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# Merging and Energy Exchange Between Optical Filaments

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**Abstract.** We investigate nonlinear interaction between collinear femtosecond laser pulses with power slightly above the critical for self-focusing  $P_{cr}$  through the processes of cross-phase modulation (CPM) and degenerate four-photon parametric mixing (FPPM). When there is no initial phase difference between the pulses we observe attraction between pulses due to CPM. The final result is merging between the pulses in a single filament with higher power. By method of moments it is found that the attraction depends on the distance between the pulses and has potential character. In the second case we study energy exchange between filaments. This process is described through FPPM scheme and requests initial phase difference between the waves.

## INTRODUCTION

In recent experiments on multi-filament parallel propagation two basic phenomena are observed: 1) the number of filaments is reduced significantly as a function of the distance [1] and 2) interflow (merging) of two, three or higher number of filaments propagating in close trajectories [2, 3, 4, 5, 6]. These mergers, with appearing of a strong filament are called Rogue events during the filamentation process.

The three dimensional localization appears similar to the incoherent soliton interaction in one-dimensional system as optical fibers, and based on clamping effects due to CPM [9, 10, 11, 12] and FPPM [12].

We investigate numerically the interaction between optical pulses in the cases when: 1) the initial phase difference between pulses is not equal to zero  $\Delta\varphi \neq 0$  and 2) the initial optical pulses admit equal phases  $\Delta\varphi = 0$ . Thus, by properly selected initial conditions, we take into account the FPPM process as addition to the CPM influence. The proposed in the paper nonlinear vector model is investigated numerically on the base of the split-step Fourier method. We introduce by the moment formalism nonlinear acceleration and potentials between the weight centroms of the pulses.

## BASIC SYSTEM OF EQUATIONS

As it was pointed in [13, 14, 15], the filamentation process can be described more correctly by using the generalized nonlinear operator

$$\vec{P}^{nl} = n_2 (\vec{E} \cdot \vec{E}) \vec{E}, \quad (1)$$

which includes additional processes associated with third harmonic generation ( $n_2$  is the nonlinear refractive index). The operator (1) generalizes the case of Marker and Terhune's operator, and includes to the self-action terms, CPM terms, FPPM terms and also additional terms associated with Third-Harmonic Generation (THG). We substitute into (1) two-component electrical vector  $\vec{E} = (E_x, E_y, 0)$  at one carrying frequency  $\omega_0$

$$\vec{E} = \frac{(A_x \exp [i(\omega_0 t - k_0 z)] + c.c.)}{2} \vec{x} + \frac{(A_y \exp [i(\omega_0 t - k_0 z)] + c.c.)}{2} \vec{y}, \quad (2)$$

where  $A_x = A_x(x, y, z, t)$ ,  $A_y = A_y(x, y, z, t)$  are the amplitude functions and  $k_0$  is the carrying wave number of the laser source.

The system of non-paraxial equations of the amplitude functions  $A_x, A_y$  of the two-component electrical field (2) has the form

$$\begin{aligned}
-2i \frac{k_0}{v_{gr}} \frac{\partial A_x}{\partial t} &= \Delta_{\perp} A_x - \frac{\beta + 1}{v_{gr}} \left( \frac{\partial^2 A_x}{\partial t^2} - 2v_{gr} \frac{\partial^2 A_x}{\partial t \partial z} \right) - \beta \frac{\partial^2 A_x}{\partial z^2} \\
&+ k_0^2 \tilde{n}_2 \left[ \frac{1}{3} (A_x^2 + A_y^2) A_x \exp(2ik_0(z - (v_{ph} - v_{gr})t)) + \left( |A_x|^2 + \frac{2}{3}|A_y|^2 \right) A_x + \frac{1}{3} A_x^* A_y^2 \right] \\
-2i \frac{k_0}{v_{gr}} \frac{\partial A_y}{\partial t} &= \Delta_{\perp} A_y - \frac{\beta + 1}{v_{gr}} \left( \frac{\partial^2 A_y}{\partial t^2} - 2v_{gr} \frac{\partial^2 A_y}{\partial t \partial z} \right) - \beta \frac{\partial^2 A_y}{\partial z^2} \\
&+ k_0^2 \tilde{n}_2 \left[ \frac{1}{3} (A_x^2 + A_y^2) A_y \exp(2ik_0(z - (v_{ph} - v_{gr})t)) + \left( |A_y|^2 + \frac{2}{3}|A_x|^2 \right) A_y + \frac{1}{3} A_y^* A_x^2 \right],
\end{aligned} \tag{3}$$

where  $v_{gr}$  and  $v_{ph}$  are the group and phase velocities correspondingly,  $\beta = k_0 v_{gr}^2 k''$ ,  $k''$  is the group velocity dispersion and  $\tilde{n}_2 = \frac{3}{8} n_2$ .

