Conditionally Stable Case for Nonlinear Two-Point Boundary Value Problems with Double Singularity

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Abstract. The double singular perturbation for two-point boundary value problem (TPBVP) for nonlinear system of ordinary differential equations (ODE) in the conditionally stable case is considered. An asymptotic expansion of the solution is constructed by the method of boundary functions and generalized inverse matrices and projectors.

1. Introduction

We consider the boundary value problems

$$\varepsilon \frac{dx}{dt} = Ax + \varepsilon F(t, x, \varepsilon, f(t,\varepsilon)) + \varphi(t), \quad t \in [a,b], \quad 0 < \varepsilon \ll 1, \qquad (1)$$

with boundary conditions

$$l(x(\cdot)) = Mx(a) + Nx(b) = h, \; h \in R^m, \qquad (2)$$

where ε is a small positive parameter.

Suppose that the following conditions are satisfied:

(C1) The $n \times n$ matrix A with constant elements has p eigenvalue with negative real part, and remaining $(n-p)$ eigenvalues have positive real part, i.e. $\lambda_i \in \sigma(A)$, $R(\lambda_i) < 0$, $i = \overline{1,p}$ and $R(\lambda_i) > 0$, $i = \overline{p+1,n}$;

(C2) The vector-function $\varphi(t)$ is an n -dimensional of the class $C^\infty([a,b])$;

(C3) The function $F(x,t,\varepsilon, f(t,\varepsilon))$ is an n -dimensional vector-function, having arbitrary order continuous partial derivatives with respect to all arguments in the domain $G = [a,b] \times D_x \times [0,\bar{\varepsilon}] \times D_f$ - where $D_x \in R^n$ is some neighborhood of the solution $x_0(t)$ of the degenerate system $Ax_0(t) + \varphi(t) = 0$, $D_f \in R^p$ is bounded and closed domain, $0 < \bar{\varepsilon} < \varepsilon$. The function $f(t,\varepsilon)$ is smooth of arbitrary order with respect to all argument in the domain $G = [a,b] \times (0,\bar{\varepsilon}]$ and its values belongs to D_f;

(C4) The vector functional l has the form (2), where the $(m \times n)$ matrices M and N have constant elements.

We assume that the function $f(t,\varepsilon)$ of (1) contain singular elements (for example, $f(t,\varepsilon) = f(\exp(-t/\varepsilon), \sin(t/\varepsilon))$. Thus the system (1) is withe double singularity. On one side the small parameter ε appear before the derivative, and on the other hand bring in a singularity of the function f.

The constructing of the asymptotic solution of the problem (1), (2) is based on the boundary functions method (see, for example, [12]). The initial research for a Cauchy problem with double singularity is made in [9] in the case $\lambda_i \in \sigma(A)$, $R(\lambda_i) < 0, \forall i$.

Primary research of the problem (1), (2) with conditions (C1)-(C4) are considered in [3]. This paper is continuation of the article [3] and [7]. The proof of the asymptoticaly of the formal asymptotic order obtained in [7], is done in work [8]. Here , we consider a private random two-point boundary problem that uses a generalized inverse matrices and projectors [1], [10].

The problem (1), (2) with condition $R(\lambda_i) < 0, \forall i, \lambda_i \in \sigma(A)$ and (C1)-(C4) in diverse cases are research in [4], [5], [7], [11].

2. Auxiliary results and Notations

First, we point out the results obtained in articles [2] and [3].

Lemma 1. [2] Let the matrix A satisfy the condition (C1), P is a spectral projection on the left half plane of the matrix A and functions $g(\tau) \in C(0,+\infty)$, $\bar{g}(\nu) \in C(-\infty,0)$ satisfy the inequalities

$$\|g(\tau)\| \le C^* \exp(-\alpha^* \tau), C^* > 0, \alpha^* > 0, \tau \ge 0;$$
$$\|\bar{g}(\nu)\| \le \bar{C}^* \exp(-\bar{\alpha}^* \nu), \bar{C}^* > 0, \bar{\alpha}^* > 0, \nu \le 0.$$

Then the systems $\frac{dx}{d\tau} = Ax + g(\tau),\ \tau \ge 0$ and $\frac{dy}{d\nu} = Ay + \bar{g}(\nu),\ \nu \le 0$ have particular solutions $\left(L_\tau g\right)(\tau)$ and $\left(L_\nu \bar{g}\right)(\nu)$ respectively in the form

$$\left(L_\tau g\right)(\tau) = \int_0^{+\infty} K(\tau,s) g(s) ds \quad \text{and} \quad \left(L_\nu \bar{g}\right)(\nu) = \int_{-\infty}^{0} \bar{K}(\nu,s) \bar{g}(s) ds \qquad (3)$$

satisfying the inequalities

$$\left\|\left(L_\tau g\right)(\tau)\right\| \le C \exp(-\gamma\tau), \tau \ge 0;$$
$$\left\|\left(L_\tau g\right)(\tau)\right\| \le C \exp(-\gamma\tau), \tau \ge 0;$$

where $C, \bar{C}, \gamma, \bar{\gamma}$ are certain positive constants, and

$$K(\tau,s) = \begin{cases} X(\tau) P X^{-1}(s), & 0 \le s \le \tau < +\infty \\ -X(\tau)\left(I-P\right) X^{-1}(s), & 0 \le \tau \le s < +\infty \end{cases} \qquad (4)$$

$$\bar{K}(\nu,s) = \begin{cases} -X(\nu)\left(I-P\right) X^{-1}(s), & -\infty < \nu \le s \le 0 \\ X(\nu) P X^{-1}(s), & -\infty < s \le \nu \le 0. \end{cases}$$

Let a linear system $\frac{dx}{dt} = Ax$ have a fundamental solution the matrix $X(t) = \exp(At),\ X(0) = E_n$ and B be an $\left(n \times n\right)$ nonsingular constant matrix such that $B^{-1}AB = diag\left(A_+, A_-\right)$, where A_+ is an $\left(p \times p\right)$ - matrix having eigenvalues with negative real part, $R(\lambda_i) < 0,\ i = \overline{1,p}$ and A_- is $\left((n-p)\times(n-p)\right)$ matrix having eigenvalues with positive real part $R(\lambda_i) > 0,\ i = \overline{p+1,n}$.

