)$, $X(0) = E_n$ and B be an $(n \times n)$ nonsingular constant matrix such that $B^{-1}AB = \text{diag}(A_+, A_-)$, where A_+ is an $(p \times p)$ -matrix corresponding for eigenvalues with negative real part $R(\lambda_i) < 0, i = \overline{1, p}$ and A_- is $((n-p) \times (n-p))$ matrix corresponding for eigenvalues with positive real part $R(\lambda_i) > 0, i = \overline{p+1, n}$.

The system $\frac{dx}{dt} = Ax$ has stable manifold S^+ in the form $S^+ : \bar{x} = H\bar{x}$, where $H = B_{21}B_{11}^{-1}$ is an $((n-p) \times p)$ -matrix, and unstable manifold S^- in the form $S^- : \bar{x} = \bar{H}\bar{x}$, where $\bar{H} = B_{12}B_{22}^{-1}$ is an $(p \times (n-p))$ -matrix. Cells $B_{ij}, i, j = 1, 2$, are elements of the block representation of the matrix $B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$.

$$\text{Let } X_p(\tau) = X(\tau) \begin{pmatrix} E_p \\ H \end{pmatrix} - (n \times p) - \text{matrix};$$

$$X_{n-p}(v) = X(v) \begin{pmatrix} \bar{H} \\ E_{n-p} \end{pmatrix} - (n \times (n-p)) - \text{matrix}.$$

The following denotations

$$D_1(\varepsilon) = l \left(X_p \left(\frac{(\cdot) - a}{\varepsilon} \right) \right) - (n \times p) - \text{matrix};$$

$$D_2(\varepsilon) = l \left(X_{n-p} \left(\frac{b - (\cdot)}{\varepsilon} \right) \right) - (n \times (n-p)) - \text{matrix};$$

$$D(\varepsilon) = (D_1(\varepsilon), D_2(\varepsilon)) - (n \times n) - \text{matrix}$$

introduce.

In this case, we assume that the matrix $D(\varepsilon)$ has the presentation

$$D(\varepsilon) = D_0 + D_1 \exp(-\alpha_1 / \varepsilon) + D_2 \exp(-\alpha_2 / \varepsilon) + \dots + D_s \exp(-\alpha_s / \varepsilon),$$

$D_k - (n \times n)$ - constant matrices, $\alpha_i, i = \overline{1, s}$ are positive constants such, that $0 < \alpha_1 < \alpha_2 < \dots < \alpha_s$.

The case where $D(\varepsilon)$ has the form $D(\varepsilon) = D_0 + O(\varepsilon^q \exp(-\alpha / \varepsilon))$, $q \in N, \alpha > 0$, $D_0 - (n \times n)$ - matrix, is considered by [3] and [9].

We apply functional l to $\Pi_k(\tau, \mu)$ and $Q_k(\nu, \mu)$:

$$l\left(\Pi_k\left(\frac{(\cdot) - a}{\varepsilon}, \mu\right) + Q_k\left(\frac{(\cdot) - b}{\varepsilon}, \mu\right)\right) = \begin{cases} h - l(z_0(\cdot)), & k = 0 \\ l(z_k(\cdot, \mu)), & k \geq 1. \end{cases} \quad (7)$$

Denote

$$\begin{aligned} h_0 &= h + l(A^{-1}\varphi(t)) \\ h_k &= -l(z_k(\cdot, \mu)) - l(L_\tau \overline{\Pi F_k})\left(\frac{(\cdot) - a}{\varepsilon}, \mu\right) - l(L_\nu \overline{Q F_k})\left(\frac{(\cdot) - b}{\varepsilon}, \mu\right) \end{aligned} \quad (8)$$