This model describes the ionization-free filamentation regime, where the pulse intensities are close to the critical one for self-focusing. The first nonlinear term in (3) corresponds to coherent GHz generation [15]. The system (3) is written in Galilean frame ( $z' = z - vt$ ;  $t' = t$ ). The last nonlinear term in (3) describes degenerate four-photon parametric mixing. To satisfy the Manley-Rowe relations of the truncated equations with a generalized nonlinear polarization of the type  $\vec{P}^{nl} = n_2 (\vec{E} \cdot \vec{E}) \vec{E}$ , some restrictions on the components of the electrical field are imposed. The conservation laws are satisfied when the components  $A_x$  and  $A_y$  are orthogonal.

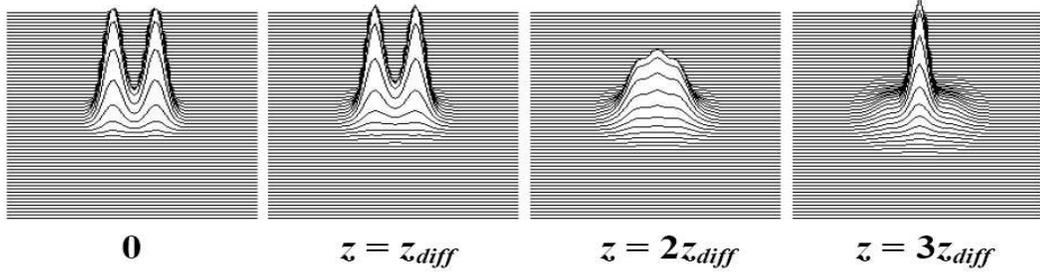
The system of equations (3) written in dimensionless form becomes

$$\begin{aligned}
-2i\alpha\delta^2 \frac{\partial A_x}{\partial t} &= \Delta_{\perp} A_x - \delta^2 (\beta + 1) \left( \frac{\partial^2 A_x}{\partial t^2} - \frac{\partial^2 A_x}{\partial t \partial z} \right) - \delta^2 \beta \frac{\partial^2 A_x}{\partial z^2} \\
&+ \gamma \left[ \frac{1}{3} (A_x^2 + A_y^2) A_x \exp(2i\alpha(z - \Delta\tilde{v}_{nl}t)) + \left( |A_x|^2 + \frac{2}{3}|A_y|^2 \right) A_x + \frac{1}{3} A_x^* A_y^2 \right] \\
-2i\alpha\delta^2 \frac{\partial A_y}{\partial t} &= \Delta_{\perp} A_y - \delta^2 (\beta + 1) \left( \frac{\partial^2 A_y}{\partial t^2} - \frac{\partial^2 A_y}{\partial t \partial z} \right) - \delta^2 \beta \frac{\partial^2 A_y}{\partial z^2} \\
&+ \gamma \left[ \frac{1}{3} (A_x^2 + A_y^2) A_y \exp(2i\alpha(z - \Delta\tilde{v}_{nl}t)) + \left( |A_y|^2 + \frac{2}{3}|A_x|^2 \right) A_y + \frac{1}{3} A_y^* A_x^2 \right],
\end{aligned} \tag{4}$$

where  $x = x/r_0$ ,  $y = y/r_0$ ,  $z = z/r_0$  are the dimensionless coordinates,  $r_0$  is the pulse waist,  $z_0 = v_{gr}t_0$  is the spatial pulse length,  $\alpha = k_0 z_0$ ,  $\delta = r_0/z_0$ ,  $\gamma = k_0^2 r_0^2 \tilde{n}_2 |A_0|^2 / 2$  is the nonlinear coefficient and  $\Delta\tilde{v}_{nl} = (v_{ph} - v_{gr})/v_{gr}$  is the normalized group-phase velocity difference.

