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Periodic solutions of the non-integrable convective fluid equation

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In the present paper we have obtained and analyzed a family of exact periodic solutions of the nonlinear evolution partial differential convecting fluid equation (CFE) by applying a modification of the bilinear transformation method. This modification is used in view of the circumstance that CFE is a non-integrable nonlinear equation. A detailed consideration has been given to the quite important case of balance between the dispersion and nonlinear effects establishing that this balance changes the structure of the equation itself. The exact periodic solutions of CFE have been also found in the important case when CFE is identical with the Kuramoto–Sivashinsky equation.
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I. INTRODUCTION

The nonlinear evolution partial differential equation (PDE): $u_t + \alpha_1 uu_x + \alpha_2 u_{xx} + \alpha_3 u_{xxx} + \alpha_4 u_{xxxx} + \alpha_5 (uu_x)_x = 0$, where $u = u(x, t)$ is an unknown but continuously differentiable to fourth order function in the two-dimensional domain $\Omega = \{(x, t) \in R^2: x \in R, t \geq 0\}$ and $\alpha_j, j = 1,2, \ldots 5$ are real parameters, was introduced by Aspe and Depassier[1] and later by Garazo and Velarde[2] for describing the evolution of long surface waves in a shallow convecting fluid layer. That is why it was originally called the convecting fluid equation (CFE). Subsequently Christov and Velarde[3,4] employing a heuristic approach, included the Rayleigh–Marangoni effect into the unidirectional long waves case. They obtained hereby the same evolution equation, which they called KDV (Korteweg and De Vries)–KSVE (Kuramoto–Sivashinsky–Velarde Equation). We will use here the initial name of this equation, namely, CFE, and will study it for the presence of exact periodic solutions.

The numerical studies in Refs. 3 and 4 show that this nonlinear evolution equation can be a successful model for qualitative description of the Benard–Marangoni waves introduced by Lin and Young.[5] Actually the CFE incorporates a number of evolution equations, important for the mathematical physics and the applied mathematics. For $\alpha_4 = \alpha_5 = 0$ this equation corresponds (under relevant rescaling of the variables t, x, u) to the classic equation of Korteweg and De Vries[6] (KDV). For $\alpha_5 = 0$ we obtain the nonlinear evolution equation of Kuramoto–Sivashinsky (KS), which is studied in two cases:

- when $\alpha_3 = 0$ (Ref. 7) for vertically falling viscous film;
- when $\alpha_3 \neq 0$ (Ref. 8)—in case of a film flowing down an inclined plane, also known as the generalized equation of wave dynamics derived by Kuramoto and Tsuzuki.[7]

The presence of two nonlinear terms in the evolution CFE distinguishes it from the majority of popular nonlinear PDEs. It has second degree of singularity and belongs to the non-integrable PDEs. According to the formalism of Ablowitz and Segur,[6] a nonlinear PDE is considered to be integrable if the inverse scattering transform (IST) method is applicable to it, i.e., either it has a Bäcklund transformation, or a non-Abelian pseudo potential, or an N-soliton solution (perhaps $N \geq 3$ is sufficient). The requirement, a nonlinear PDE to have the Painleve property, has been proposed

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in Ref. 6 as a necessary condition for integrability by the IST method. When at least one of the above-listed conditions has not been satisfied, the equation is non-integrable. The non-integrable PDEs do not possess a number of "nice" properties of the integrable equations, such as N-soliton solutions, conservation laws, etc. Three conservation laws for CFE have been found in Ref. 3, but this fact does not make the equation integrable. There are not too many analytical methods for studying the non-integrable nonlinear PDEs. For example, two cnoidal localized solutions found in Ref. 9 are known for CFE ($\alpha_j \neq 0$), as well as the solitary-wave solutions obtained in Ref. 10. So far, other exact solutions of this equation in the general case have not been announced, although in the special cases (considered in Sec. VI) there are other analytical solutions obtained by Conte and Musett, Berloff and Howard, Kudryashov, Eremenko, and others.[11–14] The authors mentioned above apply two theoretical approaches in their analysis—the singular manifold method[15] and the Nevanlinna theory[16] of meromorphic functions.

The bilinear transformation method has been used by some authors as Nakamura, Matsuno, Parker[17–21] and others, as an alternative of the IST method and has been applied mainly to integrable nonlinear PDEs. In the present paper we use a modification of this method, consisting in a suitable presentation of the spatial displacements or the wave numbers, thus allowing a successful adaptation of this powerful method to the non-integrable equations. In the context of that method, any integrable nonlinear PDE subject to bilinear transformation can be reduced to a bilinear equation $F(D_t, D_x)f\cdot f = 0$, where F is a polynomial or exponential function with regard to the Hirota bilinear differential operators D_t, D_x.[25] When a given nonlinear equation can be reduced to two (or more) equations, one of them being necessarily bilinear, while the second one called residual: $G(D_x, D_x^2, \ldots) f\cdot f = 0$ has a different structure, and if its structure is also bilinear, then the original nonlinear equation is partially integrable. Such is the regularized long-wave equation (RLW) studied by Parker.[19] In case that the residual equation does not have a bilinear structure, then the original nonlinear equation is non-integrable. Such are the evolution equations: CFE, KS, KE (Kawahara equation), sixth-order generalized Boussinesq equation (SGBE),[22] etc.