where $(L_\tau \overline{\Pi F_k})(\tau, \mu)$ and $(L_\nu \overline{Q F_k})(\nu, \mu)$ they are defined in the following manner: then the systems $\frac{dx}{d\tau} = Ax + g(\tau)$, $\tau \geq 0$ and $\frac{dy}{d\nu} = Ay + \bar{g}(\nu)$, $\nu \leq 0$ have particular solutions $(L_\tau g)(\tau)$ and $(L_\nu \bar{g})(\nu)$ respectively in the form

$$(L_\tau g)(\tau) = \int_0^{+\infty} K(\tau, s)g(s)ds \text{ and } (L_\nu \bar{g})(\nu) = \int_{-\infty}^0 \bar{K}(\nu, s)\bar{g}(s)ds$$

satisfying the inequalities $\|(L_\tau g)(\tau)\| \leq C \exp(-\gamma\tau)$, $\tau \geq 0$;

where $C, \bar{C}, \gamma, \bar{\gamma}$ are certain positive constants, and

$$\begin{aligned} K(\tau, s) &= \begin{cases} X(\tau)PX^{-1}(s), & 0 \leq s \leq \tau < +\infty \\ -X(\tau)(I - P)X^{-1}(s), & 0 \leq \tau \leq s < +\infty \end{cases} \\ \bar{K}(\nu, s) &= \begin{cases} -X(\nu)(I - P)X^{-1}(s), & -\infty < \nu \leq s \leq 0 \\ X(\nu)PX^{-1}(s), & -\infty < s \leq \nu \leq 0. \end{cases} \end{aligned} \quad (\text{look Lemma 1 [2]})$$

We denote the $(qm \times (s+1)n)$ blok matrix $Q(\varepsilon)$, where $2s+1 \leq q \leq \frac{(s+1)(s+2)}{2}$, $s \in N$. The elements of the matrix $Q(\varepsilon)$ depend on ε , but not have exponentially small terms. For example for $s=1$, we have

$2 \cdot 1 + 1 \leq q \leq \frac{(1+1)(1+2)}{2} \Rightarrow q = 3$, then the matrix $Q(\varepsilon)$ has the form $Q(\varepsilon) = \begin{pmatrix} D_0 & 0 \\ D_1 & D_0 \\ 0 & D_1 \end{pmatrix}$. If $\alpha_k = k\alpha_1$, $k = \overline{2, s}$,

then $q = 2s+1$ and $Q(\varepsilon)$ has the form

Consider systems (11), (12) and boundary conditions (7) for $k=0$. The equation $\frac{\partial \Pi_0(\tau)}{\partial \tau} = A\Pi_0(\tau)$ has a solution $\Pi_0(\tau) = X_p(\tau)\bar{c}_0$, $\bar{c}_0 \in R^p, \tau \in [0, +\infty)$ and $\frac{\partial Q_0(v)}{\partial v} = AQ_0(v)$ has a solution

$$Q_0(v) = X_{n-p}(v)\bar{\bar{c}}_0, \quad \bar{\bar{c}}_0 \in R^{n-p}, v \in (-\infty, 0].$$

Substituting into the boundary conditions for $c_0 = (\bar{c}_0, \bar{\bar{c}}_0)^T$ obtaining the algebraic system

$$D_1(\varepsilon)\bar{c}_0 + D_2(\varepsilon)\bar{\bar{c}}_0 = h_0 \Leftrightarrow D(\varepsilon)c_0 = h_0.$$

Keeping in mind the structure of the matrix $D(\varepsilon)$, we seek the unknown column vector c_0 in the form

$$c_0 = c_0(\varepsilon) = c_{00} + c_{01}e^{-\frac{\alpha_1}{\varepsilon}} + \dots + c_{0s}e^{-\frac{\alpha_s}{\varepsilon}}, \quad c_{0i} > 0, i = \overline{0, s}, \alpha_i > 0, i = \overline{1, s}.$$