## NUMERICAL SIMULATIONS

In this paper we investigate numerically the influence of CPM and FPPM on the nonlinear interaction between filaments propagating in parallel and close trajectories. We think that the both phenomena: 1) the reducing of number of the filaments as a function of the distance [1] and 2) the interflow between several filaments in one (Rogue) wave [2, 3, 4, 5, 6] are connected and they are results of different types of nonlinear interaction. By control of the initial phase difference between the pulses it is possible to include or exclude the process of CPM or FPPM during the interaction. When the initial phase difference of the pulses is equal to zero the FPPM practically does not work and the nonlinear interaction is due the CPM. When the initial phase difference between the pulses is not equal to zero dominates the process of FPPM and an intensive exchange of energy between the laser pulses is observed [16]. The



**FIGURE 1.** Fusion between two collinear filaments  $\vec{A}_1$  and  $\vec{A}_2$  with power slightly above the critical  $P_{cr}$  ( $\gamma = 1.5$ ). The pulses are separated initially at distance  $2a = 3.4$  and the evolution is governed by the system of equations (4). The pulses admit equal initial phases, *i. e.*,  $\Delta\varphi = 0$ . The similar picture is seen when the FPPM terms are excluded from the equations (4) and only interaction due to CPM is investigated. With  $z_{diff}$  is denoted the diffraction length  $z_{diff} = k_0 r_0^2$ . The  $(x, y)$  projection of the intensities of the pulses is plotted

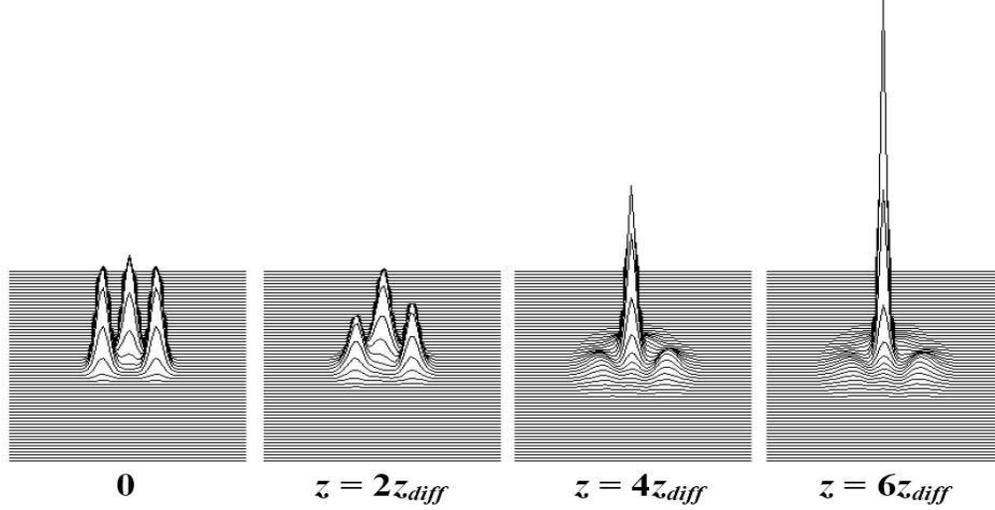
numerical simulations are performed with two, three and four laser pulses. The initial conditions are in the form of 120fs Gaussian bullets with waist and spatial length  $r_0 = z_0 = 72\mu\text{m}$  and power slightly above  $P_{cr}$ . In this case  $\alpha = 90\pi$ ,  $\delta = 1$ ,  $\Delta\tilde{v}_{nl} = 0.00023$  and  $\gamma \in [1.5, 3]$ . In order to satisfy the conservation laws the phase difference between the components  $A_x$  and  $A_y$  of the electrical field  $\vec{E}$  is initially  $\pi/2$ .

We present the laser pulses as vector fields. Let us consider the case of two pulses  $\vec{A}_1$  and  $\vec{A}_2$  propagating at small distance  $a$  between them. Each of the pulses admits  $\vec{x}$  and  $\vec{y}$  components:  $\vec{A}_j = A_{j,x}\vec{x} + A_{j,y}\vec{y}$ ,  $j = 1, 2$ . The initial conditions for numerical solution of the system of equations (4) have the form

