ON THE LAUWERIER FORMULATION OF THE TEMPERATURE FIELD PROBLEM IN OIL STRATA

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Dedicated to Professor Dr. Ivan Dimovski, Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Bulgaria, on the occasion of his 70th birthday

The paper is concerned with the fractional extension of the Lauwerier formulation of the problem related to the temperature field description in a porous medium (sandstone) saturated with oil (strata). The boundary value problem for the fractional heat equation is solved by means of the Caputo differintegration operator $D_*^{(\alpha)}$ of order $0 < \alpha \le 1$ and the Laplace transform. The solution is obtained in an integral form, where the integrand is expressed in terms of a convolution of two special functions of Wright type.

1. Introduction

The problem of describing the temperature field $u = u(x, y, z, t)$ of the strata arises when a hot water or steam is injected into the strata in order to facilitate the oil extraction process.

Two cases of fluid injection are mainly considered: linear and radial injections. In the linear case, a hot fluid is forced into the strata in the positive and negative x-directions with constant velocity through an infinitely long vertical gallery. In the radial case, a hot fluid is forced through an infinitely thin well, which is considered as a linear source of incompressible fluid with positive volume rate.

The heat equation for a porous medium is derived in [1] under several generally accepted assumptions on the model. Beside the exact formulation of the temperature field problem in oil strata, the following three approximate formulations are also treated:

  (i) *the lumped formulation*, where the thermal conductivity of the strata is infinitely large in the vertical direction;
 (ii) *the incomplete lumped formulation*, where the horizontal heat transfer in the cap and base rocks surrounding the strata is neglected;
(iii) *the formulation of Lauwerier*, where the horizontal heat transfer in the strata is also neglected.

Our interest in this paper is particularly addressed to the fractional generalization of the Lauwerier formulation since analogous problems concerning the lumped formulations as well as the incomplete lumped formulation have been recently studied in [2]. The Lauwerier formulation relates to the temperature field of a single layer stratum in the case the velocity of the heat transfer between the fluid and the skeleton is finite. In this

case, one has to consider separately the temperatures $u(x,t)$ and $\Theta(x,z,t)$ of the fluid and the cap rock, respectively. The heat equation for the regions containing the fluid and the skeleton, respectively, is derived [1, Chapter 8, Section 8.7] under the main assumption that instead of having two regions containing the fluid and the skeleton separately, there is just a single region which is taken to be a porous medium. For a sufficiently large filtration velocity, one can neglect the heat transferred to the cap rock and stratum in the x-direction in comparison with the heat transfer in the z-direction. In this case, the Lauwerier formulation for the linear fluid injection reads as

$$\frac{\partial\Theta}{\partial t} = \frac{\partial^2\Theta}{\partial z^2}, \quad 0 < x,z,t < \infty,$$

$$z = 0 : \frac{\partial u}{\partial t} = -\gamma\frac{\partial u}{\partial x} - \alpha(u - \Theta), \quad 0 < x,t < \infty, \tag{1.1}$$

$$z = 0 : \frac{\partial\Theta}{\partial t} = \mu\frac{\partial\Theta}{\partial z} + k(u - \Theta), \quad 0 < x,t < \infty,$$

$$x = 0 : u = 1, \tag{1.2a}$$

$$u,\Theta \longrightarrow 0 \quad \text{as } x^2 + z^2 \longrightarrow \infty, \tag{1.2b}$$

$$t = 0 : u = \Theta = 0. \tag{1.2c}$$

The physical meaning of the constants involved can be briefly described as follows:

(i) the constant $\gamma > 0$ depends on the volume rate of the hot fluid forced into the strata, the porosity specified as the ratio of the pores volume to the whole volume, and the coefficient of thermal conductivity of the cap rock;

(ii) the constant $\alpha > 0$ depends on the porosity, the coefficients of thermal conductivity of the cap rock, and the volumic heat capacity of the fluid;

(iii) the constant $\mu > 0$ depends on the coefficient of thermal conductivity of the cap rock, the volumic heat capacity of the skeleton, and the coefficient of thermal conductivity of the cap rock;

(iv) the constant $k > 0$ depends on the porosity and the volumic heat capacity of the fluid and the skeleton.