The system $\frac{dx}{dt} = Ax$ has stable manifold S^+ in the form $S^+ : \bar{\bar{x}} = H\bar{x}$, where $H = B_{21}B_{11}^{-1}$ is an $\left((n-p)\times p\right)$ - matrix, and unstable manifold S^- in the form $S^- : \bar{x} = \bar{H}\bar{\bar{x}}$, where $\bar{H} = B_{12}B_{22}^{-1}$ is an $\left(p \times (n-p)\right)$ - matrix. Cells $B_{ij}, i, j = 1, 2$, are elements of the block representation of the matrix $B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$.

Let

$$X_p(\tau) = X(\tau)\begin{pmatrix} E_p \\ H \end{pmatrix} \text{ - } (n \times p) \text{ - matrix;}$$

$$X_{n-p}(\nu) = X(\nu)\begin{pmatrix} \bar{H} \\ E_{n-p} \end{pmatrix} \text{ - } (n \times (n-p)) \text{ - matrix.}$$

The following denotations

$$D_1(\varepsilon) = l\left(X_p(\cdot)\right) = MX_p\left(\frac{(\cdot)-a}{\varepsilon}\right) + NX_p\left(\frac{(\cdot)-b}{\varepsilon}\right) =$$

$$= MX_p(0) + NX_p\left(\frac{a-b}{\varepsilon}\right) \qquad - \ (n\times p) - matrix;$$

$$D_2(\varepsilon) = l\left(X_{n-p}(\cdot)\right) = MX_{n-p}\left(\frac{a-(\cdot)}{\varepsilon}\right) + NX_{n-p}\left(\frac{b-(\cdot)}{\varepsilon}\right) =$$

$$= MX_{n-p}\left(\frac{a-b}{\varepsilon}\right) + NX_{n-p}(0) \qquad - \ (n\times(n\times p)) - matrix;$$

$$D(\varepsilon) = \left(D_1(\varepsilon), D_2(\varepsilon)\right) \text{ - } (n\times n) \ - matrix \quad (5)$$

introduce.

In this case, we assume that the matrix $D(\varepsilon)$ has the presentation $D(\varepsilon) = D_0 + O\left(\varepsilon^q \exp\left(-\alpha/\varepsilon\right)\right)$, $q\in N$, $\alpha > 0$, $D_0 - (n\times n) \ - matrix$.

Instead of (1), (2) we consider TPBVP with two parameters $\varepsilon \in (0, \bar{\varepsilon}]$ and $\mu \in (0, \bar{\varepsilon}]$, $0 < \bar{\varepsilon} \ll 1$

$$\varepsilon\frac{dz}{dt} = Az + \varepsilon F(t, z, \varepsilon, f(t,\mu)) + \varphi(t), \quad t\in[a,b], \quad 0<\varepsilon\ll 1, \qquad (6)$$

with boundary conditions

$$l(z(\cdot)) = Mz(a) + Nz(b) = h, \ h\in R^m, \qquad (7)$$

The solution of the TPBVP (6), (7) we look of the form

$$z(t,\varepsilon,\mu) = \sum_{k=0}^{\infty}\left[z_k(t,\mu) + \Pi_k(\tau,\mu) + Q_k(\nu,\mu)\right]\varepsilon^k, \ \tau = \frac{t-a}{\varepsilon}, \nu = \frac{t-b}{\varepsilon} \qquad (8)$$

In (8) the functions $z_k(t,\mu)$ are the elements of the regular series, $\Pi_k(\tau,\mu)$ and $Q_k(\nu,\mu)$ are boundary functions in a right neighborhood of the point $t=a$ and left neighborhood of the point $t=b$ respectively. The boundary functions are the elements of the singular series.

After the determination of $z_k(t,\mu)$, $\Pi_k(\tau,\mu)$ and $Q_k(\nu,\mu)$ the solution of (1), (2) has the form

$$x(t,\varepsilon) = \sum_{k=0}^{\infty}\left[z_k(t,\varepsilon) + \Pi_k(\tau,\varepsilon) + Q_k(\nu,\varepsilon)\right]\varepsilon^k$$

We substitute the series (8) in the system (6) and we represent the function $F(t,z,\varepsilon, f(t,\mu))$ in the form

$$F(t, \sum_{k=0}^{\infty}\left(z_k(t,\mu) + \Pi_k(\tau,\mu) + Q_k(\nu,\mu)\right)\varepsilon^k, \varepsilon, f(t,\mu))$$
$$= \bar{F}(t,\varepsilon,\mu) + \Pi F(\tau,\varepsilon,\mu) + QF(\nu,\varepsilon,\mu),$$

where

$$\begin{aligned}
&\bar{F}(t,\varepsilon,\mu) = F(t, \sum_{k=0}^{\infty} z_k(t,\mu)\varepsilon^k, \varepsilon, f(t,\mu)),\\
&\Pi F(\tau,\varepsilon,\mu) = F(a+\varepsilon\tau, \sum_{k=0}^{\infty}\left(z_k(a+\varepsilon\tau,\mu) + \Pi_k(\tau,\mu)\right)\varepsilon^k, \varepsilon, f(a+\varepsilon\tau,\mu))\\
&\qquad - F(a+\varepsilon\tau, \sum_{k=0}^{\infty} z_k(a+\varepsilon\tau,\mu)\varepsilon^k, \varepsilon, f(a+\varepsilon\tau,\mu)),\\
&QF(\nu,\varepsilon,\mu) = F(b+\varepsilon\nu, \sum_{k=0}^{\infty}\left(z_k(b+\varepsilon\nu,\mu) + Q_k(\nu,\mu)\right)\varepsilon^k, \varepsilon, f(b+\varepsilon\nu,\mu))\\
&\qquad - F(b+\varepsilon\nu, \sum_{k=0}^{\infty} z_k(b+\varepsilon\nu,\mu)\varepsilon^k, \varepsilon, f(b+\varepsilon\nu,\mu)),
\end{aligned} \qquad (9)$$