II. MATHEMATICAL NOTES

The nonlinear evolution equation:

$$u_t + \alpha_1 u u_x + \alpha_2 u_{xx} + \alpha_3 u_{xxx} + \alpha_4 u_{xxxx} + \alpha_5 (u u_x)_x = 0, \tag{1}$$

which describes the evolution of long waves in a shallow convecting fluid can also be used to describe the nonlinear behaviour of a convecting fluid in the vicinity of the Rayleigh number critical value. In this case the parameters α_j are given as follows:

$$\alpha_1 = \frac{3}{2g\,\mathrm{Pr}}(10 + \mathrm{Pr}\cdot g), \quad \alpha_2 = \frac{\sqrt{g}}{2}\left(\frac{1}{3} + \frac{34}{21}\mathrm{Pr}\right), \quad \alpha_3 = \frac{\varepsilon_0\,\mathrm{Pr}}{2079}(682 g\,\overset{2}{\mathrm{Pr}} + 717),$$
$$\alpha_4 = \frac{8\varepsilon_0}{\sqrt{g}}, \quad \alpha_5 = \frac{\mathrm{Pr}\,R_0}{15}\varepsilon_0 \quad ,$$

where $\mathrm{Pr} = \nu/a_0$ is the Prandtl number, g is the Galileo number, ε_0 is a small parameter, such that the value of the Rayleigh number ($\mathrm{Ra} = \mathrm{Gr}\cdot\mathrm{Pr}$), exceeding its critical value, is $\varepsilon_0^2 R_0$, Gr is the Grashov number.

It will be assumed further that $\alpha_j \neq 0$, $j = 1, 2, \ldots 5$. It is obvious that (1) is a bidirectional equation if $\alpha_2 = \alpha_5 = 0$, otherwise it is unidirectional. Let us note that the convecting fluid equation CFE is invariant under the Galilean transformation: $x' = x + \alpha_1 u_0$, $t' = t$, $U = u - u_0$, $u_0 = const$. This actually means that if the function $U\,(x - Vt)$ is a solution of (1) of traveling-wave type, with constant phase velocity, i.e., $V = const.$, then the function

$$u(x, t) = u_0 + U[x - (V + \alpha_1 u_0)t],$$

is also a solution of the nonlinear evolution equation (1), but now with increased phase velocity:

$$V \rightarrow V + \alpha_1 u_0$$

since the free constant u_0 can be chosen so that $signu_0 = sign\alpha_1$.

III. PERIODIC SOLUTIONS

Let us represent the solution of Eq. (1) by the Hirota–Satsuma transformation[23]

$$u(x,t) = \zeta_0 + 2\lambda(\ln\zeta(x,t))_{xx}, \tag{2}$$

where ζ_0 and λ are for now unknown parameters, and $\zeta(x, t)$ is a periodic, continuously differentiable to sixth order function, defined in the two-dimensional domain $\Omega = \{(x, t) \in R^2: x \in R, t \geq 0\}$. Substituting (2) in (1) we obtain a semi-conservative bidifferential form of the evolution equation CFE:

$$\frac{1}{2\zeta^2}[D_tD_x + \alpha_3 D_x^4 + \alpha_1\zeta_0 D_x^4 - B]\zeta\cdot\zeta + (\lambda\alpha_1 - 6\alpha_3)\left(\frac{D_x^2\zeta\cdot\zeta}{2\zeta^2}\right)^2$$
$$+\frac{\partial}{\partial x}\left[\alpha_4\left(\frac{D_x^4\zeta\cdot\zeta}{2\zeta^2}\right) + (\alpha_2+\alpha_5\zeta_0)\frac{D_x^2\zeta\cdot\zeta}{2\zeta^2} + (\lambda\alpha_3 - 6\alpha_4)\left(\frac{D_x^2\zeta\cdot\zeta}{2\zeta^2}\right)^2 - B_0\right] = 0 \tag{3}$$

where $D_z \equiv \partial/\partial z - \partial/\partial z'$ is the Hirota's bilinear differential operator,[24] which satisfies on the diagonals the following identity (see Appendix A):

$$D_t^n D_x^m \varphi(x,t)\cdot\psi(x',t') = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^n\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m \varphi\psi\Bigg|_{\substack{t=t'\\ x=x'}}, \quad m, n \in \mathrm{N}.$$

In Eq. (3) B denotes the integration constant (after single integration on x, which means that B could be dependent on t), while B_0 will be provisionally called a differential constant because of the nature of its appearance in (3). If in the same equation we choose the parameter λ so that

$$\lambda = 6\left(\frac{\alpha_3}{\alpha_1}\right), \tag{4}$$

then the function $\zeta(x, t)$ will be a solution of (3) if it satisfies simultaneously both equations:

$$\left[D_tD_x + \alpha_3 D_x^4 + \alpha_1\zeta_0 D_x^2 - B\right]\zeta\cdot\zeta = 0, \tag{5}$$

$$\alpha_4\zeta^2 D_x^4\zeta\cdot\zeta + (\alpha_2+\alpha_5\zeta_0)\zeta^2 D_x^2\zeta\cdot\zeta + \frac{3}{\alpha_1}(\alpha_3\alpha_5 - \alpha_1\alpha_4)(D_x^2\zeta\cdot\zeta)^2 = B_0\zeta^4 \tag{6}$$

Equation (5) is called bilinear as it is a polynomial with regard to the bilinear differential operators D_t and D_x, while the second one (6) is called residual. In the general case of non-integrable PDEs, the residual equation does not have a bilinear structure. It is obvious that the problem for solving the evolution equation (1) is related to the opportunity of finding such a periodic function $\zeta(x, t)$, satisfying both Eqs. (5) and (6). The appearance of the second integration constant in (6) is due to the derivative with respect to x in front of the last term in the bilinear reduction (3). Let us represent the unknown function $\zeta(x, t)$ in the form:

$$\zeta(x,t) = \theta_3(\xi, q) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{2in\xi}, \tag{7}$$

where $\theta_3(\xi, q)$ is the third Jacobi function (see Ref. 25), which is a uniformly convergent series provided that the parameter $q = e^{i\pi\tau}$ fulfills $0 < |q| < 1$. The latter condition holds if $Im\tau > 0$ and the function is even, with period $T = \pi$ and is also continuously differentiable to arbitrary order. The phase variable $\xi = \xi(x, t)$ is traditionally defined by the equality:

$$\xi = kx + \omega t + \delta, \tag{8}$$

where the parameters k, ω, δ are the wave number, the phase frequency, and the phase shift, respectively, which in general may be complex.