By using the Laplace transform

$$\bar{f}(p) = L[f(t)] = \int_0^\infty e^{-pt} f(t)dt, \quad \text{Re}(p) > 0, \tag{1.3}$$

the solution of the problem (1.1) and (1.2) is given by the following formulas [1, Chapter 8, formulas (8.7.40) and (8.7.44)]:

$u(x,t)$

$$= e^{-\alpha x/\gamma} \int_0^{t-x/\gamma} e^{-k\tau} \left[\frac{\mu(2t - 2x/\gamma - \tau)}{2\sqrt{\pi}(t - x/\gamma - \tau)^{3/2}} e^{-(\tau\mu/2\sqrt{t-x/\gamma-\tau})^2} + k\,\text{erfc}\left(\frac{\tau\mu}{2\sqrt{t - x/\gamma - \tau}} \right) \right]$$

$$\times I_0\left(2\sqrt{\frac{k\alpha x}{\gamma}}\tau \right) d\tau,$$

$$\tag{1.4}$$

$$\Theta(x,z,t) = ke^{-\alpha x/\gamma} \int_0^{t-x/\gamma} e^{-k\tau} \operatorname{erfc}\left(\frac{z+\mu\tau}{2\sqrt{t-x/\gamma-\tau}}\right) I_0\left(2\sqrt{\frac{k\alpha x}{\gamma\tau}}\right) d\tau, \qquad (1.5)$$

where $I_0(z)$ is the modified Bessel function of the first kind with zero index [6] and

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt \qquad (1.6)$$

is the complementary error function [5].

The main goal of this paper is to formulate in a reasonable way and to solve the fractional generalization of the problem (1.1) and (1.2).

Further on in the text we refer to this problem as fractional Lauwerier formulation of the temperature field problem in oil strata.

2. Fractional heat equation

The concept of a noninteger differentiation and integration of a function is almost as old as calculus itself. Many famous mathematicians including Leibniz, Euler, Lagrange, Laplace, Fourier, and Abel made some contribution to it. Nowadays there exist specialized treaties where mathematical aspects and applications of the fractional calculus are extensively discussed [13, 14, 17, 19].

Fractional calculus is a significant topic in mathematical analysis as a result of its increasing range of applications. One of the main advantages of the fractional calculus is that the fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. For our purpose, we adopt in this paper the Caputo fractional derivative

$$D_*^\beta f(t) = \begin{cases} \dfrac{1}{\Gamma(m-\beta)} \displaystyle\int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\beta+1-m}} d\tau, & m-1 < \beta < m, \\ \dfrac{d^m f(t)}{dt^m}, & \beta = m, \end{cases} \qquad (2.1)$$

where m is a positive integer. This definition was introduced by Caputo in the late sixties of the twentieth century and adopted by Caputo and Mainardi in the framework of the theory of linear viscoelasticity. The method we apply makes the following rule of the Laplace transform extremely important:

$$L\left[D_*^\beta f(t)\right] = p^\beta \bar{f}(p) - \sum_{k=0}^{m-1} f^k(0) p^{\beta-1-k}, \qquad m-1 < \beta \le m. \qquad (2.2)$$

The modelling of diffusion in a specific type of porous medium is one of the most significant applications of fractional-order derivatives [12, 18]. The fractional diffusion equation deduced by Nigmatullin [15, 16] in the form

$$\frac{\partial^{2\beta} u}{\partial t^{2\beta}} = a^2 \frac{\partial^2 u}{\partial z^2}, \qquad 0 < \beta \le \frac{1}{2}, \qquad (2.3)$$

deserves special attention. This equation is also know as the fractional diffusion-wave equation [10, 11]. The reason is that when the order of the fractional derivative is $2\beta = 1$, the equation becomes the classical diffusion equation, and if $2\beta = 2$, it becomes the classical wave equation.

For $0 < 2\beta < 1$, we have the so-called ultraslow diffusion. Values $1 < 2\beta < 2$ correspond to the so-called intermediate processes [8].