We introduce $\tilde{c}_0 = (c_{00}, c_{01}, \dots, c_{0s})^T$ - $(s+1)n$ -dimensional column vector,

$$\tilde{h}_0 = (h_0, 0, \dots, 0)^T - (2s+1)m - \text{dimensional column vector.}$$

Replacing $c_0(\varepsilon)$ in $D(\varepsilon)c_0 = h_0$ we get the algebraic system $Q\tilde{c}_0 = \tilde{h}_0$. Using (J2) then we get

$$\begin{aligned} \Pi_0(\tau) &= X_p(\tau) \sum_{k=0}^s [Q^{-1}\tilde{h}_0]_{n_{k+1}}^p \exp(-\alpha_k / \varepsilon) \\ Q_0(v) &= X_{n-p}(v) \sum_{k=0}^s [Q^{-1}\tilde{h}_0]_{n_{k+1}}^{n-p} \exp(-\alpha_k / \varepsilon), \quad \alpha_0 = 0, \alpha_i > 0, i = \overline{1, s}. \end{aligned}$$

With $k \geq 1$ the system's (10), (11) have solutions

$$\Pi_k(\tau, \mu) = X_p(\tau)\bar{c}_k + (L_t \overline{\Pi F}_k)(\tau, \mu)$$

$$Q_k(v) = X_{n-p}(v)\bar{\bar{c}}_k + (L_v \overline{Q F}_k)(v, \mu),$$

$c_k = (\bar{c}_k, \bar{\bar{c}}_k)^T$ are solutions of the equations $D(\varepsilon)c_k = h_k(\varepsilon)$, where $c_k = c_k(\varepsilon) = c_{k0} + c_{k1}e^{-\frac{\alpha_1}{\varepsilon}} + \dots + c_{ks}e^{-\frac{\alpha_s}{\varepsilon}}$.

Analogously to c_0 , from the equations $Q\tilde{c}_k = \tilde{h}_k$, where $\tilde{c}_k = (c_{k0}, c_{k1}, \dots, c_{ks})^T$ - $(s+1)n$ -dimensional column vector,

$$\tilde{h}_k = (h_{k0}, h_{k1}, \dots, h_{ks})^T - (2s+1)m - \text{dimensional column vector,}$$

$$c_k = c_k(\varepsilon) = c_{k0} + c_{k1}e^{-\frac{\alpha_1}{\varepsilon}} + \dots + c_{ks}e^{-\frac{\alpha_s}{\varepsilon}}$$

$$h_k(\varepsilon) = h_{k0} + h_{k1}e^{-\frac{\alpha_1}{\varepsilon}} + \dots + h_{ks}e^{-\frac{\alpha_s}{\varepsilon}} \quad (13)$$

$$\Rightarrow \tilde{c}_k = Q^{-1}\tilde{h}_k.$$

We finally get (10)

$$\Pi_k(\tau, \mu) = X_p(\tau) \sum_{k=0}^s [Q^{-1}\tilde{h}_k]_{n_{k+1}}^p \exp(-\alpha_k / \varepsilon) + (L_t \overline{\Pi F}_k)(\tau, \mu), \quad k \geq 1,$$

$$Q_k(v, \mu) = X_{n-p}(v) \sum_{k=0}^s [Q^{-1}\tilde{h}_k]_{n_{k+1}}^{n-p} \exp(-\alpha_k / \varepsilon) + (L_v \overline{Q F}_k)(v, \mu), \quad k \geq 1. \quad \triangleleft$$

Theorem 2. We assume that conditions (C1) – (C4) are satisfied. Then the functions $z_k(t, \mu)$, $k \geq 0$ are bounded, i.e. the next in equations hold $\|z_k(t, \mu)\| \leq N_k \quad \forall t \in [a, b]$ and $\forall \mu \in (0, \bar{\varepsilon})$, where N_k are positive constants.

The proof is given in [7], [10].