Formally, the hypothetical solution (7) should simultaneously satisfy Eqs. (5) and (6), but practically the bilinear equation (5) is solved first. This is explained by the fact that the index parity principle[18] can not be applied to the residual equation (6), although it is applicable to the bilinear equation (5). Substituting (7) in (5), we obtain the infinite system $G(m) = 0$, $m = \pm 1, \pm 2, \ldots$ resulting from the equality:

$$\sum_{n=-\infty}^{\infty} G(m) e^{2i\pi m\xi} = 0, \tag{9}$$

where

$$G(m) = \sum_{n=-\infty}^{\infty} [-4k\varpi(2n-m)^2 - 4\alpha_1 k^2 \zeta_0 (2n-m)^2 + B + 16\alpha_3 k^4 (2n-m)^4] q^{n^2+(n-m)^2},$$
$$m = 0, \pm 1, \pm 2, \ldots.$$

The infinite chain of equations $G(m) = 0$, $m = \pm 1, \pm 2, \ldots$, actually reduces to two equations, which is essentially the index parity principle. Indeed, if in the infinite sum $G(m)$ we use the rescaling $n \to n + 1$, after some simple transformations we come to the following identities:

$$G(m) = G(m-2)q^{2(m-1)} = G(m-4)q^{2(2m-4)} = \ldots = \begin{cases} G(0)q^{m^2/2}, \text{ if } m \text{ is an even number} \\ G(1)q^{(m^2-1)/2}, \text{ if } m \text{ is an odd number} \end{cases} \tag{10}$$

which allow representing the series (9) in the following compact form:

$$G(0)\theta_3(2\xi, q^2) + q^{-1/2} G(1)\theta_2(2\xi, q^2) = 0, \tag{11}$$

where

$$\theta_2(\xi, q) = \sum_{n=-\infty}^{\infty} q^{(n-1/2)^2} e^{i(2n-1)\xi}$$

is the second Jacobi theta function.[25]

Hence, the infinite chain of linear algebraic equations $G(m) = 0$, $m \in \mathrm{Z}$, is equivalent to the system of two equations: $G(0) = 0$ and $G(1) = 0$, i.e.,

$$\begin{aligned} kq\theta_3'\omega + (\theta_3/8)B &= -\alpha_1 k^2 \zeta_0 q\theta_3' + 8\alpha_3 k^4 q(\theta_3' + q\theta_3''), \\ kq\theta_2'\omega + (\theta_2/8)B &= -\alpha_1 k^2 \zeta_0 q\theta_2' + 8\alpha_3 k^4 q(\theta_2' + q\theta_2''). \end{aligned} \tag{12}$$

Obviously, this system is compatible and definite since its determinant:

$$\Delta = kq(\theta_2\theta_3' - \theta_3\theta_2')/8 = \frac{1}{8} kq W(\theta_2, \theta_3) \neq 0,$$

where $\theta_j = \theta_j(0, q^2)$, $j = 2, 3$, and $W(\theta_2, \theta_3)$ is the Wronskian of θ_2 and θ_3 and hence its only solutions are

$$\omega = \alpha_1 k\zeta_0 + 8\alpha_3 k^3 [1 + q\frac{d}{dq}(\ln W(\theta_2, \theta_3))], \tag{13}$$

$$B = 64\alpha_3 k^4 q^2 \left[\frac{W_q(\theta_2', \theta_3')}{W(\theta_2, \theta_3)}\right]. \tag{14}$$

In other words, the infinite algebraic system $G(m) = 0$, $m \in \mathrm{Z}$ will be satisfied if the phase frequency ω and the integration constant B (which depends on k and q) take values as in (13) and (14), respectively. For the time being the spatial displacement ζ_0 and the "differential" constant B_0 are unknown, but the residual equation (6) remains to be made compatible. Actually, (6) is not an equation in the true sense because we do not look for a new function to satisfy it, but rather want to identify under what conditions the function known from (5)—$\theta_3(\xi, q)$ satisfies this equation. The lack of a bilinear structure for Eq. (5) does not allow applying the index parity principle, so we will represent the spatial displacement ζ_0 as a formal numeric series:

$$\zeta_0 = -\frac{\alpha_2}{\alpha_5} + \frac{4k^2}{\alpha_1} \sum_{m=-\infty}^{\infty} a_m, \tag{15}$$

where a_m are unknown parameters at this stage, and we set the constant $B_0 = 0$. Substituting ζ_0 from (15) into (6) and applying the Cauchy formula:

$$\sum_{\nu=-\infty}^{\infty} A_\nu \sum_{j=-\infty}^{\infty} E_j = \sum_{m,n=-\infty}^{\infty} A_m E_{n-m},$$

we obtain the following infinite chain of algebraic equations:

$$a_m \sum_{n=-\infty}^{\infty} (2n-3m)^2 q^{6n^2-16mn+11m^2} = \sum_{n=-\infty}^{\infty} [\alpha_1\alpha_4 n^4 - 3\alpha_0 n^2 (3n-m)^2] q^{2n^2+(2n-m)^2}, \tag{16}$$

where we have denoted for convenience: $\alpha_0 = \alpha_3\alpha_5 - \alpha_1\alpha_4$. Hence, for $a_m (m \in \mathrm{Z})$ we obtain:

$$a_m = \frac{\sum_{n=-\infty}^{\infty} \left[\alpha_1\alpha_4 n^4 - 3\alpha_0 n^2 (3n-m)^2\right] q^{6n^2-4mn}}{q^{10m^2} \sum_{n=-\infty}^{\infty} (2n-3m)^2 q^{6n^2-16mn}}, \tag{17}$$

so that the residual equation (6) to be satisfied for the parameter ζ_0, defined by (15), i.e., a_m are fully defined parameters (depending on q). Figure 1 shows the discrete values of the spatial displacements a_m, $m = 0, \pm 1, \pm 2, \ldots$ for an average value of the perturbation parameter q ($q = e^{-\pi}$).