In this paper, we consider the fractional diffusion equation (2.3) when the fractional time derivative is determined by the Caputo fractional derivative (2.1). Thus we extend the problem (1.1) and (1.2) to the fractional Lauwerier formulation of the temperature field problem in oil strata that reads as

$$D_*^{2\beta}\Theta = \frac{\partial^2\Theta}{\partial z^2}, \quad 0 < x, z, t < \infty, \ 0 < \beta \le \frac{1}{2}, \tag{2.4}$$

subject to the boundary conditions

$$
\begin{aligned}
z = 0 : D_*^{2\beta}u &= -\gamma\frac{\partial u}{\partial x} - \alpha(u - \Theta), \quad 0 < x, t < \infty, \\
z = 0 : D_*^{2\beta}\Theta &= \mu\frac{\partial\Theta}{\partial z} + k(u - \Theta), \quad 0 < x, t < \infty,
\end{aligned}
\tag{2.5}
$$

and the conditions

$$x = 0, \qquad z = 0 : u = 1, \tag{2.6a}$$

$$u, \Theta \longrightarrow 0 \qquad \text{as } x^2 + z^2 \longrightarrow \infty, \tag{2.6b}$$

$$t = 0 : u = \Theta = 0. \tag{2.6c}$$

3. Auxiliary results

With the idea to make this paper relatively self-contained, we summarize in this section some important facts that enable us to solve the fractional problem stated above.

(i) First we mention that a key role in obtaining the solutions (1.4) and (1.5) is given to the following theorem [3, pages 35–36].

THEOREM 3.1 (Efros theorem). *Assume there exist analytic functions $G(p)$ and $q(p)$ and the relations*

$$F(p) = L[f(t)], \qquad e^{-\tau(p)}G(p) = L[g(t, \tau)]. \tag{3.1}$$

Then

$$G(p)F(q(p)) = L\left[\int_0^\infty f(\tau)g(t, \tau)d\tau\right]. \tag{3.2}$$

Theorem 3.1 is simply a generalization of the convolutional theorem for the Laplace transform. Indeed, if we take $q(p) = p$, then

$$L[g(t, \tau)] = e^{-p\tau}G(p), \tag{3.3}$$

and by the p-shift theorem, $g(t,\tau) = g(t-\tau)$. Thus formula (3.9) becomes

$$G(p)F(p) = L\left[\int_0^\infty f(\tau)g(t,\tau)d\tau\right] = L\left[\int_0^t f(t)g(t-\tau)d\tau\right], \tag{3.4}$$

since for original functions we have $g(t-\tau) = 0$ as $t < \tau$.

(ii) The fundamental solution of the basic Cauchy problem for the time fractional diffusion equation (2.3) can be expressed in terms of an auxiliary function defined as [11]

$$M(z;\beta) = \frac{1}{2\pi i}\int_{H_a} e^{\sigma-z\sigma^\beta}\frac{d\sigma}{\sigma^{1-\beta}}, \quad 0 < \beta < 1, \tag{3.5}$$

where H_a denotes the Hankel path of integration that begins $\sigma = -\infty - ib_1$ ($b_1 > 0$), encircles the branch cut that lies along the negative real axis, and ends up at $\sigma = -\infty + ib_2$ ($b_2 > 0$). It is also proved that the following relation takes place:

$$M(z;\beta) = W(-z; -\beta; 1-\beta), \tag{3.6}$$

where

$$W(z;\lambda,\mu) = \sum_{n=0}^\infty \frac{z^n}{n!\Gamma(\lambda n + \mu)} = \frac{1}{2\pi i}\int_{H_a} e^{\sigma+z\sigma^{-\lambda}}\frac{d\sigma}{d^\mu}, \quad \lambda > -1, \mu > 0, \tag{3.7}$$

is an entire function of z referred to as the Wright function (cf. [7, Chapter 18]). In the particular case $\beta = 1/2$, it holds that

$$M\left(z;\frac{1}{2}\right) = \frac{1}{\sqrt{\pi}}\sum_{m=0}^\infty (-1)^m \left(\frac{1}{2}\right)^m \frac{z^{2m}}{(2m)!} = \frac{1}{\sqrt{\pi}}e^{-z^{2/4}}. \tag{3.8}$$

Further, we introduce the following auxiliary function:

$$N(\xi;\beta;\lambda) = \frac{1}{2\pi i}\int_{H_a} e^{\sigma-\xi\sigma^{2\beta}}\frac{d\sigma}{\sigma^\lambda}, \quad \lambda > 0, 0 < \beta \le \frac{1}{2}, \tag{3.9}$$

and prove the following statement.

