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On the number of pairs of positive integers $x, y \leq H$ such that $x^2 + y^2 + 1$, $x^2 + y^2 + 2$ are square-free

by

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Dedicated to Professor Doychin Tolev
on the occasion of his 60th birthday

1. Notations. Let H be a sufficiently large positive number. We denote by ε an arbitrary small positive number, not the same in all appearances. The letters d, h, k, l, q, r, x, y with or without subscript will denote positive integers. By the letters D_1, D_2, H_0 and t we denote real numbers and by m, n integers. As usual $\mu(n)$ is the Möbius function and $\tau(n)$ denotes the number of positive divisors of n. Further $[t]$ and $\{t\}$ denote the integer part, respectively, the fractional part of t. Instead of $m \equiv n \pmod d$ we write for simplicity $m \equiv n\ (d)$. Moreover (l, m) is the greatest common divisor of l and m, and (l, m, n) is the greatest common divisor of l, m and n. The letter p will always denote a prime number. We put $\psi(t) = \{t\} - 1/2$. As usual $e(t)$=$\exp(2\pi i t)$. For any odd q we denote by $\left(\frac{\cdot}{q}\right)$ the Jacobi symbol. For any n and q such that $(n, q) = 1$ we denote by $\overline{(n)}_q$ the inverse of n modulo q. If the value of the modulus is understood from the context then we simply write $\bar{n}$.

We shall consider the Gauss sums

$$G(q, m, n) = \sum_{x=1}^{q} e\left(\frac{mx^2 + nx}{q}\right), \qquad G(q, m) = G(q, m, 0), \tag{1.1}$$

and the Kloosterman sums

$$K(q, m, n) = \sum_{\substack{x=1 \\ (x,q)=1}}^{q} e\left(\frac{mx + n\bar{x}}{q}\right). \tag{1.2}$$

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2. Introduction and statement of the result. The problem of consecutive square-free numbers arose in 1932 when Carlitz [1] proved that

$$\sum_{n\le H}\mu^2(n)\mu^2(n+1)=\prod_p\left(1-\frac{2}{p^2}\right)H+\mathcal{O}\big(H^{\theta+\varepsilon}\big), \tag{2.1}$$

where $\theta=2/3$. Formula (2.1) was subsequently improved by Heath-Brown [6] to $\theta=7/11$ and by Reuss [9] to $\theta=(26+\sqrt{433})/81$.

In 2018 the author [2] showed that for any fixed $1<c<22/13$ there exist infinitely many consecutive square-free numbers of the form $[n^c]$, $[n^c]+1$.

Recently the author [3] proved that there exist infinitely many consecutive square-free numbers of the form $[\alpha n]$, $[\alpha n]+1$, where n is natural and $\alpha>1$ is an irrational number with bounded partial quotient or irrational algebraic number.

Also recently the author [4] showed that there exist infinitely many consecutive square-free numbers of the form $[\alpha p]$, $[\alpha p]+1$, where p is prime and $\alpha>0$ is an irrational algebraic number.

On the other hand, in 2012 Tolev [11] proved ingeniously that there exist infinitely many square-free numbers of the form x^2+y^2+1. More precisely he established the asymptotic formula

$$\sum_{1\le x,y\le H}\mu^2(x^2+y^2+1)=cH^2+\mathcal{O}(H^{4/3+\varepsilon}),$$

where

$$c=\prod_p\left(1-\frac{\lambda(p^2)}{p^4}\right)\quad\text{and}\quad\lambda(q)=\sum_{\substack{1\le x,y\le q\\ x^2+y^2+1\equiv 0\,(q)}}1.$$

Define

$$\Gamma(H)=\sum_{1\le x,y\le H}\mu^2(x^2+y^2+1)\,\mu^2(x^2+y^2+2) \tag{2.2}$$

and

$$\lambda(q_1,q_2,m,n)=\sum_{x,y\,:\,(2.4)}e\left(\frac{mx+ny}{q_1q_2}\right), \tag{2.3}$$

where the summation is taken over the integers x,y satisfying the conditions

$$\begin{aligned}&1\le x,y\le q_1q_2,\\ &x^2+y^2+1\equiv 0\ (q_1),\\ &x^2+y^2+2\equiv 0\ (q_2).\end{aligned} \tag{2.4}$$

We also define

$$\lambda(q_1,q_2)=\lambda(q_1,q_2,0,0). \tag{2.5}$$

Motivated by these results and following the method of Tolev [11] we shall prove the following theorem.

THEOREM 2.1. *For the sum $\Gamma(H)$ defined by* (2.2) *we have the asymptotic formula*

$$\Gamma(H) = \sigma H^2 + \mathcal{O}(H^{8/5+\varepsilon}), \tag{2.6}$$

where

$$\sigma = \prod_p \left(1 - \frac{\lambda(p^2,1)+\lambda(1,p^2)}{p^4}\right). \tag{2.7}$$

From Theorem 2.1 it follows that there exist infinitely many consecutive square-free numbers of the form x^2+y^2+1, x^2+y^2+2.