We decompose the function $\bar{F}(t,\varepsilon,\mu)$ in Taylor series in a neighborhood of the point $(t, z_0(t), 0, f)$

$$\bar{F}(t,\varepsilon,\mu) = \sum_{k=0}^{\infty}\bar{F}_k(t,\mu)\varepsilon^k,$$

where

$$\bar{F}_k(t,\mu) = \begin{cases} F\left(t, z_0(t), 0, f(t,\mu)\right) & k=0 \\ F_z\left(t, z_0(t), 0, f(t,\mu)\right) z_k(t,\mu) & \\ + g_k\left(z_0(t),\ldots,z_{k-1}(t,\mu), f(t,\mu)\right), & k=1,2,\ldots \end{cases} \qquad (10)$$

In (10) the functions g_k contain derivative up to $(k-1)$ -th order of the function $F(t,z,\varepsilon, f(t,\mu))$ with respect to z and ε, calculated in the point $(t, z_0(t), 0, f)$.

The function $\Pi F(\tau,\varepsilon,\mu)$ we decompose also in Taylor series in a neighborhood of the point $(a, z_0(a)+\Pi_0(\tau), 0, f)$ and $z_k(t,\mu)$ in a neighborhood of the point (a,μ). Then $\Pi F(\tau,\varepsilon,\mu)$ will be written in the form

$$\Pi F(\tau,\varepsilon,\mu)=\sum_{k=0}^{\infty}\Pi F_k(\tau,\mu)\varepsilon^k$$

where

$$\Pi F_k(\tau,\mu)=\begin{cases} F(a,z_0(a)+\Pi_0(\tau),0,f(a,\mu)) \\ \quad -F(a,z_0(a),0,f(a,\mu)), & k=0 \\ F_z(a,z_0(a)+\Pi_0(\tau),0,f(a,\mu))\Pi_k(\tau,\mu) \\ \quad +G_k(\tau,\Pi_0(\tau),...,\Pi_{k-1}(\tau,\mu),f(a,\mu)), & k=1,2,... \end{cases} \tag{11}$$

The functions G_k contain derivative up to $(k-1)$ -th order of the function $F(t,z,\varepsilon,f(t,\mu))$ with respect to t,z and ε, in the point $(a,z_0(a)+\Pi_0(\tau),0,f)$ and derivative up to $(k-1)$ -th order of the function $z_k(t,\mu)$ with respect t in the point (a,μ).

The function $QF(\nu,\varepsilon,\mu)$ we decompose by analogy $\Pi F(\tau,\varepsilon,\mu)$ in Taylor series in a neighborhood of the point $(b,z_0(b)+Q_0(\nu),0,f)$ and $z_k(t,\mu)$ in a neighborhood of the point (b,μ). Then $QF(\nu,\varepsilon,\mu)$ will be written in the form

$$QF(\nu,\varepsilon,\mu)=\sum_{k=0}^{\infty}QF_k(\nu,\mu)\varepsilon^k,$$

where

$$QF_k(\nu,\mu)=\begin{cases} F(b,z_0(b)+Q_0(\nu),0,f(b,\mu)) \\ \quad -F(b,z_0(b),0,f(b,\mu)), & k=0 \\ F_z(b,z_0(b)+Q_0\nu,0,f(b,\mu))Q_k(\nu,\mu) \\ \quad +R_k(\nu,Q_0(\nu),...,Q_{k-1}(\nu,\mu),f(b,\mu)), & k=1,2,... \end{cases} \tag{12}$$

The functions R_k contain derivative up to $(k-1)$ -th order of the function $F(t,z,\varepsilon,f(t,\mu))$ with respect to t,z and ε, in the point $(b,z_0(b)+Q_0(\nu),0,f)$ and derivative up to $(k-1)$ -th order of the function $z_k(t,\mu)$ with respect t in the point (b,μ).

3. Main results

3.1.The case n=m

3.1.1. $rangD_0=n$

In this case D_0^{-1} exist. In the case of a general linear function, the finding of the elements of the regular $z_k(t,\mu)$ and singular order $\Pi_k(\tau,\mu)$ and $Q_k(\nu,\mu)$ is obtained in [3]. We apply functional l to $\Pi_k(\tau,\mu)$ and $Q_k(\nu,\mu)$:

$$\begin{aligned} l\left(\Pi_k\left(\frac{(\cdot)-a}{\varepsilon},\mu\right)+Q_k\left(\frac{(\cdot)-b}{\varepsilon},\mu\right)\right) &= M\Pi_k(0,\mu)+N\Pi_k\left(\frac{b-a}{\varepsilon},\mu\right) \\ &\quad +MQ_k\left(\frac{a-b}{\varepsilon},\mu\right)+NQ_k(0,\mu) \\ &= \begin{cases} h-Mz_0(a)-Nz_0(b), & k=0 \\ Mz_k(a,\mu)+Nz_k(b,\mu), & k\ge 1 \end{cases} \end{aligned} \tag{13}$$

Denote

$$h_0 = h - MA^{-1}\varphi(a) - NA^{-1}\varphi(b)$$

$$\begin{aligned} h_k = &- Mz_k(a,\mu) - Nz_k(b,\mu) \\ &- M\left(L_\tau \overline{\Pi F_k}\right)(0,\mu) - N\left(L_\tau \overline{\Pi F_k}\right)\left(\frac{b-a}{\varepsilon},\mu\right) \\ &- M\left(L_\nu \overline{QF_k}\right)\left(\frac{a-b}{\varepsilon},\mu\right) - N\left(L_\nu \overline{QF_k}\right)(0,\mu) \end{aligned} \qquad (14)$$

where $\left(L_\tau \overline{\Pi F_k}\right)(\tau,\mu)$ and $\left(L_\nu \overline{QF_k}\right)(\nu,\mu)$ they are defined in (3),(4).