LEMMA 3.2. *If $0 < \beta \le 1/2$ and $0 < t, \tau, z < \infty$, then the following relations hold:*
(i)

$$e^{-(z+\mu\tau)p^\beta} = L[g_1(t,\tau,z;\beta)], \quad \mu > 0; \tag{3.10}$$

(ii)

$$\frac{1}{p^\lambda}e^{-(x/\gamma+\tau)p^{2\beta}} = L[g_2(t,\tau,\beta;\lambda)], \quad \lambda > 0, x > 0, \gamma > 0, \tag{3.11}$$

where

$$g_1(t,\tau,z,\beta) = \frac{(z+\mu\tau)\beta}{t^{\beta+1}} M\left(\frac{z+\mu\tau}{t^\beta};\beta\right),\tag{3.12}$$

$$g_2(t,\tau;\beta;\lambda) = t^{\lambda-1} N\left(\frac{x/\gamma+\tau}{t^{2\beta}};\beta;\lambda\right).\tag{3.13}$$

Proof. The validity of (i) is a direct consequence of [11, formulas (3.4) and (3.5)].
   To prove part (ii), we consider the Laplace transform

$$L[g_2(p,\tau;\beta;\lambda)] = \frac{1}{p^\lambda} e^{-(x/\gamma+\tau)p^{2\beta}}, \quad \lambda > 0.\tag{3.14}$$

According to the inversion formula for the Laplace transform,

$$g_2(t,\tau;t;\beta;\lambda) = \frac{1}{2\pi i} \int_{H_a} e^{pt}\left[\frac{1}{p^\lambda} e^{-(x/\gamma+\tau)p^{2\beta}}\right] dp.\tag{3.15}$$

Setting $\sigma = pt$ and $\xi = (x/\gamma+\tau)/t^{2\beta}$, we obtain

$$g_2(t,\tau;\beta;\lambda) = \frac{t^{\lambda-1}}{2\pi i} \int_{H_a} e^{\sigma-\xi\sigma^{2\beta}} \frac{d\sigma}{\sigma^\lambda}.\tag{3.16}$$

Hence, if $0 < 2\beta < 1$,

$$g_2(t,\tau;\beta;\lambda) = t^{\lambda-1} W(-\xi;-2\beta;\lambda) = t^{\lambda-1} N(\xi;\beta;\lambda).\tag{3.17}$$

In particular,

$$g_2\left(t,\tau;\frac{1}{2};\lambda\right) = \frac{t^{\lambda-1}}{2\pi i} \int_{H_a} e^{\sigma-\xi\sigma} \frac{d\sigma}{\sigma^\lambda} = \frac{1}{2\pi i} \int_{H_a} e^{pt}\left[\frac{1}{p^\lambda} e^{-(x/\gamma+\tau)p}\right] dp.\tag{3.18}$$

Referring to [4, formulas (3) and (27)], we conclude that

$$g_2\left(t,\tau;\frac{1}{2};\lambda\right) = \int_0^t \frac{s^{-\lambda-1}}{\Gamma(\lambda)} \delta\left(t - \frac{x}{\gamma} - \tau - s\right) ds,\tag{3.19}$$

where $\Gamma(\lambda)$ is the Euler gamma function and $\delta(t)$ is the Dirac delta function. It is also worth mentioning that if $\lambda = 1$,

$$g_2\left(t;\tau;\frac{1}{2},1\right) = H\left(t - \frac{x}{\gamma} - \tau\right),\tag{3.20}$$

where $H(t-a)$ is the Heaviside function.   □

4. Fractional Lauwerier formulation

We apply the Laplace transform method to prove the following main results.