Theorem 3. The boundary functions $\Pi_k(\tau, \mu)$ and $Q_k(v, \mu)$, $k \geq 0$ decrease exponentially at $\tau \rightarrow \infty$ and $v \rightarrow -\infty$ respectively, $0 < \mu \leq \bar{\varepsilon}$.

The proof is similar to what was done in [3].

2.2. II case $\text{rang}Q = r < \min((2s+1)m, (s+1)n)$.

Theorem 4. We assume that conditions

(J3) (C1) – (C4)

(J4) $\text{rang}Q = r < \min((2s+1)m, (s+1)n)$ are satisfied.

Then in the domain G the BVP (3) has a one-parametric solution, continuous derivatives with respect to $t \in [a, b]$ and continuous solution for $\mu \in (0, \varepsilon]$. The series (4) is then formal asymptotic series for this solution, where the functions $z_k(t, \varepsilon)$, $k \geq 0$ have the form (9) for $\mu = \varepsilon$ and $\Pi_k(\tau, \mu)$ and $Q_k(\nu, \mu)$ have the form (14), (15) for $\mu = \varepsilon$, where ξ_k is arbitrary vector.

$$\Pi_0(\tau) = X_p(\tau) \sum_{k=0}^s \left[P_{Q_0} \xi_0 \right]_{n_{k+1}}^p \exp(-\alpha_k / \varepsilon) + X_p(\tau) \sum_{k=0}^s \left[Q^+ \tilde{h}_0 \right]_{n_{k+1}}^p \quad (14)$$

$$Q_0(\nu) = X_{n-p}(\nu) \sum_{k=0}^s \left[P_{Q_0} \xi_0 \right]_{n_{k+1}}^{n-p} \exp(-\alpha_k / \varepsilon) + X_{n-p}(\nu) \sum_{k=0}^s \left[Q^+ \tilde{h}_0 \right]_{n_{k+1}}^{n-p}$$

$$\begin{aligned} \Pi_k(\tau, \mu) &= X_p(\tau) \sum_{i=0}^s \left[P_{Q_i} \xi_k \right]_{n_{i+1}}^p \exp(-\alpha_i / \varepsilon) + X_p(\tau) \sum_{i=0}^s \left[Q^+ \tilde{h}_k \right]_{n_{i+1}}^p \\ &\quad + \left(L_\tau \overline{\Pi F}_k \right)(\tau, \mu), \end{aligned} \quad (15)$$

$$\begin{aligned} Q_k(\nu, \mu) &= X_{n-p}(\nu) \sum_{i=0}^s \left[P_{Q_i} \xi_k \right]_{n_{i+1}}^{n-p} \exp(-\alpha_i / \varepsilon) + X_{n-p}(\nu) \sum_{i=0}^s \left[Q^+ \tilde{h}_k \right]_{n_{i+1}}^{n-p} \\ &\quad + \left(L_\nu \overline{Q F}_k \right)(\nu, \mu) \end{aligned}$$

Proof. In this case Q^{-1} not exist. We note with P_Q and P_{Q^*} the matrix ortho projectors $P_Q : R^n \rightarrow \ker(Q)$, $P_{Q^*} : R^n \rightarrow \ker(Q^*)$, $Q^* = Q^T$. With Q^+ we note the unique Moor-Penrose inverse $((2s+1)m, (s+1)n)$ -matrix of the matrix Q . Let P_{Q_q} be a $((s+1)n \times q)$ -matrix with $q = (s+1)n - (2s+1)m = (s+1)(n-m)$ linear independent columns from the matrix P_Q . In order to be the system $Qc_0 = h_0$ is always possible us to set the condition $P_{Q_q} h_0 = 0$ and $c_0 = P_{Q_q} \xi_0 + Q^+ h_0$, $\xi_0 \in R^q$, $c_0 = (\bar{c}_0, \bar{c}_0)^T$. Then the solution of the system (11) and (12) for $k = 0$ is

