

COMPONENT MODE SYNTHESIS OF BEAM AND SHELL FINITE ELEMENTS

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Abstract: A method for synthesising of beam or shell elements is proposed. It can be used to describe beams with complex cross sections and plates and shells with variable thickness and other special features. Starting point is usually a finite element model by 3D elements of desired component. Sometimes this model is combined with other types of finite elements, such classical beams, plates and shells. This allows describing of complex structures like airplane fuselages and wings, helicopter rotor blades. Using deformation and vibrating modes that are typical for beams and shells the transformation matrix is synthesized. This matrix is then used to transform stiffness matrix and mass matrix of model into matrices with much smaller size. These matrices then can be used in static or dynamic analysis as a usual beam or shell element. Simple examples are shown and results are discussed.

1. Introduction

Substructuring is well known technique in finite element analysis for modeling part of the given structure. This part of the structure, also known as substructure or component, has reduced number of degrees of freedom (DOFs). The reduction of DOFs can be as much as hundred or thousand times. Usually component retains *boundary* nodes and DOFs that are common between component and the rest of the structure, while *internal* nodes and DOFs are removed. Generated in this way components can be used for solving large problems or in multybody dynamic simulations.

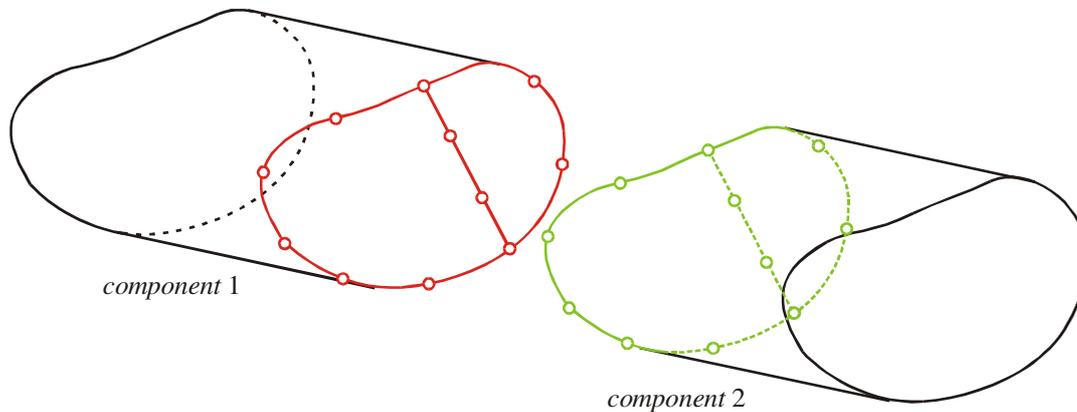


Fig. 1

However, in practice there exist problems for which existing substructuring techniques are not very efficient. For example in modeling of thin walled structures such as airplane's fuselage or wing, the number of boundary nodes and DOFs of the component can be very large (Fig. 1). Because stiffness and mass matrices of components are dense matrices, the solution of the whole problem can be inefficient.

If the shape of the structure or component is close to beam or plate it is possible further to reduce DOFs of the component and size of its matrices. The goal of present work is to introduce new substructuring technique for synthesis of beam or plate component with small number DOFs while the number of boundary nodes can be large.

2. Existing substructuring techniques

Substructuring techniques usually define relation between displacements in original structure and synthesized component. Displacements of boundary and internal nodes can be regarded as a linear combination of modes

$$\mathbf{u} = \sum_i \bar{u}_i \mathbf{q}_i \quad (1)$$

where \mathbf{u} is vector of displacements of size n , \mathbf{q}_i are modes and n is the total DOFs of original structure. Modes \mathbf{q}_i determine properties of the synthesized component. They form so called *reduced basis*. Different substructuring techniques use different sets of modes. Multipliers \bar{u}_i can be regarded as displacements of component. The number of modes is equal to the number of DOFs of the component.