Theorem 1. We assume that the conditions are fulfilled:

(J1) (C1) – (C4)

(J2) $rangD_0 = n = m$.

Then there is a unique solution in the domain G the TPBVP (6), (7) which is continuously differentiable with respect to $t \in [a,b]$ and continuous for $\mu \in (0,\bar{\varepsilon}]$. The series (8) is formally asymptotic series for this solution, where the functions $z_k(t,\mu),\ k \ge 0$ have the form

$$z_k(t,\mu) = \begin{cases} -A^{-1}\varphi(t) & k = 0 \\ A^{-1}\left(\dfrac{\partial}{\partial t} z_{k-1}(t,\mu) - \bar{F}_{k-1}(t,\mu)\right), & k \ge 1, \end{cases} \qquad (15)$$

where $\bar{F}_k(t,\mu)$ are defined in (10) and the elements of the singular series $\Pi_k(\tau,\mu)$ and $Q_k(\nu,\mu)$ are of the type

$$\Pi_k(\tau,\mu) = \begin{cases} X_p(\tau)\left[D_0^{-1}h_0\right]_p & k = 0 \\ X_p(\tau)\left[D_0^{-1}h_k\right]_p + \left(L_\tau \overline{\Pi F_k}\right)(\tau,\mu), & k \ge 1, \end{cases} \qquad (16)$$

$$Q_k(\nu,\mu) = \begin{cases} X_{n-p}(\nu)\left[D_0^{-1}h_0\right]_{n-p} & k = 0 \\ X_{n-p}(\nu)\left[D_0^{-1}h_k\right]_p + \left(L_\nu \overline{QF_k}\right)(\nu,\mu), & k \ge 1, \end{cases}$$

where h_0 and h_k are defined in (14).

Proof. It is clear that the elements $z_k(t,\mu)$ of the regular series have the form (15). It is obvious that the function $z_0(t)$ no dependent on μ. The boundary functions $\Pi_k(\tau,\mu)$ and $Q_k(\nu,\mu)$ satisfy the linear differential systems (17) and (18) respectively with boundary conditions (13).

$$\frac{\partial \Pi_k(\tau,\mu)}{\partial \tau} = A\Pi_k(\tau,\mu) + \overline{\Pi F_k}(\tau,\mu), \qquad \tau \in \left[0, \frac{b-a}{\varepsilon}\right], \quad \mu \in (0,\varepsilon], \qquad (17)$$

$$\frac{\partial Q_k(\nu,\mu)}{\partial \nu} = AQ_k(\nu,\mu) + \overline{QF_k}(\nu,\mu), \qquad \nu \in \left[\frac{a-b}{\varepsilon}, 0\right], \quad \mu \in (0,\varepsilon], \qquad (18)$$

$$\overline{\Pi F_k}(\tau,\mu) = \begin{cases} 0, & k = 0 \\ \Pi F_{k-1}(\tau,\mu), & k = 1,2,\ldots. \end{cases},$$

$$\overline{QF_k}(\nu,\mu) \begin{cases} 0, & k = 0 \\ QF_{k-1}(\nu,\mu), & k = 1,2,\ldots. \end{cases},$$

where $\Pi F_k(\tau,\mu)$ and $QF_k(\nu,\mu)$ are the notations (11) (12).

Consider systems (17), (18) and boundary conditions (13) for $k = 0$.

The equation $\dfrac{\partial \Pi_0(\tau)}{\partial \tau} = A\Pi_0(\tau)$ has a solution

$$\Pi_0(\tau) = X_p(\tau)\bar{c}_0, \quad \bar{c}_0 \in R^p, \tau \in [0,+\infty)$$

and $\dfrac{\partial Q_0(\nu)}{\partial \nu} = AQ_0(\nu)$ has a solution

$$Q_0(\nu) = X_{n-p}(\nu)\bar{\bar{c}}_0, \quad \bar{\bar{c}}_0 \in R^{n-p}, \nu \in (-\infty, 0].$$

Substituting into the boundary conditions for $c_0 = \left(\bar{c}_0, \bar{\bar{c}}_0\right)^T$ obtaining the algebraic system

$$D_1(\varepsilon)\bar{c}_0 + D_2(\varepsilon)\bar{\bar{c}}_0 = h_0 \;\Leftrightarrow\; D_0 c_0 = h_0$$

with solution

$$c_0 = D_0^{-1} h_0 .$$

Then we get

$$\Pi_0(\tau) = X_p(\tau)\left[D_0^{-1}h_0\right]_p$$

$$Q_0(\nu) = X_{n-p}(\nu)\left[D_0^{-1}h_0\right]_{n-p} .$$

with $k \ge 1$ the system's (17), (18) have solutions

$$\Pi_k(\tau,\mu) = X_p(\tau)\bar{c}_k + \left(L_\tau \overline{\Pi F_k}\right)(\tau,\mu)$$

$$Q_k(\nu) = X_{n-p}(\nu)\bar{\bar{c}}_k + \left(L_\nu \overline{QF_k}\right)(\nu,\mu),$$

$c_k = \left(\bar{c}_k, \bar{\bar{c}}_k\right)^T$ are solutions solutions of the equations $D_0 c_k = h_k \;\Rightarrow\; c_k = D_0^{-1} h_k$.