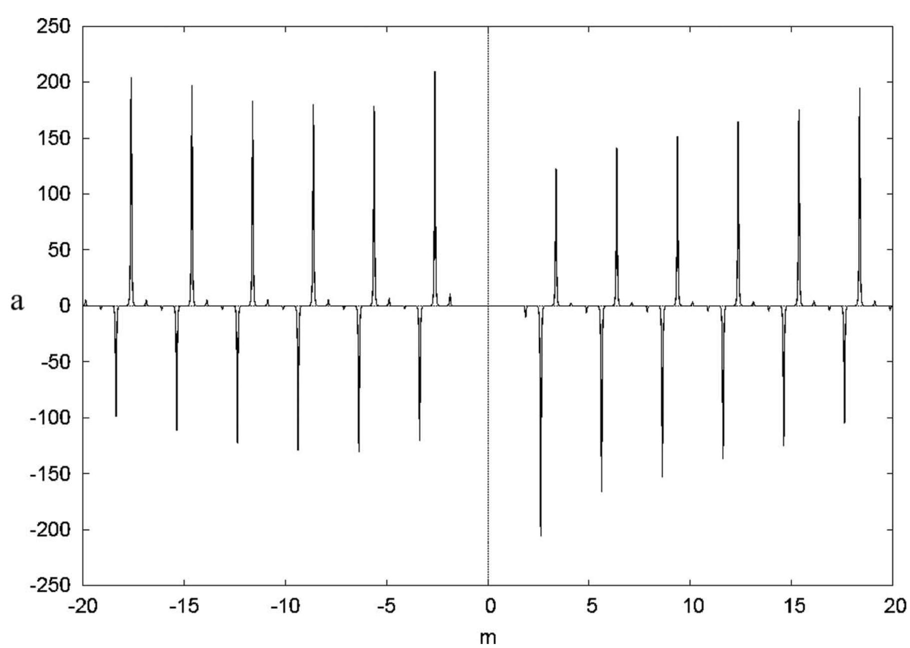

FIG. 1. Discrete values of the spatial displacements $a_m, m = 0, \pm 1, \pm 2, \ldots$ for an average value of the perturbation parameter $q = e^{-\pi}$.

The absolute convergence of the series $\sum_{\nu=-\infty}^{\infty} a_\nu$ follows from relation (16) since its left-hand side is the product:

$$-4k^2\left(\sum_{\nu=-\infty}^{\infty} a_\nu\right)\left(\sum_{n=-\infty}^{\infty} q^{n^2+(n-m)}\right)\left(\sum_{n=-\infty}^{\infty} (2n-m)^2 q^{n^2+(n-m)}\right)$$

$$= -4k^2\left(\sum_{\nu=-\infty}^{\infty} a_\nu\right)\left(\sum_{n=-\infty}^{\infty} (2n-m)^2 q^{2n^2+(2n-m)^2}\right).$$

The absolute convergence of the second series in the last relation for each $m \in Z$ and $0 < |q| < 1$, as well as the absolute convergence of the series in the right-hand side of (16), also imply the absolute convergence of the infinite series (15). We could also express the spatial displacements a_m obtained in (17) in terms of theta functions, which would be a more compact form of presentation:

$$a_m = \left(\frac{1}{q^{10m^2}}\right)\frac{[(27\alpha_0 - \alpha_1\alpha_4)\theta_3^{(4)}(z, q^6) - 36i\alpha_0 m\theta_3^{(3)}(z, q^6) + 24\alpha_0 m^2\theta_3(z, q^6)]}{[\theta_3^{(2)}(4z, q^6) + 6im\theta_3^{(1)}(4z, q^6) + 144m^2\theta_3(4z, q^6)]} \tag{18}$$

where $z = -2m\pi\tau$, $m \in Z$, and all derivatives are with respect to z.

The so presented in more compact form spatial displacements, a_m, give us reason to write that the continuously differentiable function (of arbitrary order):

$$u(x,t) = -\frac{\alpha_2}{\alpha_5} + \frac{4k^2}{\alpha_1}\sum_{m=-\infty}^{\infty} a_m + 6k^2\left(\frac{\alpha_3}{\alpha_1}\right)\frac{d^2}{d\xi^2}[\ln\theta_3(\xi, t)] \tag{19}$$

is an exact localized periodic solution of the convecting fluid equation.

IV. ANALYTICAL CONDITIONS AND REAL PERIODIC SOLUTIONS

Generally speaking, the solution (19) is a three-parametric family (with parameters: k, q, δ) of complex functions. The real solutions generated by (19) are of physical interest, so let us choose:

$$\tau = i\varepsilon,\ \ \varepsilon > 0,\ \ \text{i.e.,}\ q = e^{-\pi\varepsilon}\text{is a real number, such that } q \in (0, 1) \tag{20}$$

The non-zero wave number k could be both real and imaginary number, so we will analyze separately these two cases:

1. The wave number is real: without limiting the generality, we will assume that $k > 0$; the phase variable ξ defined by (8) will be real, for real phase shift, since the phase velocity ω from (13) is a real number for real values of the perturbation parameter q, as are those of (20).
When the phase shift δ is a real number no matter of its sign, the logarithmic derivative of (19) could be represented in a more compact form, namely,[25]

$$\frac{\theta_3^{(1)}(\xi, q)}{\theta_3(\xi, q)} = 4\sum_{m=-\infty}^{\infty}\frac{(-1)^m q^m}{1-q^{2m}}\sin(2m\xi) = 2\sum_{m=-\infty}^{\infty}(-1)^m \cos ech(\varepsilon\pi m)\cdot\sin(2m\xi) \tag{21}$$

The Jacobi theta function $\theta_3(\xi, q)$ has a grid of simple poles at the points:

$$\xi_{mn} = (m + 1/2)\pi + i\varepsilon(n + 1/2)\pi, m, n \in Z.$$

These poles can be avoided if we restrict the phase variable ξ in the horizontal strip:

$$-\pi\varepsilon < \mathrm{Im}\xi < \pi\varepsilon$$

which is, namely, the analytical condition for the solution (19). In this horizontal strip we obtain a well-defined analytical function:

$$u(x,t) = -\frac{\alpha_2}{\alpha_5} + \frac{4k^2}{\alpha_1}\sum_{m=-\infty}^{\infty}[k^2 a_m + 6(-1)^m \cos ech(\varepsilon\pi m)\cos(2m\xi)] \tag{22}$$

which is an exact, real, and periodic solution of CFE, along with conditions (13), (14), and (18). It is worth mentioning that under the condition of the hypothesis (20), the spatial displacements a_m from (18) are real.