$$\begin{aligned} \Pi_0(\tau) &= X_p(\tau) \sum_{k=0}^s \left[P_{Q_0} \xi_0 \right]_{n_{k+1}}^p \exp(-\alpha_k / \varepsilon) + X_p(\tau) \sum_{k=0}^s \left[Q^+ \tilde{h}_0 \right]_{n_{k+1}}^p \\ Q_0(\nu) &= X_{n-p}(\nu) \sum_{k=0}^s \left[P_{Q_0} \xi_0 \right]_{n_{k+1}}^{n-p} \exp(-\alpha_k / \varepsilon) + X_{n-p}(\nu) \sum_{k=0}^s \left[Q^+ \tilde{h}_0 \right]_{n_{k+1}}^{n-p} \end{aligned}$$

It should be noted that $\Pi_0(\tau)$ and $Q_0(\nu)$ does not depend on μ . Determining the ξ_0 is carried out next step when finding $\Pi_1(\tau, \mu)$ and $Q_1(\nu, \mu)$ from (11) and (12)

$$\Pi_1(\tau, \mu) = X_p(\tau) \bar{c}_1 + \left(L_\tau \overline{\Pi F}_1 \right)(\tau, \mu),$$

$$Q_1(\nu, \mu) = X_{n-p}(\nu) \bar{c}_1 + \left(L_\nu \overline{Q F}_1 \right)(\nu, \mu).$$

Then the solution of the system $Qc_1 = h_1$, $c_1 = (\bar{c}_1, \bar{c}_1)^T$ is $c_1 = P_{Q_q} \xi_1 + Q^+ h_1$. Using the permitting condition $P_{Q_q} h_1 = 0$ we get

$$\begin{aligned}\Pi_1(\tau, \mu) &= X_p(\tau) \sum_{k=0}^s \left[P_{Q_q} \xi_1 \right]_{n_{k+1}}^p \exp(-\alpha_k / \varepsilon) + X_p(\tau) \sum_{k=0}^s \left[Q^+ \tilde{h}_1 \right]_{n_{k+1}}^p \\ &\quad + \left(L_\tau \overline{\Pi F_1} \right) (\tau, \mu) \\ Q_1(\nu, \mu) &= X_{n-p}(\nu) \sum_{k=0}^s \left[P_{Q_q} \xi_1 \right]_{n_{k+1}}^{n-p} \exp(-\alpha_k / \varepsilon) + X_{n-p}(\nu) \sum_{k=0}^s \left[Q^+ \tilde{h}_1 \right]_{n_{k+1}}^{n-p} \\ &\quad + \left(L_\nu \overline{Q F_1} \right) (\nu, \mu)\end{aligned}$$

Then $\Pi_k(\tau, \mu)$ and $Q_k(\nu, \mu)$ we obtain using the method of mathematical induction

$$\begin{aligned}\Pi_k(\tau, \mu) &= X_p(\tau) \sum_{i=0}^s \left[P_{Q_q} \xi_k \right]_{n_{i+1}}^p \exp(-\alpha_i / \varepsilon) + X_p(\tau) \sum_{i=0}^s \left[Q^+ \tilde{h}_k \right]_{n_{i+1}}^p \\ &\quad + \left(L_\tau \overline{\Pi F_k} \right) (\tau, \mu), \\ Q_k(\nu, \mu) &= X_{n-p}(\nu) \sum_{i=0}^s \left[P_{Q_q} \xi_k \right]_{n_{i+1}}^{n-p} \exp(-\alpha_i / \varepsilon) + X_{n-p}(\nu) \sum_{i=0}^s \left[Q^+ \tilde{h}_k \right]_{n_{i+1}}^{n-p} \\ &\quad + \left(L_\nu \overline{Q F_k} \right) (\nu, \mu)\end{aligned}$$

where \tilde{h}_k are defined in (13). \triangleleft

Remark 1. The proof that the formally asymptotic solution is asymptotically will be further discussed.

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