We finally get

$$\Pi_k(\tau,\mu) = X_p(\tau)\left[D_0^{-1}h_k\right]_p + \left(L_\tau \overline{\Pi F_k}\right)(\tau,\mu)$$

$$Q_k(\nu,\mu) = X_{n-p}(\nu)\left[D_0^{-1}h_k\right]_p + \left(L_\nu \overline{QF_k}\right)(\nu,\mu), \qquad k \ge 1 . \quad \triangleleft$$

3.1.2. $rang D_0 = r < n$.

In work [7] has received the solution under general boundary conditions. In this work, we consider only a two-point boundary problem.

In this case D_0^{-1} not exist. But according to [10] or [1] we will us unite Mur-Penrous pseudoinverse matrix of D_0 which is written as D_0^{+} More details in connection to pseudoinverse matrices, means of determination, can be seen in the marked literature. Let P_{D_0} and $P_{D_0^*}$ are $(n \times n)$ -matrices (orthogonal projections) projecting. Having in mind that $rang D_0 = r < n$,then $rang P_{D_0} = rang P_{D_0^*} = n - r = q.$

Theorem 2. Suppose the following conditions are satisfied:

(H1) (C_1) – (C_2)

(H2) the matrix $D(\varepsilon)$ has the representation $D(\varepsilon) = D_0 + O\left(\varepsilon^q \exp(-\alpha/\varepsilon)\right)$, $q \in N$, $\alpha > 0$, $D_0 - (n \times n)$ - $-$ *matrix* and $rang D_0 = r < n$;

(H3) $P_{D_0^*} h_0 = 0$, where $h_0 = h - MA^{-1}\varphi(a) - NA^{-1}\varphi(b)$;

(H4) the nonlinear equation $P_{D_0^*} h_1(\varepsilon,\mu,\xi_0) = 0$ for all $0 \le \varepsilon \le \bar{\varepsilon}$, $0 < \mu \le \bar{\varepsilon}$ have unique bounded solution withe respect to $\xi_0 = \psi(\varepsilon,\mu) \in R^q$, where

$$\begin{aligned} h_1(\varepsilon,\mu,\xi_0) = &-Mz_1(a,\mu) - Nz_1(b,\mu) \\ &- M\int_0^{+\infty} K(\tau,s)\Pi F_0(s)ds \qquad (19) \\ &- N\int_{-\infty}^{0} \bar{K}(\nu,s) QF_0(s)ds \end{aligned}$$

For $z_0(t)$ we have the form

$$z_0(t) = -A^{-1}\varphi(t) \qquad (20)$$

and for boundary functions $\Pi_0(\tau)$, $Q_0(\nu)$ representation

$$\begin{aligned} \Pi_0(\tau) &= X_p(\tau)\left[P_{D_{0_q}}\right]_p \xi_0 + X_p(\tau)\left[D_0^{+}h_0\right]_p \\ Q_0(\nu) &= X_{n-p}(\nu)\left[P_{D_{0_q}}\right]_{n-p} \xi_0 + X_{n-p}(\nu)\left[D_0^{+}h_0\right]_{n-p} \end{aligned} \qquad (21)$$

respectively.

Proof with minor modifications is the same as in work [7], using condition H3 and 19. $\triangleleft$

We introduce the following denotations:

$$h_k\left(\varepsilon,\mu,\xi_{k-1}\right)=D_1(\varepsilon,\mu)\xi_{k-1}+S_{k-1}(\varepsilon,\mu)\text{, where } \xi_{k-1}\in R^q,\ k\geq 2,$$

$$D_1\left(\varepsilon,\mu\right)=-MF_z\left(a,z_0(a)+\Pi_0(0),0,f(a,\mu)\right)X_p\left(a\right)\left[D_{0_q}\right]_p$$
$$-NF_z\left(b,z_0(b)+Q_0(0),0,f(b,\mu)\right)X_{n-p}\left(b\right)\left[D_{0_q}\right]_{n-p}$$

$$\begin{aligned}S_k\left(\varepsilon,\mu\right)=&-Mz_{k-1}(a,\mu)-Nz_{k-1}(b,\mu)\\&-MF_z\left(a,z_0(a)+\Pi_0(0),0,f(a,\mu)\right)\bar{\Phi}_k\left(a,\varepsilon,\mu\right)\\&-NF_z\left(b,z_0(b)+Q_0(0),0,f(b,\mu)\right)\bar{\bar{\Phi}}_k\left(b,\varepsilon,\mu\right)\\&-MG_k\left(a,\Pi_0(a),...,\Pi_{k-1}\left(a,\mu\right),f\left(a,\mu\right)\right)\\&-NR_k\left(b,Q_0(b),...,Q_{k-1}\left(b,\mu\right),f\left(b,\mu\right)\right),\qquad k\geq 1,\end{aligned}\tag{22}$$

$$\bar{\Phi}_k\left(\tau,\varepsilon,\mu\right)=X_p\left(\tau\right)\left[D_0^+\bar{h}_k\right]_p+\left(L_\tau\overline{\Pi F_k}\right)\left(\tau,\mu\right),$$

$$\bar{\bar{\Phi}}_k\left(\nu,\varepsilon,\mu\right)=X_{n-p}\left(\nu\right)\left[D_0^+h_0\right]_{n-p}+\left(L_\nu\overline{QF_k}\right)\left(\nu,\mu\right),$$

$$\bar{h}_k=\bar{h}_k\left(\varepsilon,\mu\right)=\bar{h}_k\left(\varepsilon,\mu,\xi_{k-1}\right)=\bar{h}_k\left(\varepsilon,\mu,\psi_k\left(\varepsilon,\mu\right)\right),$$

$$\bar{D}_1\left(\varepsilon,\mu\right)=P_{D_{0_q}}D_1\left(\varepsilon,\mu\right),$$

$$\bar{S}_k\left(\varepsilon,\mu\right)=-P_{D^*_{0_q}}S_k\left(\varepsilon,\mu\right),\qquad k\geq 1.$$

Theorem 3. Suppose the following conditions are satisfied:
(H1)-(H4) of the Theorem 2 and the condition
(H5) $rang\,\bar{D}_1\left(\varepsilon,\mu\right)\neq 0,\quad 0\leq\varepsilon\ll 1\ \ and\ \ 0<\mu\ll 1$ be fulfilled.