By combining modes into columns of matrix \mathbf{Q} , equation 1 can be expressed in matrix form

$$\mathbf{u} = \mathbf{Q}\bar{\mathbf{u}} \quad (2)$$

Matrix \mathbf{Q} is called *transformation matrix*. It relates displacements in two different basis – original and reduced. Equations of motion in original basis is given by

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{R} \quad (3)$$

where \mathbf{M} and \mathbf{K} are mass and stiffness matrices and \mathbf{R} is load vector. By using traditional technique for finite elements when basis is changed, matrices and load vector can be expressed in reduced basis by

$$\bar{\mathbf{K}} = \mathbf{Q}^T \mathbf{K} \mathbf{Q}, \quad \bar{\mathbf{M}} = \mathbf{Q}^T \mathbf{M} \mathbf{Q}, \quad \bar{\mathbf{R}} = \mathbf{Q}^T \mathbf{R}$$

and equations of motion in reduced basis are

$$\bar{\mathbf{M}}\ddot{\bar{\mathbf{u}}} + \bar{\mathbf{K}}\bar{\mathbf{u}} = \bar{\mathbf{R}} \quad (4)$$

2.1. Static condensation

Static condensation uses m boundary DOFs in boundary nodes [1]. Every mode in \mathbf{Q} is equal to displacements due to unit displacement in one DOFs:

$$\mathbf{q}_1 = \begin{bmatrix} \mathbf{v}_1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{q}_2 = \begin{bmatrix} \mathbf{v}_2 \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \dots, \quad \mathbf{q}_m = \begin{bmatrix} \mathbf{v}_m \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \quad (5)$$

In equation 5 vector \mathbf{v}_i contains displacements of internal nodes due to unit displacement at i^{th} boundary DOF. Because modes correspond only to deformation of original structure, they can be called *deformation modes*. Deformation modes are also known as *constraint modes*. Static condensation gives good results for static problems but is not suitable for dynamic problems.

2.2. Craig-Bampton method

To improve dynamic properties of synthesized component in addition to vibration modes Craig-Bampton method uses l *vibration modes* [2]:

$$\mathbf{q}_1 = \begin{bmatrix} \mathbf{v}_1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{q}_2 = \begin{bmatrix} \mathbf{v}_2 \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \dots, \quad \mathbf{q}_m = \begin{bmatrix} \mathbf{v}_m \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}, \quad \mathbf{q}_{m+1} = \begin{bmatrix} \phi_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots, \quad \mathbf{q}_{m+l} = \begin{bmatrix} \phi_l \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (3)$$

where ϕ_i is the i^{th} vibration mode. Vibration modes are obtained from the solution of the eigenproblem for original structure with fixed boundary DOFs. By increasing number of vibration modes it is possible to achieve good results for dynamic problems.

3. Synthesis of Beam Component

Proposed methods for synthesizing beam component also uses deformation and vibration modes. Nodes in original structure are divided into internal nodes and *dependent nodes*. Dependent nodes are boundary nodes that lie in end sections, but their displacements depend on displacements in *master nodes* (Figure 2).

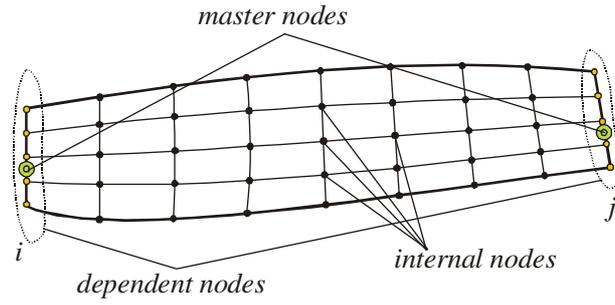


Fig. 2. 2D FEM model of beam.

Master nodes are the only nodes in beam component. They define DOFs for component. In every master nodes there are usual DOFs for beam element: 3 translations and 3 rotations for 3D beam and 2 translations and 1 rotation for 2D beam. The number of master nodes is usually 2 per component. Dependent nodes are divided in groups. Nodes from one group lie in one plane and during deformation they lie in one plane (Bernoulli hypothesis for beams' sections). Displacements of nodes from one group are defined from translations and rotations of the corresponding master node.

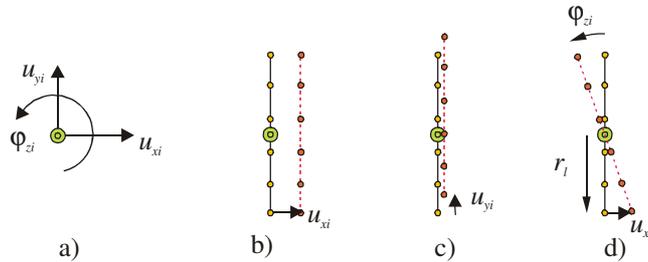


Fig. 3. Displacements of dependent nodes.