THEOREM 4.1. *If $0 < \beta \leq 1/2$ and $\lambda > 0$, the solutions of the fractional Lauwerier formulation of the temperature field problem in oil strata (2.4), (2.5), and (2.6) are given by the integrals*

$$u(x,t) = \int_0^\infty e^{-(\alpha x/\gamma + \tau k)} [\varphi(t,\tau;\beta) * g_1(t,\tau,0;\beta)] I_0\left(2\sqrt{\frac{\alpha k x}{\gamma}}\,\tau\right) d\tau, \tag{4.1}$$

$$\Theta(x,z,t) = k \int_0^\infty e^{-(\alpha x/\gamma + \tau k)} [g_1(t,\tau;z;\beta) * g_2(t,\tau,\beta;1)] I_0\left(2\sqrt{\frac{\alpha k x}{\gamma}}\,\tau\right) d\tau, \tag{4.2}$$

where $g_1(t,\tau,z;\beta)$ and $g_2(t,\tau;\beta;1)$ are defined by (3.12) and (3.13), respectively, and

$$\varphi(t,\tau;\beta) = g_2(t,\tau;\beta;1-2\beta) + \mu g_2(t,\tau;\beta;1-\beta) + k g_2(t,\tau;\beta;1). \tag{4.3}$$

Proof. We denote

$$\bar{u}(x,p) = L[u(x,t)], \qquad \bar{\Theta}(x,z,p) = L[\Theta(x,z,t)], \tag{4.4}$$

where L is the Laplace transformation operator. Applying the Laplace transform to (2.4), (2.5), and (2.6), we obtain, in accordance with (2.2) and (2.6c),

$$p^{2\beta}\bar{\Theta}(x,z,p) = \frac{\partial^2 \bar{\Theta}(x,z,p)}{\partial z^2}, \quad 0 < x,\, z < \infty, \tag{4.5}$$

$$z = 0: p^{2\beta}\bar{u}(x,p) = -\gamma\frac{\partial \bar{u}(x,p)}{\partial x} - \alpha[\bar{u}(x,p) - \bar{\Theta}(x,z,p)], \quad 0 < x < \infty,$$

$$z = 0: p^{2\beta}\bar{u}(x,z,p) = \mu\frac{\partial \bar{\Theta}(x,z,p)}{\partial x} + k[\bar{u}(x,p) - \bar{\Theta}(x,z,p)], \quad 0 < x < \infty, \tag{4.6}$$

$$x = 0: \bar{u}(0,p) = \frac{1}{p}, \tag{4.7a}$$

$$\bar{u}, \Theta \longrightarrow 0 \quad \text{as } x^2 + z^2 \longrightarrow \infty. \tag{4.7b}$$

The solution of (4.5) which remains bounded as $z \to \infty$ is

$$\bar{\Theta}(x,z,p) = c(x,p)e^{-zp^\beta}, \tag{4.8}$$

where $c(x,p)$ is still unknown function. Substituting (4.8) into (4.6), we have

$$p^{2\beta}\bar{u}(x,p) = -\gamma\frac{\partial \bar{u}(x,p)}{\partial x} - \alpha[\bar{u}(x,p) - c(x,p)], \tag{4.9}$$

$$p^{2\beta}c(x,p) = -\mu p^\beta c(x,p) + k[\bar{u}(x,p) - c(x,p)]. \tag{4.10}$$

Solving (4.10) for $c(x,p)$, we get

$$c(x,p) = \frac{k\bar{u}(x,p)}{p^{2\beta} + \mu p^\beta + k}. \tag{4.11}$$

The substitution of this representation of $c(x,p)$ into (4.9) leads to the equation

$$\frac{\partial \bar{u}(x,p)}{\partial x} = -\frac{1}{\gamma}\left(p^{2\beta} + \alpha - \frac{\alpha k}{p^{2\beta} + \mu p^{\beta} + k}\right)\bar{u}(x,p). \tag{4.12}$$

The solution of this equation that conforms to (4.7a) is

$$\bar{u}(x,p) = \frac{1}{p}\exp\left[\frac{\alpha k x}{\gamma} \cdot \frac{1}{p^{2\beta} + \mu p^{\beta} + k} - \frac{(p^{2\beta} + \alpha)x}{\gamma}\right]. \tag{4.13}$$

Then obviously

$$c(x,p) = \frac{k}{p^{2\beta} + \mu p^{\beta} + k} \cdot \frac{1}{p}\exp\left[\frac{\alpha k x}{\gamma} \cdot \frac{1}{p^{2\beta} + \mu p^{\beta} + k} - \frac{(p^{2\beta} + \alpha)x}{\gamma}\right], \tag{4.14}$$

and from (4.8) it follows that

$$\Theta(x,z,p) = \frac{k}{p}\frac{e^{-zp^{\beta}}}{p^{2\beta} + \mu p^{\beta} + k}\exp\left[\frac{\alpha k x}{\gamma} \cdot \frac{1}{p^{2\beta} + \mu p^{\beta} + k} - \frac{(p^{2\beta} + \alpha)x}{\gamma}\right]. \tag{4.15}$$