Then the coefficients of the series (8) $z_k(t,\mu)$, $\Pi_k(\tau,\mu)$ and $Q_k(\nu,\mu)$ for $k\geq 1$ have the form

$$z_k(t,\mu)=-\left(A^{-1}\right)^{k+1}\frac{d^k\varphi(t)}{dt^k}-\sum_{i=k}^{1}\left(A^{-1}\right)^i\frac{d^{i-1}}{dt^{i-1}}\bar{F}_{k-i}\left(t,\mu\right),$$

$$\Pi_k(\tau,\mu)=-X_p\left(\tau\right)\left[P_{D_{0q}}\right]_p\bar{D}^{-1}\left(\varepsilon,\mu\right)\bar{S}_k\left(\varepsilon,\mu\right)+\bar{\Phi}_k\left(\tau,\varepsilon,\mu\right),$$

$$Q_k(\nu,\mu)=-X_{n-p}\left(\nu\right)\left[P_{D_{0q}}\right]_{n-p}\bar{D}^{-1}\left(\varepsilon,\mu\right)\bar{S}_k\left(\varepsilon,\mu\right)+\bar{\bar{\Phi}}_k\left(\nu,\varepsilon,\mu\right)\tag{23}$$

The proof is analogous to that of Theorem 2 by [7]. $\lhd$

3.2. The case m<n

3.2.1. $rangD_0=m<n$

Theorem 4. We assume that conditions
(J1) (C1) – (C4)
(J2) $rangD_0=m<n$ are satisfied.

Then in the domain G the TPBVP (6), (7) has a one-parametric solution, continuous derivatives with respect to $t\in[a,b]$ and continuous solution for $\mu\in(0,\bar{\varepsilon}]$. The series (8) is then formal asymptotic series for this solution, where the functions $z_k(t,\varepsilon),\ k\geq 0$ have the form (15) for $\mu=\varepsilon$ and $\Pi_k(\tau,\mu)$ and $Q_k(\nu,\mu)$ have the form (24), (25) for $\mu=\varepsilon$, where η_k is arbitrary q - dimensional vector.

$$\begin{aligned}&\Pi_0(\tau)=X_p(\tau)\left[P_{D_{0q}}\right]_p\eta_0+X_p(\tau)\left[D_0^+h_0\right]_p,\\&Q_0(\nu)=X_{n-p}(\nu)\left[P_{D_{0q}}\right]_{n-p}\eta_0+X_{n-p}(\nu)\left[D_0^+h_0\right]_{n-p}\end{aligned}\tag{24}$$

$$\begin{aligned}&\Pi_k(\tau,\mu)=X_p(\tau)\left[P_{D_{0q}}\right]_p\eta_k+X_p(\tau)\left[D_0^+h_k\right]_p+\left(L_\tau\overline{\Pi F_k}\right)\left(\tau,\mu\right),\\&Q_k(\nu,\mu)=X_{n-p}(\nu)\left[P_{D_{0q}}\right]_{n-p}\eta_k+X_{n-p}(\nu)\left[D_0^+h_k\right]_{n-p}+\left(L_\nu\overline{QF_k}\right)\left(\nu,\mu\right)\end{aligned}\tag{25}$$

Proof. In this case D_0^{-1} exist. We note with P_{D_0} and $P_{D_0^*}$ the matrix orthoprojectors $P_{D_0}:\mathbb{R}^n\to\ker(D_0)$, $P_{D_0^*}:\mathbb{R}^m\to\ker(D_0^*)$, $D_0^*=D_0^T$. With D_0^+ we note the unique Moor-Penrose inverse $(n\times m)$-matrix of the matrix D_0. Let $P_{D_{0q}}$ be a $(n\times q)-$ matrix with $q=n-m$ linear independent columns from the matrix P_{D_0}. In this case the system $D_0 c_0 = h_0$ is always possible to solve ($P_{D_0^*} = m - m = 0$) and $c_0 = P_{D_{0q}}\eta_0 + D_0^+ h_0$, $\eta_0 \in R^q$, $c_0 = \left(\overline{c}_0, \overline{\overline{c}}_0\right)^T$. Then the solution of the system (17) and (18) for $k = 0$ is

$$\Pi_0(\tau) = X_p(\tau)\left[P_{D_{0q}}\right]_p \eta_0 + X_p(\tau)\left[D_0^+ h_0\right]_p,$$

$$Q_0(\nu) = X_{n-p}(\nu)\left[P_{D_{0q}}\right]_{n-p} \eta_0 + X_{n-p}(\nu)\left[D_0^+ h_0\right]_{n-p}$$

It should be noted that $\Pi_0(\tau)$ and $Q_0(\nu)$ does not depend on μ. Determining the η_0 is carried out next step when finding $\prod_1(\tau,\mu)$ and $Q_1(\nu,\mu)$ from (17), (18)

$$\prod_1(\tau,\mu) = X_p(\tau)\overline{c}_1 + \left(L_\tau \overline{\Pi F_1}\right)(\tau,\mu),$$

$$Q_1(\nu,\mu) = X_{n-p}(\nu)\overline{\overline{c}}_1 + \left(L_\nu \overline{QF_1}\right)(\nu,\mu).$$