2. The wave number is purely imaginary: let in this case $k \to ik$, assuming that $k > 0$. For such purely imaginary values of the wave number it is possible to obtain real periodic solutions $u(x, t)$ if the phase velocity $\omega(k)$ from (13) is a purely imaginary number and the phase shift δ is chosen in a proper way. Formula (18) shows that for purely imaginary values of k, the spatial displacements a_m are real and hence $\zeta_0 \in R$ and consequently the phase velocity $\omega(k)$ will be a purely imaginary number. Now, for the phase shift let us choose one imaginary number: $\delta \to \delta + \pi\varepsilon$ then $i\xi \to i\xi + \tau\pi$, $(\tau = i\varepsilon)$. If we take advantage of the quasi-periodic property of the theta function θ_3, namely,

$$\theta_2(i\xi, q) = q^{1/4} e^{i\xi} \theta_3(i\xi + \pi\tau/2, q),$$

this would allow us to express the logarithmic derivative from (19) in a more convenient form:

$$\frac{\theta_3^{(1)}(i\xi + \pi\tau/2, q)}{\theta_3(i\xi + \pi\tau/2, q)} = \frac{\theta_2^{(1)}(i\xi, q)}{\theta_2(i\xi, q)} - i.$$

Let us apply to the above relation the following identity:

$$\frac{\theta_2^{(1)}(z, q)}{\theta_2(z, q)} = i \sum_{m=-\infty}^{\infty} \tanh[i(z - \varepsilon\pi m)], \tag{23}$$

which is obtained from the infinite product:

$$\theta_2(z, q) = 2q^{1/4} \cos z \prod_{n=1}^{\infty} (1 - q^{2n})(1 + 2q^{2n} \cos 2z + q^{2n}).$$

In result, we can again obtain a well-defined function in the strip $-\pi\varepsilon < \mathrm{Im}\xi < \pi\varepsilon$, which will be an exact, real, and periodic solution of the nonlinear evolution equation CFE:

$$u(x, t) = -\frac{\alpha_2}{\alpha_5} + \frac{k^2}{\alpha_1} \sum_{m=-\infty}^{\infty} [4k^2 a_m + 6k^2 \alpha_3 \sec h^2(\xi - m\pi\varepsilon)]. \tag{24}$$

Let us specify in this case the domain of convergence of the infinite series (24) in the context of the inequalities $-\pi\varepsilon < \mathrm{Im}\xi < \pi\varepsilon$. Taking into account that $\mathrm{Im}(i\xi) = kx + \omega t + \varepsilon$, then the domain of convergence is the open interval:

$$-\pi\varepsilon < kx + \omega t + \delta < \pi\varepsilon.$$

As seen from (24), for purely imaginary values of the wave number k are generated waves, which represent an infinite set of periodic solitary-wave profiles having phase velocities $\omega = \omega(k, q)$, but different spatial displacements, $a_m(q)$, given by the equality (18). The solutions we obtained above in (22) and (24) are actually dynamically equivalent.

This property of a nonlinear periodic wave to be represented as an exact sum of solitary-wave profiles is called a nonlinear superposition principle. Toda[26] was the first to show that the cnoidal wave in the KDV equation: $u_t - 6uu_x + u_{xxx} = 0$ can be represented as a double infinite sum of reiterating sech^2 solitary-wave profiles. It has been shown that a majority of the known nonlinear evolution equations in applied mathematics and mathematical physics have this property. These are mainly integrable equations.

Let us also mention the important peculiarity of the interpretation of the nonlinear superposition principle noticed by Parker,[22] namely, that practically there is no superposition of solitary-wave solutions in the conventional sense of the linear theory, but rather superposition of solitary-wave profiles. That is true since in the general case, the velocity of the periodic wave differs from the one of the solitary wave, whose form is exactly doubled in order to generate periodic solutions. The exact real periodic solutions of CFE obtained in (22) and (24) also confirm the nonlinear superposition principle for one non-integrable nonlinear evolution equation.

For larger values of the perturbation parameter q, i.e., $q \to 1$ (or $\varepsilon \to 0$), the solitary-wave profiles have an increasing zone of overlapping and their sum generates a sinusoidal wave with small amplitude. For increased wavelengths (i.e.,$q \to 0$) the separate solitary-wave profiles gradually split and differentiate.

V. THE CASE: $\alpha_1\alpha_4 = \alpha_3\alpha_5$

The authors of Ref. 13 have obtained an exact solitary solution of CFE as well as a cnoidal exact solution was obtained in Ref. 12 in two separate cases: $\alpha_1\alpha_4 = \alpha_3\alpha_5$ and $\alpha_1\alpha_4 = 6\alpha_3\alpha_5$. The authors have not analyzed why the case

$$\alpha_1\alpha_4 = \alpha_3\alpha_5 \tag{25}$$

is more special for this equation. In the present section, we will try to answer this question by using the bilinear transformation method. Let us note that the Painleve analysis [15] of Eq. (1), under the assumption (25), showed that it has second degree of singularity with three mobile critical points. In case that the phase velocity of the traveling wave is $-\alpha_2\alpha_3/\alpha_4$, then the equation has the same degree of singularity, but with two mobile critical points.