To apply Theorem 3.1, we represent

$$\bar{u}(x,p) = F[q(p;\beta)]G(x,p;\beta), \tag{4.16}$$

where

$$F[q(p;\beta)] = \frac{1}{q(p;\beta)}\exp\left[\frac{\alpha k x}{\gamma} \cdot \frac{1}{q(p;\beta)}\right], \tag{4.17}$$

$$G(x,p;\beta) = \frac{p^{2\beta} + \mu p^{\beta} + k}{p}\exp\left[-\frac{x}{\gamma}(p^{2\beta} + \alpha)\right]. \tag{4.18}$$

The well-known formula [4, page 247, (2.4.82)] enables us to get

$$F(p) = L[f(t)] = L\left[I_0\left(2\sqrt{\frac{\alpha k x}{\gamma}t}\right)\right]. \tag{4.19}$$

Further, we consider the function

$$e^{-\tau q(p;\beta)}G(x,p;\beta) = \frac{p^{2\beta} + \mu p^{\beta} + k}{p}\exp\left[-\left(\frac{x}{\gamma} + \tau\right)p^{2\beta} - \tau \mu p^{\beta} - \left(\frac{\alpha x}{\gamma} + \tau k\right)\right]. \tag{4.20}$$

From Lemma 3.2 and the convolution theorem, we obtain

$$\frac{1}{p^{1-2\beta}}e^{-(x/\gamma+\tau)p^{2\beta}}e^{-\tau \mu p^{\beta}} = L[g_2(t,\tau;\beta;1-2\beta) * g_1(t,\tau,0;\beta)],$$
$$\frac{1}{p^{1-\beta}}e^{-(x/\gamma+\tau)p^{2\beta}}e^{-\tau \mu p^{\beta}} = L[g_2(t,\tau;\beta;1-\beta) * g_1(t,\tau,0;\beta)], \tag{4.21}$$
$$\frac{1}{p}e^{-(x/\gamma+\tau)p^{2\beta}}e^{-\tau \mu p^{\beta}} = L[g_2(t,\tau;\beta;1) * g_1(t,\tau,0;\beta)].$$

Taking into account (4.20) as well as the formulas (4.21), we can write

$$e^{-\tau q(p;\beta)}G(x,p;\beta) = L[g(t,\tau)], \tag{4.22}$$

where

$$g(t,\tau) = e^{-(\alpha x/\gamma + \tau k)}[\varphi(t,\tau;\beta) * g_1(t,\tau,0;\beta)]. \tag{4.23}$$

The application of Theorem 3.1 leads directly to the solution (4.1).

Following the same idea, we represent $\bar{\Theta}(x,z,p)$ as the product

$$\bar{\Theta}(x,z,p) = F[q(p;\beta)]G_1(x,p;\beta), \tag{4.24}$$

where $F[q(p,\beta)]$ is defined by (4.17) and

$$G_1(x,p;\beta) = \frac{k}{p}e^{-zp^\beta}e^{-(x/\gamma)p^{2\beta}}e^{-\alpha x/\gamma}. \tag{4.25}$$

Obviously

$$e^{-\tau q(p;\beta)}G_1(x,p;\beta) = ke^{-(\alpha x/\gamma + k\tau)}e^{-(z+\mu\tau)p^\beta}\frac{1}{p}e^{-(x/\gamma+\tau)p^{2\beta}}. \tag{4.26}$$

Then as a direct consequence of Lemma 3.2 and the convolution theorem, we obtain

$$e^{-\tau q(p;\beta)}G_1(x,p;\beta) = ke^{-(\alpha x/\gamma + k\tau)}[g_1(t,\tau,z,\beta) * g_2(t,\tau;\beta;1)]. \tag{4.27}$$

Then (4.24), (4.27), and Theorem 3.1 lead to the solution (4.2) and this completes the proof of the theorem. $\square$