3. Lemmas. The first lemma we need gives us the basic properties of the Gauss sum.

LEMMA 3.1 ([5, 7]).

(i) *If* $(q_1,q_2)=1$ *then*

$$G(q_1q_2, m_1q_2+m_2q_1, n) = G(q_1, m_1q_2^2, n)G(q_2, m_2q_1^2, n).$$

(ii) *If* $(q,m)=d$ *then*

$$G(q,m,n) = \begin{cases} dG(q/d, m/d, n/d) & \text{if } d \mid n, \\ 0 & \text{if } d \nmid n. \end{cases}$$

(iii) *If* $(q,2m)=1$ *then*

$$G(q,m,n) = e\left(\frac{-\overline{4m}\,n^2}{q}\right)\left(\frac{m}{q}\right)G(q,1).$$

(iv) *If* $(q,2)=1$ *then*

$$G^2(q,1) = (-1)^{(q-1)/2}q.$$

The next lemma gives us A. Weil's estimate for the Kloosterman sum.

LEMMA 3.2 ([8]).

$$|K(q,m,n)| \le \tau(q)q^{1/2}(q,m,n)^{1/2}.$$

The next lemma is the central moment in the proof of Theorem 2.1. Right here we apply the properties of the Gauss sum and A. Weil's estimate for the Kloosterman sum.

LEMMA 3.3. *Let $8 \nmid q_1$, $8 \nmid q_2$ and $(q_1,q_2)=1$. Then for the function λ defined by* (2.3) *we have*

$$|\lambda(q_1,q_2,m,n)| \le 16\tau^2(q_1q_2)(q_1q_2)^{1/2}(q_1q_2,m,n)^{1/2}. \tag{3.1}$$

In particular,

$$\lambda(q_1,q_2) \ll (q_1q_2)^{1+\varepsilon}. \tag{3.2}$$

Remark 3.4. An estimate of type (3.1) is valid for any positive integers q_1,q_2. We introduce the restrictions $8\nmid q_1$, $8\nmid q_2$ since in this case the proof is slightly simpler and only such q_1,q_2 appear in our work.

Proof. We consider three cases.

Case 1: $2\nmid q_1q_2$. Using (1.1), (2.3), (2.4) and Lemma 3.1 we get

$$\begin{aligned}
(3.3)\qquad &\lambda(q_1,q_2,m,n)\\
&=\frac{1}{q_1q_2}\sum_{1\le x,y\le q_1q_2} e\left(\frac{mx+ny}{q_1q_2}\right)\sum_{1\le h_1\le q_1} e\left(\frac{h_1(x^2+y^2+1)}{q_1}\right)\\
&\quad\times\sum_{1\le h_2\le q_2} e\left(\frac{h_2(x^2+y^2+2)}{q_2}\right)\\
&=\frac{1}{q_1q_2}\sum_{1\le h_1\le q_1} e\left(\frac{h_1}{q_1}\right)\sum_{1\le h_2\le q_2} e\left(\frac{2h_2}{q_2}\right)G(q_1q_2,h_1q_2+h_2q_1,m)\\
&\quad\times G(q_1q_2,h_1q_2+h_2q_1,n)\\
&=\frac{1}{q_1q_2}\sum_{1\le h_1\le q_1} e\left(\frac{h_1}{q_1}\right)G(q_1,h_1q_2^2,m)G(q_1,h_1q_2^2,n)\\
&\quad\times\sum_{1\le h_2\le q_2} e\left(\frac{2h_2}{q_2}\right)G(q_2,h_2q_1^2,m)G(q_2,h_2q_1^2,n)\\
&=\frac{1}{q_1q_2}\sum_{l_1\mid q_1}\sum_{\substack{1\le h_1\le q_1\\(h_1,q_1)=q_1/l_1}} e\left(\frac{h_1}{q_1}\right)G(q_1,h_1q_2^2,m)G(q_1,h_1q_2^2,n)\\
&\quad\times\sum_{l_2\mid q_2}\sum_{\substack{1\le h_2\le q_2\\(h_2,q_2)=q_2/l_2}} e\left(\frac{2h_2}{q_2}\right)G(q_2,h_2q_1^2,m)G(q_2,h_2q_1^2,n).
\end{aligned}$$

Bearing in mind (1.2), (3.3), $2\nmid q_1q_2$ and Lemma 3.1 we obtain

$$\begin{aligned}
(3.4)\qquad &\lambda(q_1,q_2,m,n)\\
&=q_1q_2\sum_{\substack{l_1\mid q_1\\(q_1/l_1)\mid(m,n)}}\frac{1}{l_1^2}\sum_{\substack{1\le r_1\le l_1\\(r_1,l_1)=1}} e\left(\frac{r_1}{l_1}\right)G(l_1,r_1q_2^2,ml_1q_1^{-1})G(l_1,r_1q_2^2,nl_1q_1^{-1})
\end{aligned}$$