Let us express the solution of (1), under the assumption (25) (i.e., $\alpha_0 = 0$), again by the Hirota–Satsuma transformation:

$$u(x,t) = \zeta_1 + 6\left(\frac{\alpha_4}{\alpha_1}\right)(\ln\zeta(\xi_1))_{xx}, \tag{26}$$

where $\zeta_1 = const.$, $\xi_1 = kx + \omega_1 t + \delta_1$. In this case the bilinear reduction of (1) leads to (5) and (6), which under the assumption (25) are expressed as follows:

$$[D_t D_x + \alpha_3 D_x^4 + \alpha_1\zeta_1 D_x^2 - B_1]\zeta\cdot\zeta = 0, \tag{27}$$

$$[\alpha_4 D_x^4 + (\alpha_2 + \alpha_5\zeta_1)D_x^2 - B_0]\zeta\cdot\zeta = 0. \tag{28}$$

While the bilinear equation (27) has not changed, the residual equation (28) has changed its structure—it has already become a bilinear. This allows applying the index parity principle to both equations since the solution of (27) is analogous to the solution (7), i.e.,

$$\zeta(\xi_1) = \theta_3(\xi_1, q).$$

The condition for $\zeta(\xi_1)$ to be a solution of (1), under the assumption (25), means that the unknown for the time being parameters ω_1, ζ_1, B_1, and B_0 should satisfy the following algebraic systems:

$$(kq\theta_3)\omega_1 + (\theta_3/8)B_1 = 8k^4\alpha_3 q(\theta_3' + q\theta_3'') - k^2\alpha_1 q\theta_3'\zeta_1,$$

$$(kq\theta_2)\omega_1 + (\theta_2/8)B_1 = 8k^4\alpha_3 q(\theta_2' + q\theta_2'') - k^2\alpha_1 q\theta_2'\zeta_1,$$

and

$$k^2 q(\alpha_2 + \alpha_5\zeta_1)\theta_3' + (\theta_3/8)B_0 = 8k^4\alpha_4(\theta_3' + q\theta_3'')$$

$$k^2 q(\alpha_2 + \alpha_5\zeta_1)\theta_2' + (\theta_2/8)B_0 = 8k^4\alpha_4(\theta_2' + q\theta_2''),$$

each one of which is compatible and definite, and their unique solutions are

$$\zeta_1 = \frac{1}{\alpha_5}(8k^2\alpha_4 - \alpha_2) + 8qk^2\left(\frac{\alpha_4}{\alpha_5}\right)[\ln W(\theta_2, \theta_3)]_q, \tag{29}$$

$$\omega_1 = \frac{k\alpha_1}{\alpha_5}(\alpha_2 - 8k^2\alpha_4) + 8k^3(\alpha_3 - \alpha_1 k^2)[\ln W(\theta_2, \theta_3)]_q, \tag{30}$$

$$B_0 = 64\alpha_4 k^4 q^2 W(\theta_2', \theta_3')/W(\theta_2, \theta_3), \quad (31)$$

$$B_1 = 64\alpha_3 k^4 q^2 W(\theta_2', \theta_3')/W(\theta_2, \theta_3). \quad (32)$$

In the context of Sec. IV we can make the conclusion that the exact periodic solution of (1) for real wave number k (which we can consider as positive) and under the assumption (25) is the following:

$$u(x,t) = \zeta_1 + 48k^2\left(\frac{\alpha_3}{\alpha_1}\right)\sum_{m=-\infty}^{\infty}\frac{(-1)^m m q^m}{1-q^{2m}}\cos 2m\xi_1, \quad (33)$$

while in the case of purely imaginary wave number, i.e., $k \to ik(k > 0)$, the periodic solution is:

$$u(x,t) = \zeta_1 + 12k^2\left(\frac{\alpha_3}{\alpha_1}\right)\sum_{m=-\infty}^{\infty}\sec h^2(\xi_1 - m\pi\varepsilon), \quad (34)$$

where $\xi_1 = kx + \omega_1 t + \delta_1$ is a real phase variable after the rescaling $k \to ik$; $\omega_1 \to i\omega_1$; $\delta_1 \to i\delta_1$.

The periodic solutions of Eq. (1), obtained in (32) and (33) under the assumption (25), differ from the corresponding solutions (22) and (24) in the general case by the fact that the spatial displacements in the first case (under the hypothesis (25)) are one and the same for each wave, i.e., $\zeta_1 = const.$ for $k = const.$ and $q = const.$ In the general case each periodic wave has individual spatial displacement a_m, $m = 0, \pm 1, \pm 2, \ldots$ In other words, the hypothesis (25), characterizing a proportional balance between the dispersion and the nonlinear effects, has turned the non-integrable equation (1) into a partially integrable equation.

VI. THE CASE $\alpha_5 = 0$: EXACT PERIODIC SOLUTION OF THE KURAMOTO–SIVASHINSKY EQUATION (KSE)

Without limiting the generality we can assume $\alpha_1 = 1$, otherwise the rescaling: $t \to \alpha_1 t$, $u \to u/\alpha_1$ transforms Eq. (1) in equation:

$$u_t + uu_x + \alpha_2 u_{xx} + \alpha_3 u_{xxx} + \alpha_4 u_{xxxx} = 0. \quad (35)$$

As mentioned in Sec. I, the above equation was suggested in Ref. 7 as a model equation describing the evolution of the dynamics in a non-vertically falling viscous film. In the case $\alpha_3 = 0$ a solitary solution of (35) is known, found in Ref. 10, and the analogous exact solitary-wave solution of (35) in the general case was found in Ref. 16. An approximate periodic solution of the KSE was suggested in Ref. 12, applying the Whitham's method of representation the periodic wave in an evolution equation as superposition of equally spaced solitons. Essentially, these are the known exact (solitary) and approximate (periodic) solutions of KSE. We will show that we can find exact periodic solutions of this equation by means of the bilinear transformation method. Indeed, presenting the solution of Eq. (35) by the function:

$$u(x,t) = 6\alpha_3(\ln\zeta(\xi_2))_{xx}, \quad (36)$$

where $\zeta(\xi_2)$ is a periodic function, continuously differentiable in the domain Ω and $\xi_2 = kx + \omega_2 t + \delta_2$, then according to (5) and (6), $\zeta(\xi_2)$ would be a solution of (35) if satisfies both equations:

$$(D_t D_x + \alpha_3 D_x^4 - B_2)\zeta.\zeta = 0, \quad (37)$$

$$\alpha_4(D_x^4\zeta\cdot\zeta)\zeta^2 + \alpha_2(D_x^2\zeta\cdot\zeta)\zeta^2 - 3\alpha_4(D_x^2\zeta\cdot\zeta)^2 = B_0\zeta^4, \quad (38)$$

resulting from (5) and (6), respectively, for $\alpha_5 = 0$, $\alpha_1 = 1$, $\zeta_0 = 0$, and $B \to B_2$. The bilinear equation (37), under the assumption that $\zeta(\xi_2) = \theta_3(\xi_2, q)$, is reduced to the linear system:

$$(kq\theta_3')\omega_2 + (\theta_3/8)B_2 = 8\alpha_3 k^4 q(\theta_3' + q\theta_3''),$$

$$(kq\theta_2')\omega_2 + (\theta_2/8)B_2 = 8\alpha_3 k^4 q(\theta_2' + q\theta_2''),$$

where we have denoted for convenience $\theta_j = \theta_j(0, q^2)$, as we did in (12). The solutions of this system are as follows:

$$\omega_2 = 8k^3\alpha_3[1 + q(\ln W(\theta_2, \theta_3))_q], \tag{39}$$

$$B_2 = 64k^4\alpha_3 q^2 \left[\frac{W(\theta_2', \theta_3')}{W(\theta_2, \theta_3)}\right]. \tag{40}$$

The assumption that there are no spatial displacements in the solution of (35) is based on the physical conditions in deriving the equation, namely, falling of a viscous fluid film on an inclined surface. The Jacobi function $\theta_3(\xi_2, q)$ will be an exact periodic solution of the nonlinear Kuramoto–Sivashinsky equation (35) if satisfies the residual equation (38), i.e., if $\zeta(\xi_2) = \theta_3(\xi_2, q)$ satisfies the infinite chain of algebraic equations (for each integer m) provided that $B_0 = 0$. If we define the formal series:

$$k^2 = \frac{1}{4}\left(\frac{\alpha_2}{\alpha_4}\right)\sum_{m=-\infty}^{\infty} k_m^2, \tag{41}$$

where k_m^2 are currently unknown parameters, then the residual equation will take the form:

$$k_m^2 \sum_{n=-\infty}^{\infty} [(n-m)^4 - 3(n-m)^2(3n-4m)^2] q^{2(n-m)^2+(2n-3m)^2} = \sum_{n=-\infty}^{\infty} (2n-m)^2 q^{2n^2+(2n-m)^2} \tag{42}$$

as these equalities are valid for any integer m, thereby k_m^2 are uniquely determined by the equality:

$$k_m^2 = \frac{\sum_{n=-\infty}^{\infty} (2n-m) q^{2n^2+(2n-m)^2}}{\sum_{n=-\infty}^{\infty} [(n-m)^4 - 3(n-m)^2(3n-4m)^2] q^{2(n-m)^2+(2n-3m)^2}}.$$

It can be proved (as in Sec. II) that thus defined parameters k_m^2 form an absolutely convergent number series (41) when $0 < |.q|. < 1$. Furthermore, let us note that the terms k_m^2 of this infinite series can be expressed by the third theta function, but this representation is rather complicated and for that reason will not be shown here.

When the wave number defined by (41) is real (we consider it positive), then for the representation $\tau = i\varepsilon(\varepsilon > 0)$ and $q = e^{-\pi\varepsilon}$ we obtain the following real periodic solution of Eq. (35):

$$u(x,t) = 12k^2\left(\frac{\alpha_2\alpha_3}{\alpha_4}\right)\sum_{m=-\infty}^{\infty}\frac{(-1)^m m q^m}{1-q^{2m}}\cos 2\xi_2 m. \tag{43}$$

If the wave number from (41) is imaginary, i.e.,$k \to ik(k > 0)$, then applying the quasi-periodic property of Jacobi's theta functions, as well as the identity (23), we can write the periodic solution of KS in the form:

$$u(x,t) = \frac{3}{2}\left(\frac{\alpha_2\alpha_3}{\alpha_4}\right)k^2\sum_{m=-\infty}^{\infty}\operatorname{sec h}^2(\xi_2 - m\pi\varepsilon), \tag{44}$$

where the wave numbers k^2, the phase velocity $\omega_2(k, q)$, and the integration constant B_2 are defined by (41) and (39), and (40), respectively. This solution is a superposition of solitary-wave profiles of the type sech^2, having wavelength $d = \pi\varepsilon/k$.

Therefore, the KS equation generates periodic waves with constant amplitudes, constant periods, and constant wavelengths, for each phase value ξ_2, such that: $-\pi\varepsilon < \mathrm{Im}\xi_2 < \pi\varepsilon$, $\varepsilon > 0$.

In conclusion, if we set $\alpha_2 = 0$ in Eq. (35), then the residual equation (38) will transform into the following:

$$B_0 = 2\alpha_4 \left[\left(\frac{D_x^4 \zeta \cdot \zeta}{2\zeta^2} \right) - 6 \left(\frac{D_x^2 \zeta \cdot \zeta}{2\zeta^2} \right)^2 \right] = 2\alpha_4 (\ln \zeta)_{xxxx},$$

i.e., it cannot be fulfilled for $\zeta = \theta_3(\xi_2, q)$ and $B_0 = const$. Hence the KS equation does not possess localized periodic solutions for $\alpha_2 = 0$, but as it was shown in Ref. 14, KSE has a solitary-wave solution for $\alpha_3 = 0$. It is not difficult to notice that for $\alpha_3 = 0$ the algebraic system (12) is changed into homogeneous ($\zeta_0 = 0$) with the trivial solution $\omega_2 = B_2 = 0$. Therefore, for $\alpha_3 = 0$ as well, the KS equation does not have a periodic solution.