Then the solution of the system $D_0 c_1 = h_1$, $c_1 = \left(\overline{c}_1, \overline{\overline{c}}_1\right)^T$ is $c_1 = P_{D_{0q}}\eta_1 + D_0^+ h_1$.

$$\prod_1(\tau,\mu) = X_p(\tau)\left[P_{D_{0q}}\right]_p \eta_1 + X_p(\tau)\left[D_0^+ h_1\right]_p + \left(L_\tau \overline{\Pi F_1}\right)(\tau,\mu),$$

$$Q_1(\nu,\mu) = X_{n-p}(\nu)\left[P_{D_{0q}}\right]_{n-p} \eta_1 + X_{n-p}(\nu)\left[D_0^+ h_1\right]_{n-p} + \left(L_\nu \overline{QF_1}\right)(\nu,\mu).$$

Then $\prod_k(\tau,\mu)$ and $Q_k(\nu,\mu)$ we obtain using the method of mathematical induction

$$\prod_k(\tau,\mu) = X_p(\tau)\left[P_{D_{0q}}\right]_p \eta_k + X_p(\tau)\left[D_0^+ h_k\right]_p + \left(L_\tau \overline{\Pi F_k}\right)(\tau,\mu),$$

$$Q_k(\nu,\mu) = X_{n-p}(\nu)\left[P_{D_{0q}}\right]_{n-p} \eta_k + X_{n-p}(\nu)\left[D_0^+ h_k\right]_{n-p} + \left(L_\nu \overline{QF_k}\right)(\nu,\mu). \quad \triangleleft$$

Remark 1. The case where $rangD_0 = n_1 < m$ is being considered in a similar way to Case 3.1.2, of course, with small modifications.

Remark 2. The case of over-determination, when $m > n$ considered further with all the features mentioned above.

Example of a double singularity.
We are looking at the problem

$$\begin{pmatrix} \varepsilon \dfrac{dx}{dt} \\ \varepsilon \dfrac{dy}{dt} \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \varepsilon \begin{pmatrix} \sin \dfrac{t}{\varepsilon} x \\ 0 \end{pmatrix} + \begin{pmatrix} 2t \\ 0 \end{pmatrix}, \quad 0 < \varepsilon \ll 1,$$

with boundary conditions

$$l(X) = \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} + \begin{pmatrix} 0 & -1 \end{pmatrix}\begin{pmatrix} x(1) \\ y(1) \end{pmatrix} = h,$$

where $X = (x,y)^T \in R^2$, $h \in R, h - const$, $t \in [0,1]$, $n = 2$, $m = 1$,

$$A = \begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix}, \quad F(x,t,\varepsilon, f(t,\varepsilon)) = \begin{pmatrix} \sin \dfrac{t}{\varepsilon} x \\ 0 \end{pmatrix}, \quad \varphi(t) = \begin{pmatrix} 2t \\ 0 \end{pmatrix}, \quad a = 0,\ b = 1.$$

We are going to a problem with two parameters:

$$\begin{pmatrix} \varepsilon \dfrac{dx}{dt} \\ \varepsilon \dfrac{dy}{dt} \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \varepsilon \begin{pmatrix} \sin \dfrac{t}{\mu} x \\ 0 \end{pmatrix} + \begin{pmatrix} 2t \\ 0 \end{pmatrix} \tag{26}$$

$Z = (x,y)^T$, $\mu \in \left(0, \overline{\varepsilon}\right]$, $\overline{\varepsilon} << 1$

with boundary conditions:

$$l(Z) = \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} + \begin{pmatrix} 0 & -1 \end{pmatrix}\begin{pmatrix} x(1) \\ y(1) \end{pmatrix} = h. \tag{27}$$

The solution of the BVP (26), (27) we look of the form

$$z(t,\varepsilon,\mu) = \sum_{k=0}^{\infty}\left[z_k(t,\mu) + \Pi_k(\tau,\mu) + Q_k(\nu,\mu)\right]\varepsilon^k, \quad \tau = \frac{t}{\varepsilon}, \nu = \frac{t-1}{\varepsilon}$$

The functions $z_k(t,\mu)$ are the elements of the regular series from the formulas (15), $\Pi_k(\tau,\mu)$ and $Q_k(\nu,\mu)$ are boundary functions in a right neighborhood of the point $t=0$ and left neighborhood of the point $t=1$ respectively from the formulas (21). The boundary functions are the elements of the singular series.

We find the matrix $A=\begin{pmatrix}0 & 2\\ 1 & -1\end{pmatrix}$ eigenvalues $\lambda_1=-2,\ \lambda_2=1$.

Then eigenvectors corresponding to λ_1 is $\alpha=\begin{pmatrix}-1\\ 1\end{pmatrix}$ and to λ_2 is $\beta=\begin{pmatrix}2\\ 1\end{pmatrix}$.