$$\times \sum_{\substack{l_2\mid q_2\\(q_2/l_2)\mid(m,n)}} \frac{1}{l_2^2} \sum_{\substack{1\le r_2\le l_2\\(r_2,l_2)=1}} e\left(\frac{2r_2}{l_2}\right) G(l_2, r_2q_1^2, ml_2q_2^{-1})G(l_2, r_2q_1^2, nl_2q_2^{-1})$$

$$= q_1q_2 \sum_{\substack{l_1\mid q_1\\(q_1/l_1)\mid(m,n)}} \frac{G^2(l_1,1)}{l_1^2} \sum_{\substack{1\le r_1\le l_1\\(r_1,l_1)=1}} e\left(\frac{r_1-\overline{4r_1q_2^2}(m^2+n^2)l_1^2q_1^{-2}}{l_1}\right)$$

$$\times \sum_{\substack{l_2\mid q_2\\(q_2/l_2)\mid(m,n)}} \frac{G^2(l_2,1)}{l_2^2} \sum_{\substack{1\le r_2\le l_2\\(r_2,l_2)=1}} e\left(\frac{2r_2-\overline{4r_2q_1^2}(m^2+n^2)l_2^2q_2^{-2}}{l_2}\right)$$

$$= q_1q_2 \sum_{\substack{l_1\mid q_1\\(q_1/l_1)\mid(m,n)}} \frac{(-1)^{(l_1-1)/2}}{l_1} K(l_1, 1, \overline{4q_2^2}(m^2+n^2)l_1^2q_1^{-2})$$

$$\times \sum_{\substack{l_2\mid q_2\\(q_2/l_2)\mid(m,n)}} \frac{(-1)^{(l_2-1)/2}}{l_2} K(l_2, 2, \overline{4q_1^2}(m^2+n^2)l_2^2q_2^{-2}).$$

From (3.4) and Lemma 3.2 it follows that

$$(3.5)\quad |\lambda(q_1,q_2,m,n)| \le q_1q_2 \sum_{\substack{l_1\mid q_1\\(q_1/l_1)\mid(m,n)}} \frac{\tau(l_1)}{l_1^{1/2}} \sum_{\substack{l_2\mid q_2\\(q_2/l_2)\mid(m,n)}} \frac{\tau(l_2)}{l_2^{1/2}}$$
$$\le q_1q_2\tau(q_1q_2) \sum_{r_1\mid(q_1,m,n)} q_1^{-1/2}r_1^{1/2} \sum_{r_2\mid(q_2,m,n)} q_2^{-1/2}r_2^{1/2}$$
$$\le \tau^2(q_1q_2)(q_1q_2)^{1/2}(q_1q_2,m,n)^{1/2}.$$

CASE 2: $q_1 = 2^hq_1'$, where $2\nmid q_1'$ and $h\le 2$, and $2\nmid q_2$. The function $\lambda(q_1,q_2,m,n)$ defined by (2.3) is such that if

$$(q_1'q_1'', q_2'q_2'') = (q_1', q_1'') = (q_2', q_2'') = 1$$

then

$$(3.6)\quad \lambda(q_1'q_1'', q_2'q_2'', m, n) = \lambda(q_1', q_2', m\overline{(q_1''q_2'')}_{q_1'q_2'}, n\overline{(q_1''q_2'')}_{q_1'q_2'})$$
$$\times \lambda(q_1'', q_2'', m\overline{(q_1'q_2')}_{q_1''q_2''}, n\overline{(q_1'q_2')}_{q_1''q_2''}).$$

(Since the proof is elementary we skip the details.)

Using (3.5), (3.6) and the trivial estimate $|\lambda(2^h,1,m,n)|\le 4^h$ we get

$$(3.7)\quad |\lambda(2^hq_1', q_2, m, n)| = \big|\lambda(2^h, 1, m\overline{(q_1'q_2)}_{2^h}, n\overline{(q_1'q_2)}_{2^h})$$
$$\times \lambda(q_1', q_2, m\overline{(2^h)}_{q_1'q_2}, n\overline{(2^h)}_{q_1'q_2})\big|$$
$$\le 16\tau^2(q_1'q_2)(q_1'q_2)^{1/2}(q_1'q_2,m,n)^{1/2}$$
$$\le 16\tau^2(q_1q_2)(q_1q_2)^{1/2}(q_1q_2,m,n)^{1/2}.$$

Case 3: $2 \nmid q_1$ and $q_2 = 2^h q_2'$, where $2 \nmid q_2'$ and $h \le 2$. By (3.5), (3.6) and the trivial estimate $|\lambda(1, 2^h, m, n)| \le 4^h$ we get

$$\begin{aligned} |\lambda(q_1, 2^h q_2', m, n)| &= \big|\lambda(1, 2^h, m\overline{(q_1 q_2')}_{2^h}, n\overline{(q_1 q_2')}_{2^h}) \\ &\qquad \times \lambda(q_1, q_2', m\overline{(2^h)}_{q_1 q_2'}, n\overline{(2^h)}_{q_1 q_2'})\big| \\ &\le 16\tau^2(q_1 q_2')(q_1 q_2')^{1/2}(q_1 q_2', m, n)^{1/2} \\ &\le 16\tau^2(q_1 q_2)(q_1 q_2)^{1/2}(q_1 q_2, m, n)^{1/2}. \end{aligned} \tag{3.8}$$