In the general case of the KS equation, i.e., $\alpha_3 \neq 0$, it was proved in Ref. 17 that it allows an elliptic solution only for the condition $\alpha_3^2 = 16\alpha_2\alpha_4$. This solution is

$$u(x,t) = A\wp(\xi_0, G_2, G_3) + B\wp'(\xi_0, G_2, G_3),$$

where $A = -60\alpha_4; B = -15\alpha_3; \xi_0 = x - V_0 t + \delta_0$, $V_0 = \alpha_2\alpha_3/4\alpha_4; G_2 = \alpha_2^2/6\alpha_4(5\alpha_4 - 3\alpha_3)$; $G_3 \in R$, and the integration constant: $E = -2880\alpha_4^2 G_3 + 975G_2/2$. We can avoid the double poles of the Weierstrass biperiodic function $\wp$ applying the vertical phase translation: $\delta_0 = \omega_2/2$, and we can also localize cnoidal solutions of KS, which are conditional:

$$u(x,t) = cn(z,\mu_0)(e_2 - e_3)[2B\sqrt{e_1 - e_3}\, sn(z,\mu_0)\, dn(z,\mu_0) - Acn(z,\mu_0)] + Ae_2,$$

where $z = \xi_0\sqrt{e_1 - e_3}$, $\mu_0^2 = (e_2 - e_3)/(e_1 - e_3)$, $e_1 > e_2 > e_3$ are the real roots of the cubic algebraic equation: $4Y^3 - G_2 Y - G_3 = 0$, and sn, cn, and dn are the Jacobi elliptic functions—sine-amplitude, cosine-amplitude, and delta-amplitude, respectively.

VII. CONCLUSIONS

The nonlinear evolution convecting fluid equation gave us an opportunity to apply the bilinear transformation method on a completely non-integrable nonlinear equation, what CFE is in the general case, and on a partially integrable equation, as CFE was found to be under the assumption (25). The representation of the common spatial displacement ζ_0 as the infinite sum (15) revealed the physical nature of the long waves described in this case with the non-integrable equation, which essentially means that each separate harmonic of the nonlinear periodic wave has its own spatial displacement $a_m(q)$. For the integrable and partially integrable nonlinear PDEs as is the case (25), these different vertical displacements converge into one common spatial displacement. For larger values of the perturbation parameter q, i.e., $q \to 1$ (or $\varepsilon \to 0$), the solitary-wave profiles have an increasing zone of overlapping and their sum generates a sinusoidal wave with small amplitude. For increased wavelengths (i.e.,$q \to 0$) the separate solitary-wave profiles gradually split and differentiate.

This physical nature of the periodic nonlinear waves, generated by the non-integrability of the model equations, is not so obvious. It is the result of the inhomogeneous structure of the bilinear representation (3) of the equations (5) and (6). In other words the bilinear differential reduction (2) and (3) of the non-integrable evolution equations does not cause the residual equations to vanish. They actually generate the above mentioned different spatial displacements.

This feature of the residual equations (having or not bilinear structure), typical of non-integrable nonlinear equations, as a rule generates infinite algebraic systems. In spite of these mathematical complications that can be overcome employing certain individual approaches, the bilinear transformation method is also applicable to non-integrable nonlinear PDE, known that it cannot be solved using standard analytical methods.

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APPENDIX A: LOGARITHMIC DERIVATIVES BY THE HIROTA'S OPERATORS

Logarithmic derivatives expressed by the Hirota's bilinear differential operators D_t, D_x

$$(\ln\zeta)_{xx} = \frac{D_x^2\zeta\cdot\zeta}{2\zeta^2},\ (\ln\zeta)_{tx} = \frac{D_t D_x\zeta\cdot\zeta}{2\zeta^2}, \tag{A1}$$

$$(\ln\zeta)_{xxxx} = \frac{D_x^4\zeta\cdot\zeta}{2\zeta^2} - 6\left(\frac{D_x^2\zeta\cdot\zeta}{2\zeta^2}\right)^2, \tag{A2}$$

$$(\ln\zeta)_{xxxxxx} = \frac{D_x^6\zeta\cdot\zeta}{2\zeta^2} - 30\left(\frac{D_x^2\zeta\cdot\zeta}{2\zeta^2}\right)\left(\frac{D_x^4\zeta\cdot\zeta}{2\zeta^2}\right) + 120\left(\frac{D_x^2\zeta\cdot\zeta}{2\zeta^2}\right)^3. \tag{A3}$$

APPENDIX B: IDENTITIES WITH THE JACOBI THETA FUNCTIONS

Identities with the Jacobi theta functions $\theta_3 = \theta_3(0, q^2)$, $\theta_2 = \theta_2(0, q^2)$

$$\sum_{n=-\infty}^{\infty} q^{2n^2} = \theta_3,\ \sum_{n=-\infty}^{\infty} q^{n^2+(n-1)^2} = q^{1/2}\theta_2, \tag{B1}$$

$$\sum_{n=-\infty}^{\infty} n^2 q^{2n^2} = q\theta_3'/2,\ \sum_{n=-\infty}^{\infty} (2n-1)^2 q^{n^2+(n-1)^2} = 2q^{3/2}\theta_2', \tag{B2}$$

$$\sum_{n=-\infty}^{\infty} n^4 q^{2n^2} = q(\theta_3' + q\theta_3'')/4,\ \sum_{n=-\infty}^{\infty} (2n-1)^4 q^{n^2+(n-1)^2} = 4q^{3/2}(\theta_2' + q\theta_2''), \tag{B3}$$

$$\sum_{n=-\infty}^{\infty} n^6 q^{2n^2} = q(q\theta_3' + q^2\theta_3'')'/8,\ \sum_{n=-\infty}^{\infty} (2n-1)^6 q^{n^2+(n-1)^2} = 8q^{3/2}(q\theta_2' + q^2\theta_2'')'. \tag{B4}$$

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