COROLLARY 4.2. *For the particular case $\beta = 1/2$, the solution (4.1) reduces to (1.4).*

Proof. Because of (3.8), we evidently can write

$$g_1\left(t,\tau,0;\frac{1}{2}\right) = \frac{\tau\mu}{2\sqrt{\pi}t^{3/2}}e^{-(\tau\mu/2\sqrt{t})^2}. \tag{4.28}$$

According to (3.13) and [9, Section 17.13, formula (27)],

$$g_2\left(t,\tau;\frac{1}{2};0\right) = \frac{1}{t}N\left(\frac{x/\gamma+\tau}{t};\frac{1}{2};0\right)$$
$$= \frac{1}{2\pi i}\int_{H_a}e^{pt}[e^{-(x/\gamma+\tau)p}]dp = \delta\left(t-\frac{x}{\gamma}-\tau\right)H\left(t-\frac{x}{\gamma}-\tau\right). \tag{4.29}$$

Likewise from (3.13), [9, Section 17.13, formula (3)], and the t-shifting theorem for the Laplace transform it follows that

$$g_2\left(t,\tau;\frac{1}{2};\frac{1}{2}\right) = \frac{1}{t^{1/2}}N\left(\frac{x/\gamma+\tau}{t};\frac{1}{2};\frac{1}{2}\right)$$
$$= \frac{1}{2\pi i}\int_{H_a}e^{pt}\left[\frac{1}{p^{1/2}}e^{-(x/\gamma+\tau)p}\right]dp = \frac{H(t-x/\gamma-\tau)}{\sqrt{\pi}\sqrt{(t-x/\gamma-\tau)}}. \tag{4.30}$$

Taking into account (3.20), we obtain that

$$\varphi\left(t,\tau;\frac{1}{2}\right) = \left[\delta\left(t - \frac{x}{y} - \tau\right) + \mu\frac{1}{\sqrt{\pi}\sqrt{(t - x/y - \tau)}} + k\right]H\left(t - \frac{x}{y} - \tau\right). \qquad (4.31)$$

Formulas (4.28) and (4.31) yield that in the case $\beta = 1/2$ the convolution into the integrand of (4.1) takes the form

$$\varphi\left(t,\tau;\frac{1}{2}\right) * g_1(t,\tau,0;\beta) = \frac{\tau\mu H(t - x/y - \tau)}{2\sqrt{\pi}}\int_0^{t-x/y-\tau}\frac{1}{s^{3/2}}e^{-(\tau\mu/2\sqrt{s})^2}\delta\left(t - \frac{x}{y} - \tau - s\right)ds$$

$$+ \frac{\tau\mu^2 H(t - x/y - \tau)}{2\pi}\int_{x/y+\tau}^t\frac{1}{(t-s)^{3/2}}e^{-(\tau\mu/2\sqrt{t-s})^2}\frac{1}{\sqrt{(s - x/y - \tau)}}ds$$

$$+ \frac{k\tau\mu}{2\sqrt{\pi}}H\left(t - \frac{x}{y} - \tau\right)\int_{x/y+\tau}^t\frac{1}{(t-s)^{3/2}}e^{-(\tau\mu/2\sqrt{t-s})^2}ds. \qquad (4.32)$$

The third integral in (4.32) can be computed as

$$\frac{\tau\mu}{2}\int_{x/y+\tau}^t(t-s)^{3/2}e^{-(t\mu/2\sqrt{t-s})^2}ds = 2\int_0^{t-x/y-\tau}e^{-(\tau\mu/2\sqrt{t-x/y-\tau-s'})^2}d\left(\frac{\tau\mu}{2\sqrt{t - x/y - \tau - s'}}\right)$$

$$= 2\int_{(\tau\mu/2\sqrt{t-x/y-\tau})}^{\infty}e^{-w^2}dw = \sqrt{\pi}\operatorname{erfc}\left(\frac{\tau\mu}{2\sqrt{t - x/y - \tau}}\right). \qquad (4.33)$$