Now (3.1) follows from (3.5), (3.7) and (3.8). As a byproduct of (3.1) we obtain (3.2). ∎

Lemma 3.5. *Assume that* $8 \nmid q_1$, $8 \nmid q_2$, $(q_1, q_2) = 1$ *and* $H_0 \ge 2$. *Then for the sums*

$$\Lambda_1 = \sum_{1\le m\le H_0} \frac{|\lambda(q_1, q_2, m, 0)|}{m}, \qquad \Lambda_2 = \sum_{1\le m,n\le H_0} \frac{|\lambda(q_1, q_2, m, n)|}{mn} \tag{3.9}$$

we have

$$\Lambda_1 \ll (q_1 q_2)^{1/2+\varepsilon} H_0^\varepsilon, \qquad \Lambda_2 \ll (q_1 q_2)^{1/2+\varepsilon} H_0^\varepsilon. \tag{3.10}$$

Proof. Using (3.9) and Lemma 3.3 we get

$$\Lambda_1 \ll (q_1 q_2)^{1/2+\varepsilon} \sum_{1\le m\le H_0} \frac{(q_1 q_2, m)^{1/2}}{m} = (q_1 q_2)^{1/2+\varepsilon} \Lambda_0, \tag{3.11}$$

where

$$\Lambda_0 = \sum_{1\le m\le H_0} \frac{(q_1 q_2, m)^{1/2}}{m}.$$

We have

$$\Lambda_0 \ll \sum_{r|q_1 q_2} r^{1/2} \sum_{\substack{m\le H_0 \\ m\equiv 0\,(r)}} \frac{1}{m} \ll (\log H_0) \sum_{r|q_1 q_2} r^{-1/2} \ll (q_1 q_2 H_0)^\varepsilon. \tag{3.12}$$

From (3.11) and (3.12) we deduce the first inequality in (3.10).

Using (3.9), (3.12) and Lemma 3.3 we obtain

$$\begin{aligned} \Lambda_2 &\ll (q_1 q_2)^{1/2+\varepsilon} \sum_{1\le m,n\le H_0} \frac{(q_1 q_2, m, n)^{1/2}}{mn} \\ &\ll (q_1 q_2)^{1/2+\varepsilon} \sum_{1\le m,n\le H_0} \frac{(q_1 q_2, m)^{1/2}(q_1 q_2, n)^{1/2}}{mn} \\ &= (q_1 q_2)^{1/2+\varepsilon} \Lambda_0^2 \ll (q_1 q_2)^{1/2+\varepsilon} H_0^\varepsilon, \end{aligned}$$

which proves the second inequality in (3.10). ∎

The final lemma we need gives us important expansions.

LEMMA 3.6 ([10]). *For any* $H_0 \geq 2$, *we have*

$$\psi(t) = -\sum_{1\leq|m|\leq H_0} \frac{e(mt)}{2\pi i m} + \mathcal{O}(f(H_0,t)),$$

where $f(H_0,t)$ *is a positive, infinitely differentiable and* 1*-periodic function of* t. *It can be expanded into the Fourier series*

$$f(H_0,t) = \sum_{m=-\infty}^{+\infty} b_{H_0}(m)e(mt),$$

with coefficients $b_{H_0}(m)$ *such that*

$$b_{H_0}(m) \ll \frac{\log H_0}{H_0} \quad \textit{for all } m$$

and

$$\sum_{|m|>H_0^{1+\varepsilon}} |b_{H_0}(m)| \ll H_0^{-A}.$$

Here $A>0$ *is arbitrarily large and the constant in the* $\ll$ *symbol depends on* A *and* ε.

4. Proof of the theorem. Using (2.2) and the well-known identity $\mu^2(n) = \sum_{d^2|n}\mu(d)$ we get

$$\Gamma(H) = \sum_{\substack{d_1,d_2\\(d_1,d_2)=1}} \mu(d_1)\mu(d_2) \sum_{\substack{1\leq x,y\leq H\\x^2+y^2+1\equiv 0\,(d_1^2)\\x^2+y^2+2\equiv 0\,(d_2^2)}} 1 = \Gamma_1(H)+\Gamma_2(H), \tag{4.1}$$

where

$$\Gamma_1(H) = \sum_{\substack{d_1d_2\leq z\\(d_1,d_2)=1}} \mu(d_1)\mu(d_2)\Sigma(H,d_1^2,d_2^2), \tag{4.2}$$

$$\Gamma_2(H) = \sum_{\substack{d_1d_2> z\\(d_1,d_2)=1}} \mu(d_1)\mu(d_2)\Sigma(H,d_1^2,d_2^2), \tag{4.3}$$

$$\Sigma(H,d_1^2,d_2^2) = \sum_{\substack{1\leq x,y\leq H\\x^2+y^2+1\equiv 0\,(d_1^2)\\x^2+y^2+2\equiv 0\,(d_2^2)}} 1, \tag{4.4}$$

$$\sqrt{H} \leq z \leq H. \tag{4.5}$$

Estimation of $\Gamma_1(H)$**.** When estimating $\Gamma_1(H)$ we will suppose that $q_1 = d_1^2$, $q_2 = d_2^2$, where d_1 and d_2 are square-free, $(q_1,q_2)=1$ and $d_1d_2\leq z$.