On Figure 3 are shown group of nodes and corresponding master node for 2D beam model. All nodes have the same x -coordinate. Figures 3b, 3c and 3d show displacements of dependent nodes defined by unit displacement in x -direction, unit displacement in y -direction and unit rotation about z -axis respectively.

By using relation between displacements of dependent nodes and master nodes deformation modes can be defined. For every unit displacements in master nodes corresponding displacements of dependent nodes are calculated, forming vectors \mathbf{w}_i . Then for every vector \mathbf{w}_i corresponding displacements \mathbf{v}_i of internal nodes are calculated. Vectors \mathbf{w}_i and \mathbf{v}_i form deformation modes for component synthesis. To improve dynamic properties of the component vibration modes can be added. They are calculated by fixing boundary DOFs:

$$\mathbf{q}_1 = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{w}_1 \end{bmatrix}, \dots, \mathbf{q}_m = \begin{bmatrix} \mathbf{v}_m \\ \mathbf{w}_m \end{bmatrix}, \mathbf{q}_{m+1} = \begin{bmatrix} \phi_1 \\ 0 \end{bmatrix}, \dots, \mathbf{q}_{m+l} = \begin{bmatrix} \phi_l \\ 0 \end{bmatrix} \quad (7)$$

On Figure 4 are shown displacements of nodes (deformation modes) for 2D beam model due to unit displacements of master node at the beginning of the beam.

The number of master nodes may be more than two but there are no need for this because increasing number of master nodes doesn't increase degree of approximation as in usual finite elements. On the contrary additional master nodes impose constraints on displacements of internal nodes and this can lead to poor results.

There is no needs unit displacements in different master nodes to have same directions. It is possible to define curvilinear beam component, where unit displacements are parallel and perpendicular to beam axis (it is curvilinear).

Also there is no need to use 6 DOFs in master nodes. If for example tension-compression effects are negligible, then corresponding DOFs can be removed from master DOFs.

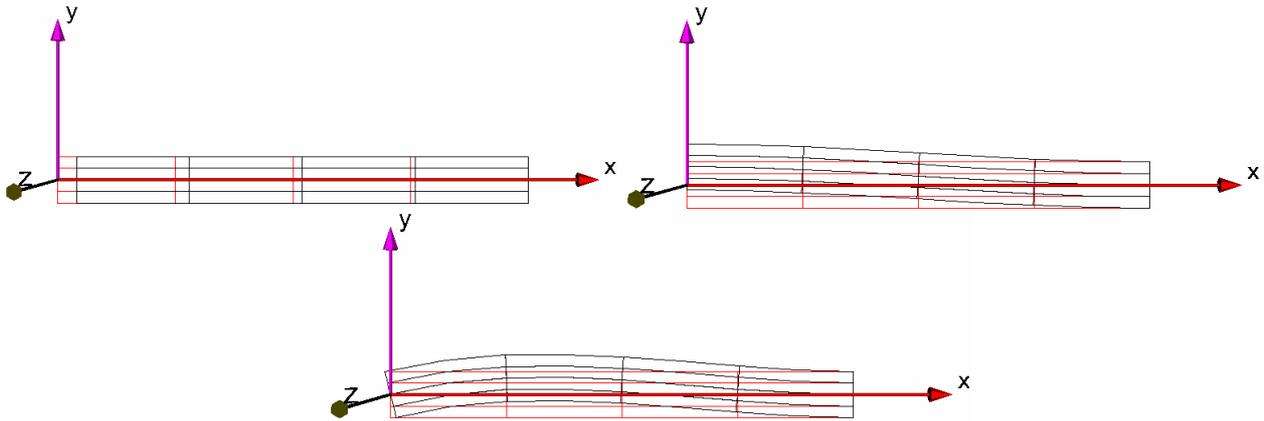


Fig. 4. Deformation modes of 2D structure for 1st master node.

4. Synthesis of Plate and Shell Component

Similar concept can be exploited to synthesize plate or shell component. Main difference is that in this case special provisions must be made to maintain displacements compatibility between elements. To achieve this displacements of dependent nodes that lie on one side of the plate (for example side 2-3 on Figure 5) must depend uniquely only on displacements of corresponding master nodes (nodes 2 and 3).