$$\begin{aligned}
 A_x &= A_{1,x} + A_{2,x} = \frac{A_1^0}{\sqrt{2}} \exp\left(-\frac{(x+a)^2 + y^2 + z^2}{2}\right) + \frac{A_2^0}{\sqrt{2}} \exp\left(-\frac{(x-a)^2 + y^2 + z^2}{2}\right) \exp(i\Delta\varphi) \\
 A_y &= A_{1,y} + A_{2,y} = \left\{ \frac{A_1^0}{\sqrt{2}} \exp\left(-\frac{(x+a)^2 + y^2 + z^2}{2}\right) + \frac{A_2^0}{\sqrt{2}} \exp\left(-\frac{(x-a)^2 + y^2 + z^2}{2}\right) \exp(i\Delta\varphi) \right\} \exp\left(i\frac{\pi}{2}\right),
 \end{aligned} \tag{5}$$

where  $A_x$  and  $A_y$  are composed of the  $x$ - and  $y$ -components of the two optical pulses propagating along different parallel trajectories. The phase difference between the pulses is denoted by  $\Delta\varphi$ . The initial conditions in the case of higher number of pulses are constructed in similar way.

By varying the phase difference  $\Delta\varphi$  we include (and exclude, when  $\Delta\varphi = 0$ ) the FPPM process. The interaction of optical pulses  $\vec{A}_1$  and  $\vec{A}_2$  for  $\gamma = 1.5$ ,  $2a = 3.4$  and  $\Delta\varphi = 0$  is shown on Figure 1. The pulses start to attract each other without energy exchange and as result a merging and self-focusing due to CPM are observed. To verify this result we also exclude the parametric step from the the system of equations (4) and increase the intensity by factor 1/4 to keep on the critical power. In the both numerical experiments (with  $\Delta\varphi = 0$  or when the parametric step is excluded from the program) the results are similar - there is no energy exchange and the fusing between the filaments is clearly seen. Similar potential type of interaction by CPM was reported in optical fibers [11, 12]. On Figure 2 the interaction between three pulses with different phases -  $\varphi_1 = 0, \varphi_2 = \pi/4, \varphi_3 = \pi/2$  - is presented. In this numerical simulation  $\gamma = 1.5$ . When the phase difference between the pulses is not equal to zero the FPPM process is included and the pulses exchange energy. We observe that one of the pulses amplifies and self-focuses itself and gets enough power to continue its propagation, while the other two pulses give out energy, enter into linear mode and vanish. In this way the number of filaments can be reduced by non-linear parametric processes in  $\chi^{(3)}$  media. The interflow between four filaments in one (Rogue) wave is shown on Figure 3. This numerical result is similar to the experiment presented in [5].



**FIGURE 2.** Energy exchange between three collinear filaments with power slightly above the critical  $P_{cr}$  ( $\gamma = 1.5$ ). The evolution is governed by the system of equations (4). The phases of the pulses (from left to right) are correspondingly  $\varphi_1 = 0, \varphi_2 = \pi/4, \varphi_3 = \pi/2$ . Due to degenerated FPPM process one of the filaments is amplified while the other two filaments enter in linear mode and vanish. With  $z_{diff}$  is denoted the diffraction length  $z_{diff} = k_0 r_0^2$ . The  $(x, y)$  projection of the intensities of the pulses is plotted

## METHOD OF MOMENTS

To obtain analytical expressions of the influence of CPM on the relative moving of optical pulses we exclude the FPPM process and GHz generation from the system of equations (4). The basic system in this case is transformed to  $(3 + 1)D$  of Manakov type

$$\begin{aligned}
 -2ik_0 \left[ \frac{1}{v_{gr}} \frac{\partial A_x}{\partial t} + \frac{\partial A_x}{\partial z} \right] &= \Delta A_x - \frac{1 + k_0 v_{gr}^2 k''}{v_{gr}^2} \frac{\partial^2 A_x}{\partial t^2} + k_0^2 \tilde{n}_2 \left( |A_x|^2 + \frac{2}{3} |A_y|^2 \right) A_x \\
 -2ik_0 \left[ \frac{1}{v_{gr}} \frac{\partial A_y}{\partial t} + \frac{\partial A_y}{\partial z} \right] &= \Delta A_y - \frac{1 + k_0 v_{gr}^2 k''}{v_{gr}^2} \frac{\partial^2 A_y}{\partial t^2} + k_0^2 \tilde{n}_2 \left( |A_y|^2 + \frac{2}{3} |A_x|^2 \right) A_y.
 \end{aligned} \tag{6}$$