Denote

$$\Omega(H,q_1,q_2,x)=\sum_{\substack{h\le H\\ h\equiv x\,(q_1q_2)}}1. \tag{4.6}$$

Apparently

$$\Omega(H,q_1,q_2,x)=Hq_1^{-1}q_2^{-1}+\mathcal{O}(1). \tag{4.7}$$

Using (2.4), (4.4) and (4.6) we obtain

$$\Sigma(H,q_1,q_2)=\sum_{x,y\,:\,(2.4)}\Omega(H,q_1,q_2,x)\,\Omega(H,q_1,q_2,y). \tag{4.8}$$

On the other hand, (4.6) gives us

$$\Omega(H,q_1,q_2,y)=\left[\frac{H-y}{q_1q_2}\right]-\left[\frac{-y}{q_1q_2}\right]=\frac{H}{q_1q_2}+\psi\left(\frac{-y}{q_1q_2}\right)-\psi\left(\frac{H-y}{q_1q_2}\right). \tag{4.9}$$

From (4.8) and (4.9) we find that

$$\Sigma(H,q_1,q_2)=\sum_{x,y\,:\,(2.4)}\Omega(H,q_1,q_2,x)\left(\frac{H}{q_1q_2}-\psi\left(\frac{H-y}{q_1q_2}\right)\right)+\Sigma', \tag{4.10}$$

where

$$\Sigma'=\sum_{x,y\,:\,(2.4)}\Omega(H,q_1,q_2,x)\,\psi\left(\frac{-y}{q_1q_2}\right).$$

To estimate Σ', we decompose it as

$$\Sigma'=\Sigma''+\Sigma''', \tag{4.11}$$

where

$$\Sigma''=\sum_{\substack{1\le x\le q_1q_2\\ x^2+1\equiv0\,(q_1)\\ x^2+2\equiv0\,(q_2)}}\Omega(H,q_1,q_2,x)\sum_{\substack{1\le y\le q_1q_2\\ y^2\equiv0\,(q_1q_2)}}\psi\left(\frac{-y}{q_1q_2}\right), \tag{4.12}$$

$$\Sigma'''=\sum_{\substack{1\le x\le q_1q_2\\ x^2+1\not\equiv0\,(q_1)\\ x^2+2\not\equiv0\,(q_2)}}\Omega(H,q_1,q_2,x)\sum_{\substack{1\le y\le q_1q_2\\ y^2\equiv-x^2-1\,(q_1)\\ y^2\equiv-x^2-2\,(q_2)}}\psi\left(\frac{-y}{q_1q_2}\right). \tag{4.13}$$

First, we consider Σ'''. We note that the sum over y in (4.13) does not contain terms with $y=q_1q_2/2$ or $y=q_1q_2$. Moreover for any y satisfying the congruences and such that $1\le y<q_1q_2/2$ the number q_1q_2-y satisfies the same congruences and we have $\psi\big(\frac{-y}{q_1q_2}\big)+\psi\big(\frac{-(q_1q_2-y)}{q_1q_2}\big)=0$. Consequently,

$$\Sigma'''=0. \tag{4.14}$$

Next we consider the sum Σ''. According to the above considerations the sum over y in (4.12) reduces to a sum with at most two terms (corresponding to

$y = q_1q_2/2$ and $y = q_1q_2$). Therefore

$$\Sigma'' \ll \sum_{\substack{1\le x\le q_1q_2\\ x^2+1\equiv 0\,(q_1)\\ x^2+2\equiv 0\,(q_2)}} \Omega(H, q_1, q_2, x). \tag{4.15}$$

Now taking into account (4.7), (4.15), the Chinese remainder theorem and the fact that the number of solutions of the congruence $x^2 \equiv a\,(q_1q_2)$ is $\mathcal{O}((q_1q_2)^\varepsilon) = \mathcal{O}(H^\varepsilon)$, we get

$$\Sigma'' \ll \sum_{\substack{1\le x\le q_1q_2\\ x^2\equiv a\,(q_1q_2)}} \Omega(H, q_1, q_2, x) \ll H^\varepsilon(Hq_1^{-1}q_2^{-1} + 1). \tag{4.16}$$

Here a depends on q_1 and q_2.