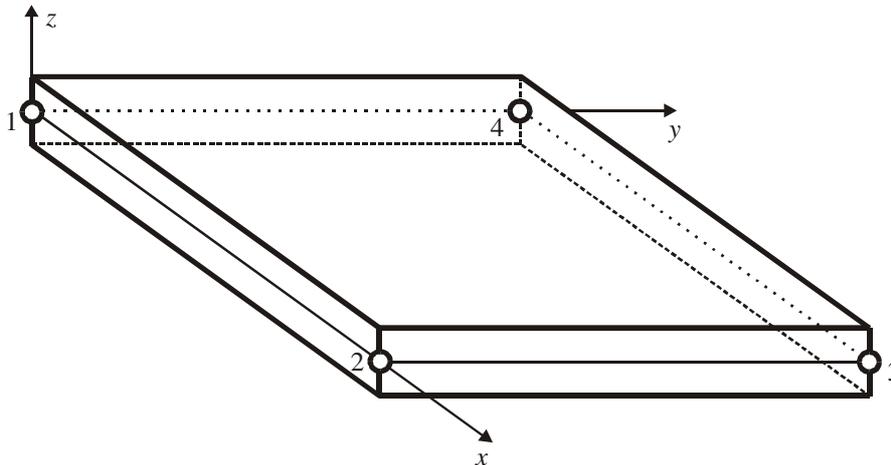


Fig. 5. Plate model.

Because there are translational and rotational DOFs in master nodes displacements of dependent nodes can be described as displacements in two node beam element with Hermite interpolation of displacements. This is reasonable choice to assure interelement displacements compatibility.

5. Example of Two Dimensional Beam

The following example validate the proposed substructuring technique. The deformation and frequencies of two dimensional beam are investigated under different conditions (Figure 6). The original beam is modeled by 40x4 9-node plain stress elements. The components are modeled by 4x4 9-node plain stress elements that have the same size as in original beam. The beam is modelled by 10 components.

The different conditions that are investigated are:

- Beams with height-to-length ratio of 1/4, 1/20 and 1/100 are examined;
- Deformation under 3 load cases are examined – tension, bending by end force and bending by end moment;
- The number of vibration modes for synthesizing component is 0, 1, 2, 4, 8 and 16;
- Two variants of Poisson's ratio are examined - $\nu=0.33$ and $\nu=0.0$.

Results are summarised in Table 1 and on Figure 7. Because the number of vibration modes has no influence on deformation, only results for zero number of vibration modes are shown in Table 1.

The variant with Poisson's ratio $\nu=0.0$ corresponds to the classical beam model. For this variant errors are much smaller than the other. Therefore as original structure is close to the beam as errors are smaller. This is also true for frequencies – for slender beams errors are smaller.

The results for frequencies show also that there are no needs to include more than 8 vibration modes for component synthesis of beam, so the total number of DOFs for beam component is 20: 2 nodes x 6 DOFs plus 8 vibration modes.

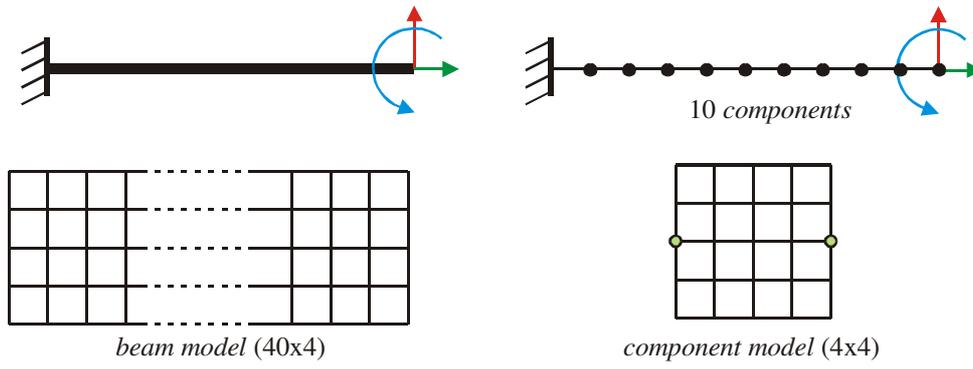


Fig. 6. Beam and component models.