The scalar case of nonlinear interaction is investigated in [12]. In this paper we will preform similar analysis applied to collinear laser pulses presented as vector fields  $\vec{A}_1$  and  $\vec{A}_2$ . We decompose as in the previous section the vectors  $\vec{A}_1$  and  $\vec{A}_2$  in  $(x, y)$  plane

$$\vec{A}_j = A_{j,x} \vec{x} + A_{j,y} \vec{y}; \quad j = 1, 2. \tag{7}$$

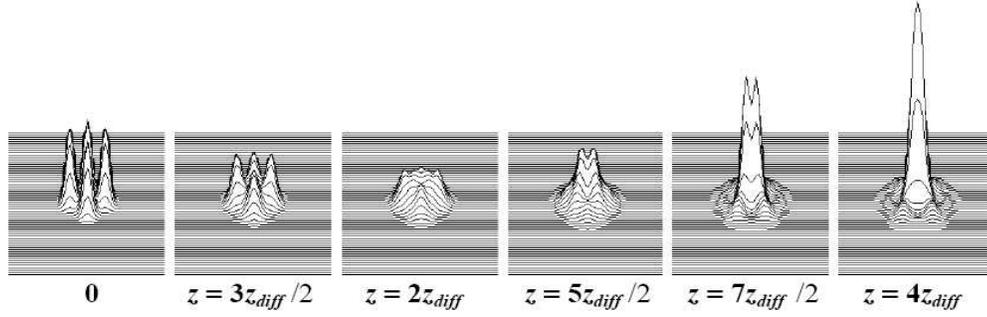
Thus, the components  $A_x$  and  $A_y$  in (6) become

$$A_x = A_{1,x} + A_{2,x}; \quad A_y = A_{1,y} + A_{2,y}. \tag{8}$$

Let us introduce the integral of energy of  $A_x$  and  $A_y$

$$p_j = \int \int \int |A_j(x, y, z, t)|^2 dU = \text{const}; \quad j = x, y \tag{9}$$

where  $dU = dx dy dz$  and also the integrals of center of weight in  $x$  direction of  $A_x$  and  $A_y$  are



**FIGURE 3.** Fusion between four collinear filaments governed by the system of equations (4). The pulses admit equal initial phases, *i. e.*,  $\Delta\varphi = 0$ . We observe the formation of one strong wave due to attraction between the pulses by CPM process. The  $(x, y)$  spot of the intensities of the pulses is plotted

$$\langle \dot{X}_j \rangle = \frac{1}{p_j} \int \int \int x |A_j(x, y, z, t)|^2 dU; \quad j = x, y, \quad (10)$$

Only the second derivative  $\frac{\partial^2}{\partial x^2}$  in the system (6) is non-commutative operator in regard to  $x$  translation. The other differential operators commute with  $x$  and therefore the velocity in  $x$  direction of the center of weight can be written as

$$\langle \dot{X}_j \rangle = \frac{iv_{gr}}{2k_0 p_j} \int \int \int \left[ A_j^*(x, y, z, t) \frac{\partial A_j(x, y, z, t)}{\partial x} - A_j(x, y, z, t) \frac{\partial A_j^*(x, y, z, t)}{\partial x} \right] dU; \quad j = x, y. \quad (11)$$

The acceleration in  $x$  direction of the center of weight can be expressed by the following convolution integral

$$\langle \ddot{X}_j(\Delta x, t) \rangle = \frac{v_{gr} k_0 \tilde{n}_2}{3p_j} \int \int \int \left[ |A_j(x + \Delta x, y, z, t)|^2 \frac{\partial}{\partial x} |A_k(x, y, z, t)|^2 \right] dU; \quad j = x, y, \quad (12)$$

where  $k \neq j$ . In the similar way we obtain the expressions of the accelerations in  $y$  and  $z$  directions. Here we investigate the simplest case of two *spherically-symmetric* pulses, located at arbitrary distance  $\Delta x$  in  $x$  direction. Therefore we calculate the acceleration in  $x$  direction (12) only. Substituting the decomposition (8) in (12) we obtain

$$\langle \ddot{a}(\Delta x, t) \rangle_{\vec{A}_1} = C_1 \int \int \int \left[ |A_{x_1}(x + \Delta x, y, z, t)|^2 \frac{\partial}{\partial x} |A_{y_2}(x, y, z, t)|^2 + |A_{y_1}(x + \Delta x, y, z, t)|^2 \frac{\partial}{\partial x} |A_{x_2}(x, y, z, t)|^2 \right] dU, \quad (13)$$