By using [9, Section 3.471, formula (3)], one can see that

$$\int_{x/y+\tau}^t(t-s)^{3/2}\left(s - \frac{x}{y} - \tau\right)^{-1/2}e^{-(t\mu/2\sqrt{t-s})^2}ds = \int_0^{t-x/y-\tau}x^{-3/2}\left(t - \frac{x}{y} - \tau - x\right)^{-1/2}e^{-(\tau\mu/2\sqrt{x})^2}dx$$

$$= \frac{2}{\tau\mu}\frac{\sqrt{\pi}}{\sqrt{(t - x/y - \tau)}}e^{-(\tau\mu/2\sqrt{t-x/y-\tau})^2}. \qquad (4.34)$$

For the first integral in (4.32), it is clear that

$$\int_0^{t-x/y-\tau}\frac{1}{s^{3/2}}e^{(\tau\mu/2\sqrt{s})^2}\delta\left(t - \frac{x}{y} - \tau - s\right)ds = \left(t - \frac{x}{y} - \tau\right)^{-3/2}e^{-(\tau\mu/2\sqrt{t-x/y-\tau})^2}. \qquad (4.35)$$

The substitution of these integrals by their expressions into (4.32) results in

$$
\varphi\left(t,\tau;\frac{1}{2}\right) * g_1(t;\tau,0;\beta)
$$

$$
= \left\{ \frac{\mu[2(t-x/\gamma)-\tau]}{2\sqrt{\pi}(t-x/\gamma-\tau)^{3/2}} e^{-(\tau\mu/2\sqrt{t-x/\gamma-\tau})^2} + k\,\mathrm{erfc}\left(\frac{\tau\mu}{2\sqrt{t-x/\gamma-\tau}}\right) \right\} \cdot H\left(t-\frac{x}{\gamma}-\tau\right);
$$

$$
\tag{4.36}
$$

that proves the statement.   □

COROLLARY 4.3. *For the particular case $\beta = 1/2$, the solution (4.2) reduces to (1.5).*

Proof. From (3.8) and (3.12), it follows that

$$
g_1\left(t,\tau,z;\frac{1}{2}\right) = \frac{z+\mu\tau}{2t^{3/2}} \cdot \frac{1}{\sqrt{\pi}} e^{-((z+\mu\tau)/2\sqrt{t})^2}.
\tag{4.37}
$$

Then according to (3.20), the convolution into the integrand of (4.2) takes the form

$$
g_1\left(t,\tau,z;\frac{1}{2}\right) * g_2\left(t,\tau;\frac{1}{2};1\right)
$$

$$
= \int_0^t \frac{z+\mu\tau}{2\sqrt{\pi}(t-s)^{3/2}} e^{-((z+\mu\tau)/2\sqrt{t-s})^2} H\left(s-\frac{x}{\gamma}-\tau\right) ds
$$

$$
= H\left(t-\frac{x}{\gamma}-\tau\right) \frac{1}{2\sqrt{\pi}} \int_{x/\gamma+\tau}^t \frac{z+\mu\tau}{(t-s)^{3/2}} e^{-((z+\mu\tau)/2\sqrt{t-s})^2} ds
$$

$$
= H\left(t-\frac{x}{\gamma}-\tau\right) \frac{1}{2\sqrt{\pi}} \int_0^{t-x/\gamma-\tau} \frac{z+\mu\tau}{(t-x/\gamma-\tau-\xi)^{3/2}} e^{-((z+\mu\tau)/2\sqrt{t-x/\gamma-\tau-\xi})^2} d\xi
$$

$$
\tag{4.38}
$$

$$
= H\left(t-\frac{x}{\gamma}-\tau\right) \frac{2}{\sqrt{\pi}} \int_0^{t-x/\gamma-\tau} e^{-((z+\mu\tau)/2\sqrt{t-x/\gamma-\tau-\xi})^2} d\left(\frac{z+\mu\tau}{2\sqrt{t-x/\gamma-\tau-\xi}}\right)
$$

$$
= H\left(t-\frac{x}{\gamma}-\tau\right) \frac{2}{\sqrt{\pi}} \int_{(z+\mu\tau)/2\sqrt{t-x/\gamma-\tau}}^\infty e^{-w^2} dw
$$

$$
= H\left(t-\frac{x}{\gamma}-\tau\right) \mathrm{erfc}\left(\frac{z+\mu\tau}{2\sqrt{t-x/\gamma-\tau}}\right),
$$

and proves the statement.   □

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