From (4.11), (4.14) and (4.16) it follows that

$$\Sigma' \ll H^\varepsilon(Hq_1^{-1}q_2^{-1} + 1). \tag{4.17}$$

By (4.10) and (4.17) we obtain

$$\begin{aligned}\Sigma(H, q_1, q_2) = \sum_{x,y\,:\,(2.4)} \Omega(H, q_1, q_2, x)\bigg(\frac{H}{q_1q_2} - \psi\bigg(\frac{H-y}{q_1q_2}\bigg)\bigg)&\\ + \mathcal{O}(H^\varepsilon(Hq_1^{-1}q_2^{-1} + 1)).&\end{aligned} \tag{4.18}$$

Proceeding in the same way with the sum $\Omega(H, q_1, q_2, x)$ we find that

$$\begin{aligned}\Sigma(H, q_1, q_2) = \sum_{x,y\,:\,(2.4)} \left(\frac{H}{q_1q_2} - \psi\left(\frac{H-x}{q_1q_2}\right)\right)\left(\frac{H}{q_1q_2} - \psi\left(\frac{H-y}{q_1q_2}\right)\right)&\\ + \mathcal{O}(H^\varepsilon(Hq_1^{-1}q_2^{-1} + 1)).&\end{aligned} \tag{4.19}$$

Bearing in mind (2.3), (2.5) and (4.19) we get

$$\begin{aligned}\Sigma(H, q_1, q_2) = \frac{H^2\lambda(q_1, q_2)}{q_1^2q_2^2} - 2\frac{H}{q_1q_2}\Sigma_1(H, q_1, q_2)&\\ + \Sigma_2(H, q_1, q_2) + \mathcal{O}(H^\varepsilon(Hq_1^{-1}q_2^{-1} + 1)),&\end{aligned} \tag{4.20}$$

where

$$\Sigma_1(H, q_1, q_2) = \sum_{x,y\,:\,(2.4)} \psi\bigg(\frac{H-x}{q_1q_2}\bigg), \tag{4.21}$$

$$\Sigma_2(H, q_1, q_2) = \sum_{x,y\,:\,(2.4)} \psi\bigg(\frac{H-x}{q_1q_2}\bigg)\psi\bigg(\frac{H-y}{q_1q_2}\bigg). \tag{4.22}$$

First we consider $\Sigma_1(H, q_1, q_2)$. Using (2.3) and Lemma 3.6 with $H_0 = H$ we obtain

$$\Sigma_1(H, q_1, q_2) = \Sigma_1'(H, q_1, q_2) + \mathcal{O}(\Sigma_1''(H, q_1, q_2)), \tag{4.23}$$

where

$$\Sigma_1'(H,q_1,q_2) = -\sum_{x,y\,:\,(2.4)}\sum_{1\le|m|\le H}\frac{e\big(m\big(\frac{H-x}{q_1q_2}\big)\big)}{2\pi im} \tag{4.24}$$
$$= -\sum_{1\le|m|\le H}\frac{e\big(\frac{mH}{q_1q_2}\big)\lambda(q_1,q_2,-m,0)}{2\pi im},$$

$$\Sigma_1''(H,q_1,q_2) = \sum_{x,y\,:\,(2.4)} f\bigg(H,\frac{H-x}{q_1q_2}\bigg). \tag{4.25}$$

Formula (4.24) and Lemma 3.5 give us

$$\Sigma_1'(H,q_1,q_2) \ll H^{\varepsilon}\,(q_1q_2)^{1/2}. \tag{4.26}$$

In order to estimate $\Sigma_1''(H,q_1,q_2)$ we use (2.3), (2.5), (4.25) and Lemmas 3.3, 3.5, 3.6 to get

$$\Sigma_1''(H,q_1,q_2) \tag{4.27}$$
$$= \sum_{x,y\,:\,(2.4)}\bigg(b_H(0)+\sum_{1\le|m|\le H^{1+\varepsilon}} b_H(m)\,e\bigg(m\bigg(\frac{H-x}{q_1q_2}\bigg)\bigg)\bigg)+\mathcal{O}(1)$$
$$= b_H(0)\,\lambda(q_1,q_2)+\sum_{1\le|m|\le H^{1+\varepsilon}} b_H(m)\,e\bigg(\frac{mH}{q_1q_2}\bigg)\lambda(q_1,q_2,-m,0)+\mathcal{O}(1)$$
$$\ll H^{\varepsilon-1}q_1q_2+1+H^{\varepsilon-1}\sum_{1\le|m|\le H^{1+\varepsilon}}|\lambda(q_1,q_2,-m,0)|$$
$$\ll H^{\varepsilon-1}q_1q_2+1+H^{\varepsilon}\sum_{1\le|m|\le H^{1+\varepsilon}}\frac{|\lambda(q_1,q_2,m,0)|}{m} \ll H^{\varepsilon-1}q_1q_2+H^{\varepsilon}(q_1q_2)^{1/2}.$$

From (4.23), (4.26) and (4.27) it follows that

$$\Sigma_1(H,q_1,q_2) \ll H^{\varepsilon-1}q_1q_2+H^{\varepsilon}(q_1q_2)^{1/2}. \tag{4.28}$$