$$\langle \ddot{a}(\Delta x, t) \rangle_{\vec{A}_2} = C_1 \int \int \int \left[ |A_{x_2}(x - \Delta x, y, z, t)|^2 \frac{\partial}{\partial x} |A_{y_1}(x, y, z, t)|^2 + |A_{y_2}(x - \Delta x, y, z, t)|^2 \frac{\partial}{\partial x} |A_{x_1}(x, y, z, t)|^2 \right] dU, \quad (14)$$

where by  $\langle \ddot{a}(\Delta x, t) \rangle_{\vec{A}_1}$  and  $\langle \ddot{a}(\Delta x, t) \rangle_{\vec{A}_2}$  are denoted *the accelerations of the pulses* (not of the components) with condition  $\langle \ddot{a}(\Delta x, t) \rangle_{\vec{A}_1} + \langle \ddot{a}(\Delta x, t) \rangle_{\vec{A}_2} = 0$  and  $C_1 = \frac{(p_x + p_y) v_{gr} k_0 \tilde{n}_2}{3p}$ ;  $p = p_x p_y$ . In the case of spherically-symmetric functions and circular polarization ( $A_{x_i} = A_{y_i}$ ) the acceleration of the center weights can be presented in spherical coordinates

$$\langle \ddot{a}(\Delta r, t) \rangle_{\vec{A}_1} = 2C_1 \int \int \int \left[ |A_1(r + \Delta r, t)|^2 \frac{\partial}{\partial r} |A_2(r, t)|^2 r^2 \sin \theta \right] dr d\theta d\varphi, \quad (15)$$

$$\langle \ddot{a}(\Delta r, t) \rangle_{\vec{A}_2} = 2C_1 \int \int \int \left[ |A_2(r + \Delta r, t)|^2 \frac{\partial}{\partial r} |A_1(r, t)|^2 r^2 \sin \theta \right] dr d\theta d\varphi.$$

where  $\Delta r$  is the distance between the centers of weight of the pulses. Since the acceleration depends on  $\Delta r$  (15) we can introduce the nonlinear potential

$$V(\Delta r, t) = V(0, t) - \int_0^{\Delta r} \langle \ddot{a}_i(\Delta r, t) \rangle (d\Delta r). \quad (16)$$

Let us suppose that the pulses do not change their shape and spectrum during the propagation process – as it can be seen on Figure 1 this assumption is correct, when the pulses are at a sufficient distance from each other. At close distances the acceleration and potential depend significantly on time. We use trial functions with Gaussian profile (as in the numerical experiments above)

$$A = A_1 = A_2 = A_0 \exp\left(-\frac{x^2 + y^2 + z^2}{2}\right) = A_0 \exp\left(-\frac{r^2}{2}\right). \quad (17)$$

The exact formulae of the nonlinear acceleration  $\ddot{a}(\Delta r)$  (15) and the potential (16) between the centers of weight for normalized constant  $2\pi^2 C_1 A_0^4 = 1$  are

$$\langle \ddot{a}_i^{Gauss}(\Delta r) \rangle = -2\sqrt{2\pi}\Delta r(3 + \Delta r^2)\exp(-\Delta r/2); \quad V^{Gauss}(\Delta r) = 2\sqrt{2\pi}(5 + \Delta r^2)\exp(-\Delta r/2). \quad (18)$$

In the general case, *the acceleration and the potential are not stationary* and as it can be seen from the expressions (15) and (16) depend in addition on the time. That is why the spatial forms and the spectrums of the pulses at short distances are modulated significantly. The numerical experiments demonstrate, that if the pulses are separated along  $x$  direction, the forms and the  $k_x$  spectrums of the both pulses become asymmetric.

## CONCLUSIONS

We have developed a vector model to describe the phenomena of interflow of filaments and Rogue events as well as the reduction of number of the filaments as a function of the distance during multi-filament propagation. The results of the numerical analysis show that the investigated above processes are result of nonlinear interactions due to CPM and FPPM mechanisms. The numerical experiments are performed with respect the initial phase difference between the optical pulses. The merging between spherically-symmetric filaments is investigated analytically by using the method of moments.

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