$$diag\left(A_+,A_-\right)=\begin{pmatrix}-2 & 0\\ 0 & 1\end{pmatrix}=C$$

We find the matrix B such that $B^{-1}AB=C$

$$B=\begin{pmatrix}1 & 2\\ -1 & 1\end{pmatrix},\qquad B^{-1}=\frac{1}{3}\begin{pmatrix}1 & -2\\ 1 & 1\end{pmatrix},\qquad B=\begin{pmatrix}1 & 2\\ -1 & 1\end{pmatrix}=\begin{pmatrix}B_{11} & B_{12}\\ B_{21} & B_{22}\end{pmatrix},$$

$$H=B_{21}B_{11}^{-1}=-1.1^{-1}=-1,\qquad \bar{H}=B_{12}B_{22}^{-1}=2.1^{-1}=2\ .$$

$$X(t)=\exp(At)=\Phi(t)=B.\begin{pmatrix}e^{-2t} & 0\\ 0 & e^{t}\end{pmatrix}.B^{-1}$$

$$=\frac{1}{3}\begin{pmatrix}e^{-2t}+2e^{t} & -2e^{-2t}+2e^{t}\\ -e^{-2t}+e^{t} & 2e^{-2t}+e^{t}\end{pmatrix},$$

$$\det X(t)=3e^{-t}\ ,$$

$$X^{-1}(t)=\frac{1}{9}\begin{pmatrix}2e^{-t}+2e^{2t} & 2e^{-t}-2e^{2t}\\ e^{-t}-e^{2t} & e^{-t}+2e^{2t}\end{pmatrix}.$$

We find
$$X_1(\tau)=X_p(\tau)=X(\tau)\begin{pmatrix}E_1\\ H\end{pmatrix}=\begin{pmatrix}e^{-2\tau}\\ -e^{-2\tau}\end{pmatrix},$$

$$X_2(\nu)=X_{n-p}(\nu)=X(\nu)\begin{pmatrix}\bar{H}\\ E_1\end{pmatrix}=\begin{pmatrix}2e^{\nu}\\ e^{\nu}\end{pmatrix}.$$

$$D_1(\varepsilon)=l\left(X_p(\cdot)\right)=\begin{pmatrix}1 & 0\end{pmatrix}X_p\left(\frac{0}{\varepsilon}\right)+\begin{pmatrix}0 & -1\end{pmatrix}X_p\left(\frac{1}{\varepsilon}\right)$$

$$=\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}1\\ -1\end{pmatrix}+\begin{pmatrix}0 & -1\end{pmatrix}\begin{pmatrix}e^{-\frac{2}{\varepsilon}}\\ -e^{-\frac{2}{\varepsilon}}\end{pmatrix}=1+e^{-\frac{2}{\varepsilon}}\qquad,$$

$$D_2(\varepsilon)=l\left(X_{n-p}(\cdot)\right)=\begin{pmatrix}1 & 0\end{pmatrix}X_{n-p}\left(-\frac{1}{\varepsilon}\right)+\begin{pmatrix}0 & -1\end{pmatrix}X_{n-p}(0)$$

$$=\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}2e^{-\frac{1}{\varepsilon}}\\ e^{-\frac{1}{\varepsilon}}\end{pmatrix}+\begin{pmatrix}0 & -1\end{pmatrix}\begin{pmatrix}2\\ 1\end{pmatrix}=2e^{-\frac{1}{\varepsilon}}-1\qquad,$$

$$D(\varepsilon)=\left(D_1(\varepsilon),D_2(\varepsilon)\right)=\begin{pmatrix}1+e^{-\frac{2}{\varepsilon}} & -1+2e^{-\frac{1}{\varepsilon}}\end{pmatrix}$$

Denote $\quad D_0=D_0(\varepsilon)=\begin{pmatrix}1 & -1\end{pmatrix}+\mathrm{O}\left(-1/\varepsilon\right)\quad\Rightarrow D_0=\begin{pmatrix}1 & -1\end{pmatrix}$

$$D^{*}=D^{T}=\begin{pmatrix}1\\ -1\end{pmatrix}$$

We find the pseudoinverse matrix $\quad D_0^{+}=D^{*}\left(DD^{*}\right)^{-1}=\begin{pmatrix}\frac{1}{2}\\ -\frac{1}{2}\end{pmatrix}$

and ortho-projectors $P_{D_0} = E_m - D_0 D_0^+ = 1 - \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} = 0$. Therefore the problem is only solution.

$$A = \begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix} \qquad \Rightarrow A^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$$

$$z_0(t) = -A^{-1}\varphi(t) = -\frac{1}{2}\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2t \\ 0 \end{pmatrix} = \begin{pmatrix} -t \\ -t \end{pmatrix}$$

$$\begin{aligned} h_0 &= h - MA^{-1}\varphi(a) - NA^{-1}\varphi(b) \\ &= h + \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = h - 1 \quad (const) \end{aligned}$$

The boundary functions are obtained from the formulas (21)

$$\Pi_0(\tau) = \begin{pmatrix} \frac{1}{2}e^{-2\tau}h \\ -\frac{1}{2}e^{-2\tau}h \end{pmatrix},$$

$$Q_0(\nu) = \begin{pmatrix} -e^{\nu}h \\ -\frac{1}{2}e^{\nu}h \end{pmatrix}$$

$$\begin{aligned} z(t,\varepsilon,\mu) &= \sum_{k=0}^{\infty}\left[z_k(t,\mu) + \Pi_k(\tau,\mu) + Q_k(\nu,\mu)\right]\varepsilon^k \\ &= z_0(t,\mu) + \Pi_0(\tau) + Q_0(\nu) + O(\varepsilon) \\ &= \begin{pmatrix} -t \\ -t \end{pmatrix} + \begin{pmatrix} \frac{1}{2}e^{-2\tau}h \\ -\frac{1}{2}e^{-2\tau}h \end{pmatrix} + \begin{pmatrix} -e^{\nu}h \\ -\frac{1}{2}e^{\nu}h \end{pmatrix} + O(\varepsilon)\,. \end{aligned}$$

After the determination of $z_k(t,\mu)$, $\Pi_k(\tau,\mu)$ and $Q_k(\nu,\mu)$ the solution of (1), (2) has the form

$$x(t,\varepsilon) = \sum_{k=0}^{\infty}\left[z_k(t,\varepsilon) + \Pi_k(\tau,\varepsilon) + Q_k(\nu,\varepsilon)\right]\varepsilon^k$$

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