Next we consider the sum $\Sigma_2(H,q_1,q_2)$. Bearing in mind (4.22), (4.25), (4.27) and Lemmas 3.5 and 3.6 we find that

$$\Sigma_2(H,q_1,q_2) = \sum_{x,y\,:\,(2.4)}\sum_{1\le|m|,|n|\le H}\frac{e\big(\frac{(m+n)H}{q_1q_2}\big)e\big(-\frac{mx+ny}{q_1q_2}\big)}{(2\pi i)^2mn} \tag{4.29}$$
$$+\mathcal{O}(H^{\varepsilon}\Sigma_1''(H,q_1,q_2))$$
$$= \sum_{1\le|m|,|n|\le H}\frac{e\big(\frac{(m+n)H}{q_1q_2}\big)}{(2\pi i)^2mn}\lambda(q_1,q_2,-m,-n)$$
$$+\mathcal{O}(H^{\varepsilon-1}\,q_1q_2+H^{\varepsilon}\,(q_1q_2)^{1/2})$$

$$\ll \sum_{1\le|m|,|n|\le H}\frac{|\lambda(q_1,q_2,m,n)|}{|mn|}+H^{\varepsilon-1}q_1q_2+H^{\varepsilon}(q_1q_2)^{1/2}$$
$$\ll H^{\varepsilon-1}q_1q_2+H^{\varepsilon}(q_1q_2)^{1/2}.$$

Taking into account (4.20), (4.28) and (4.29) we get

$$\Sigma(H,q_1,q_2)=H^2\frac{\lambda(q_1,q_2)}{q_1^2q_2^2}+\mathcal{O}\big(H^{\varepsilon}(Hq_1^{-1/2}q_2^{-1/2}+q_1^{1/2}q_2^{1/2}+H^{-1}q_1q_2)\big). \tag{4.30}$$

From (4.2), (4.5) and (4.30) we obtain

$$\begin{aligned}\Gamma_1(H)&=H^2\sum_{\substack{d_1d_2\le z\\(d_1,d_2)=1}}\frac{\mu(d_1)\mu(d_2)\lambda(d_1^2,d_2^2)}{d_1^4d_2^4}+\mathcal{O}\big(H^{\varepsilon}z^2\big)\\&=\sigma H^2-H^2\sum_{\substack{d_1d_2> z\\(d_1,d_2)=1}}\frac{\mu(d_1)\mu(d_2)\lambda(d_1^2,d_2^2)}{d_1^4d_2^4}+\mathcal{O}\big(H^{\varepsilon}z^2\big),\end{aligned} \tag{4.31}$$

where

$$\sigma=\sum_{\substack{d_1,d_2=1\\(d_1,d_2)=1}}^{\infty}\frac{\mu(d_1)\mu(d_2)\lambda(d_1^2,d_2^2)}{d_1^4d_2^4}. \tag{4.32}$$

Using (3.2) we find that

$$\sum_{\substack{d_1d_2> z\\(d_1,d_2)=1}}\frac{\mu(d_1)\mu(d_2)\lambda(d_1^2,d_2^2)}{d_1^4d_2^4}\ll\sum_{\substack{d_1d_2> z\\(d_1,d_2)=1}}\frac{(d_1d_2)^{2+\varepsilon}}{(d_1d_2)^4}\ll\sum_{n>z}\frac{\tau(n)}{n^{2-\varepsilon}}\ll z^{\varepsilon-1}. \tag{4.33}$$

It remains to see that the product (2.7) and the sum (4.32) coincide. From (2.5), (3.6) and $(d_1,d_2)=1$ it follows that

$$\lambda(d_1^2,d_2^2)=\lambda(d_1^2,1)\lambda(1,d_2^2). \tag{4.34}$$

Bearing in mind (4.32) and (4.34) we get

$$\sigma=\sum_{d_1=1}^{\infty}\frac{\mu(d_1)\lambda(d_1^2,1)}{d_1^4}\sum_{d_2=1}^{\infty}\frac{\mu(d_2)\lambda(1,d_2^2)}{d_2^4}f_{d_1}(d_2), \tag{4.35}$$

where

$$f_{d_1}(d_2)=\begin{cases}1 & \text{if } (d_1,d_2)=1,\\ 0 & \text{if } (d_1,d_2)>1.\end{cases}$$

Clearly the function

$$\frac{\mu(d_2)\lambda(1,d_2^2)}{d_2^4}f_{d_1}(d_2)$$

is multiplicative with respect to d_2 and the series

$$\sum_{d_2=1}^{\infty}\frac{\mu(d_2)\lambda(1,d_2^2)}{d_2^4}f_{d_1}(d_2)$$

is absolutely convergent.