Table 1. Errors for vertical displacements of end node.

load case	1/4		1/20		1/100	
	$\nu=0.33$	$\nu=0.0$	$\nu=0.33$	$\nu=0.0$	$\nu=0.33$	$\nu=0.0$
1	0.58%	0.0%	1.3%	0.0%	5.4%	0.0%
2	0.49%	0.0004%	0.68%	0.0025%	2.8%	0.2%
3	0.51%	0.00002%	0.71%	0.0%	2.6%	0.0%

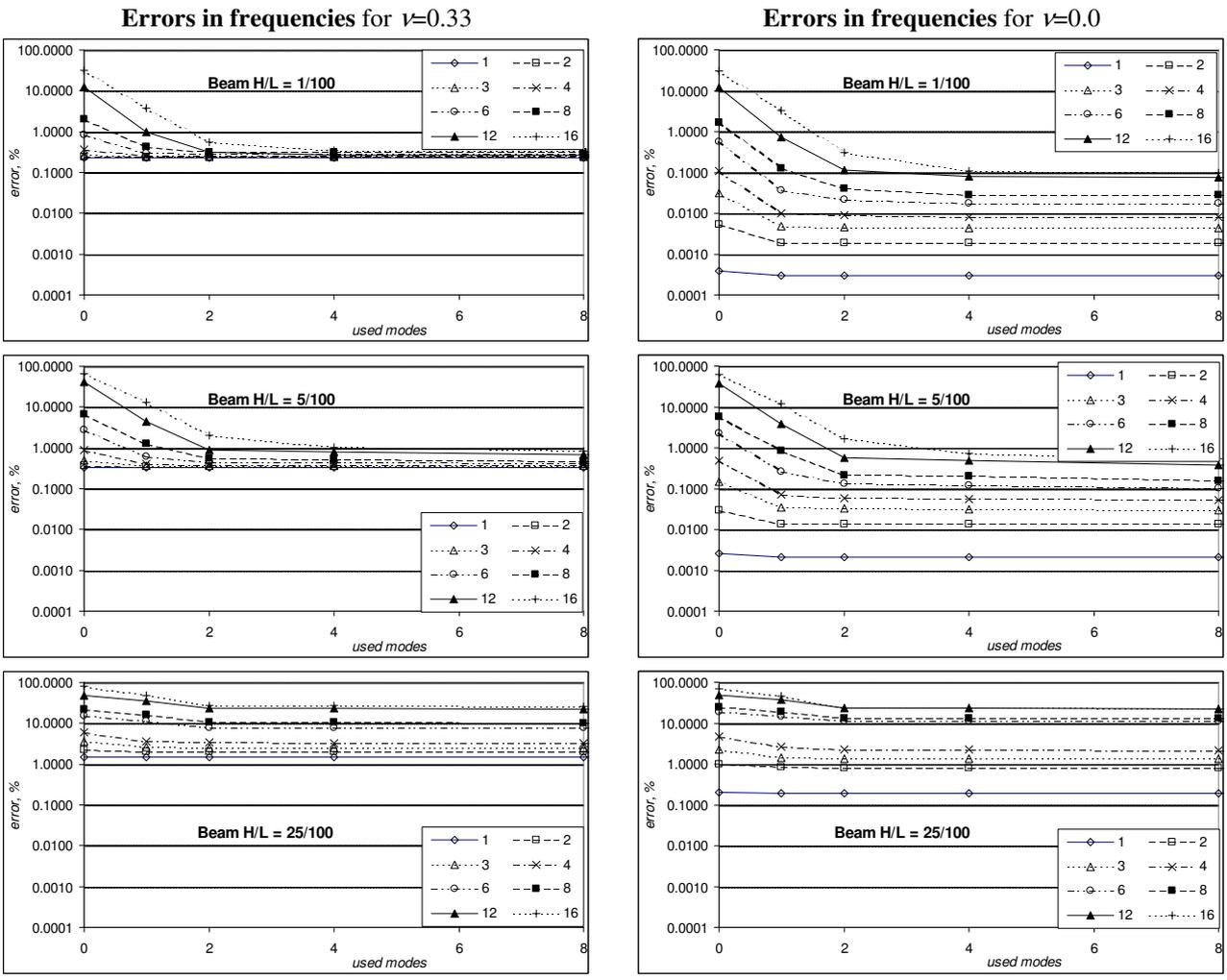


Fig. 7. Errors in frequencies for different conditions.

6. Conclusions

The new technique for component synthesis of beams and plates is proposed. It is based on special constraints imposed on boundary nodes accordingly to beam theory. The new technique allow to reduce the number of DOFs for component to a no more than 20.

The investigation of two dimensional beam under different conditions shows that by using this technique resonable results can be obtained. The level of errors is acceptable for many applications. Further investigations must be made for other types of structures including thinwalled structures.

Acknowledgement

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