Applying the Euler product we obtain

$$\begin{aligned}(4.36)\qquad &\sum_{d_2=1}^{\infty}\frac{\mu(d_2)\lambda(1,d_2^2)}{d_2^4}f_{d_1}(d_2)\\ &=\prod_{p\nmid d_1}\left(1-\frac{\lambda(1,p^2)}{p^4}\right)=\prod_{p}\left(1-\frac{\lambda(1,p^2)}{p^4}\right)\prod_{p\mid d_1}\left(1-\frac{\lambda(1,p^2)}{p^4}\right)^{-1}.\end{aligned}$$

From (4.35) and (4.36) it follows that

$$\begin{aligned}(4.37)\qquad \sigma &=\sum_{d_1=1}^{\infty}\frac{\mu(d_1)\lambda(d_1^2,1)}{d_1^4}\prod_{p}\left(1-\frac{\lambda(1,p^2)}{p^4}\right)\prod_{p\mid d_1}\left(1-\frac{\lambda(1,p^2)}{p^4}\right)^{-1}\\ &=\prod_{p}\left(1-\frac{\lambda(1,p^2)}{p^4}\right)\sum_{d_1=1}^{\infty}\frac{\mu(d_1)\lambda(d_1^2,1)}{d_1^4}\prod_{p\mid d_1}\left(1-\frac{\lambda(1,p^2)}{p^4}\right)^{-1}.\end{aligned}$$

Obviously the function

$$\frac{\mu(d_1)\lambda(d_1^2,1)}{d_1^4}\prod_{p\mid d_1}\left(1-\frac{\lambda(1,p^2)}{p^4}\right)^{-1}$$

is multiplicative with respect to d_1 and the series

$$\sum_{d_1=1}^{\infty}\frac{\mu(d_1)\lambda(d_1^2,1)}{d_1^4}\prod_{p\mid d_1}\left(1-\frac{\lambda(1,p^2)}{p^4}\right)^{-1}$$

is absolutely convergent. Applying again the Euler product from (4.37) we obtain

$$\begin{aligned}(4.38)\qquad \sigma &=\prod_{p}\left(1-\frac{\lambda(1,p^2)}{p^4}\right)\prod_{p}\left(1-\frac{\lambda(p^2,1)}{p^4}\left(1-\frac{\lambda(1,p^2)}{p^4}\right)^{-1}\right)\\ &=\prod_{p}\left(1-\frac{\lambda(p^2,1)+\lambda(1,p^2)}{p^4}\right).\end{aligned}$$

Bearing in mind (4.5), (4.31), (4.33) and (4.38) we get

$$(4.39)\qquad \Gamma_1(H)=\sigma H^2+\mathcal{O}(H^{\varepsilon}(z^2+H^2z^{-1})),$$

where σ is given by the product (2.7).

Estimation of $\Gamma_2(H)$. Using (4.3) we write

$$|\Gamma_2(H)| \ll (\log H)^2 \sum_{D_1\le d_1<2D_1} \sum_{D_2\le d_2<2D_2} \sum_{\substack{k\le(2H^2+1)d_1^{-2} \\ kd_1^2+1\equiv 0\,(d_2^2)}} \sum_{\substack{1\le x,y\le H \\ x^2+y^2=kd_1^2-1}} 1, \tag{4.40}$$

where

$$1/2 \le D_1, D_2 \le \sqrt{2H^2+2}, \quad D_1D_2 \ge z/4. \tag{4.41}$$

On the one hand, (4.40) gives us

$$\begin{aligned} |\Gamma_2(H)| &\ll H^{\varepsilon} \sum_{D_1\le d_1<2D_1} \sum_{k\le(2H^2+1)D_1^{-2}} \sum_{D_2\le d_2<2D_2} \sum_{\substack{l\le(2H^2+2)D_2^{-2} \\ kd_1^2+1=ld_2^2}} 1 \\ &\ll H^{\varepsilon} \sum_{D_1\le d_1<2D_1} \sum_{k\le(2H^2+1)D_1^{-2}} \tau(kd_1^2+1) \\ &\ll H^{\varepsilon} \sum_{D_1\le d_1<2D_1} \sum{}_{k\le(2H^2+1)D_1^{-2}} 1 \ll H^{2+\varepsilon} D_1^{-1}. \end{aligned} \tag{4.42}$$

On the other hand, (4.40) implies

$$\begin{aligned} |\Gamma_2(H)| &\ll H^{\varepsilon} \sum_{D_2\le d_2<2D_2} \sum_{l\le(2H^2+2)D_2^{-2}} \sum_{D_1\le d_1<2D_1} \sum_{\substack{k\le(2H^2+1)D_1^{-2} \\ kd_1^2=ld_2^2-1}} 1 \\ &\ll H^{\varepsilon} \sum_{D_2\le d_2<2D_2} \sum_{l\le(2H^2+2)D_2^{-2}} \tau(ld_2^2-1) \\ &\ll H^{\varepsilon} \sum_{D_2\le d_2<2D_2} \sum_{l\le(2H^2+2)D_2^{-2}} 1 \ll H^{2+\varepsilon} D_2^{-1}. \end{aligned} \tag{4.43}$$

By (4.41)–(4.43) it follows that

$$|\Gamma_2(H)| \ll H^{2+\varepsilon} z^{-1/2}. \tag{4.44}$$

End of the proof. Bearing in mind (4.1), (4.39) and (4.44) and choosing $z = H^{4/5}$ we obtain the asymptotic formula (2.6).

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