Small Parameter Method Application to the Energy Integral of an Autonomous Oscillating System

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Abstract. An asymptotic method applied directly to the energy integral of an autonomous oscillating system is used. Using a formal method, the formulas for finding the zero and first approximations are derived. Two cases of perturbing potentials, of the fourth and third degree, respectively, are considered. The solution in the first approximation is found as a power function of the solution in the zero approximation.

INTRODUCTION

One of the difficulties in finding the solution of a nonlinear ordinary differential equation in analytical form is that in most cases this could only be done approximately. A number of asymptotic methods have been developed, based on generally divergent series, which, however, provide a good approximation from the point of view of practice [1]. Poincaré [2] and Van der Paul [3] made great contributions to this field of mathematics. Poincaré developed techniques for finding a periodic solution by decomposing the solution into series by parameter characterizing the system, smaller than 1. Van der Paul's method is to find approximate solutions by averaging, using that the rate of change of the amplitude is small, that is, proportional to a small parameter. One of the best developed methods is that of Krylov - Bogolyubov - Mitropolski [4] (KBM), which summarizes the ideas of the two scientists. There are methods using the presence of a large parameter. One of the most popular is the Wentzel-Kramers-Brillouin method (WKB) [5].

In this work, an autonomous oscillating system for which the energy integral is known, is considered. The method of the small parameter in a way different from its classical application, is used. Using a formal method, the energy integral for finding an approximate solution in the order of the small parameter, is attacked directly. The calculations are limited to the first approximation, which is often of the greatest practical importance.

FORMAL METHOD FOR DECOMPOSITION OF THE ENERGY INTEGRAL BY THE POWERS OF THE SMALL PARAMETER

Consider a dynamic system whose energy integral can be written as:

$$E = \frac{\omega^2 x^2 + \dot{x}^2}{2} + \omega^2 V(\varepsilon, x), \tag{1}$$

where ε is a small parameter, ω^2 is a constant. Let:

$$V(0,x)=0, V(\varepsilon,0)=0. \quad (2)$$

Let the function $V(\varepsilon,x)$ be continuous and differentiable by n^{th} order with respect on both arguments. Consider new variable:

$$\Omega t=\tau, \frac{d}{dt}=\Omega\frac{d}{d\tau}, \Omega=const. \quad (3)$$

Then:

$$\mathcal{E}=\frac{E}{\omega^2}=\frac{x^2+\sigma^2 x'^2}{2}+V(\varepsilon,x), \sigma^2=\left(\frac{\Omega}{\omega}\right)^2. \quad (4)$$

The quantities that characterize the system into series according to the small parameter are decomposed:

$$x=x_0+\varepsilon x_1+\cdots+\ \varepsilon^n x_n+\cdots, \quad (5)$$
$$\sigma=\sigma_0+\varepsilon\sigma_1+\cdots+\ \varepsilon^n\sigma_n+\cdots, \sigma_0=1, \quad (6)$$
$$\mathcal{E}=\mathcal{E}_0+\varepsilon\mathcal{E}_1+\cdots+\ \varepsilon^n\mathcal{E}_n+\cdots. \quad (7)$$

The series are broken up to the n^{th} term. Asymptotic approximation to the solution is finding by these series.
Consider the decomposition of some arbitrary quantity y by the powers of the small parameter:

$$y=y_0+\varepsilon y_1+\cdots+\ \varepsilon^n y_n. \quad (8)$$

Then:

$$\begin{gathered} y|_{\varepsilon=0}=y_0, \\ \left.\frac{dy}{d\varepsilon}\right|_{\varepsilon=0}=y_\varepsilon|_{\varepsilon=0}=y_1, \\ \left.\frac{d^2y}{d\varepsilon^2}\right|_{\varepsilon=0}=y_{\varepsilon\varepsilon}|_{\varepsilon=0}=2y_{2,} \\ \dots\dots\dots\dots\dots\dots \\ \left.\frac{d^ny}{d\varepsilon^n}\right|_{\varepsilon=0}=y_{\varepsilon^n}|_{\varepsilon=0}=n!\,y_{n.} \end{gathered} \quad (9)$$

Thus, the relation between the functions in the expansions can be realized by differentiating the two sides of the equality in the equation for E by the small parameter, except for the zero approximation.
Let:

$$V(\varepsilon,x)=\varepsilon\,V(x). \quad (10)$$

The zero approximation is obtained directly by equating the small parameter to 0:

$$\mathcal{E}|_{\varepsilon=0}=\mathcal{E}_0=\frac{{x_0}^2+{x_0'}^2}{2}. \quad (11)$$

Differentiating the two sides of the equality of the integral $\mathcal{E}$ once and equating the small parameter to 0, the first approximation is obtained:

$$\left.\frac{d\mathcal{E}}{d\varepsilon}\right|_{\varepsilon=0}=\frac{2xx_\varepsilon+2\sigma\sigma_\varepsilon x'^2+2\sigma^2 x'x'_\varepsilon}{2}+\left.\frac{\partial\varepsilon V}{\partial\varepsilon}\right|_{\varepsilon=0}, \quad (12)$$
$$\mathcal{E}_1=\frac{2x_0x_1+2\sigma_0\sigma_1{x_0'}^2+2{\sigma_0}^2x_0'x_1'}{2}+V(x_0). \quad (13)$$

Similarly:
$$2\mathcal{E}_2=\frac{2{x_1}^2+4x_0x_2+2{\sigma_1}^2{x_0'}^2+4\sigma_0\sigma_2{x_0'}^2+4\sigma_0\sigma_1x_0'x_1'+2{\sigma_0}^2{x_1'}^2+4{\sigma_0}^2x_0'x_2'}{2}+\frac{\partial V(x_0)}{\partial x_0}x_1. \quad (14)$$

Finding other approximations is becoming more and more technically complex. In practice, due to its complexity, further solution of the problem without the use of specialized computer programs is impractical even for a second approximation.

USING THE METHOD FOR FINDING A SOLUTION IN CASE OF FOURTH DEGREE DISTURBANCE WITH RESPECT TO THE VARIABLE

In order to indicate a specific strategy for completing the task, the function $\varepsilon\, V(x)$ must be presented in a certain form. Let the perturbing potential be:

$$V(x) = -\frac{x^4}{4}. \tag{15}$$

The expression for $\mathcal{E}$ in this case is given by:

$$\frac{E}{\omega^2} = \mathcal{E} = \frac{x^2+\sigma^2 x'^2}{2} - \frac{\varepsilon x^4}{4}. \tag{16}$$

In addition, let the conditions apply:

$$x(0) = l, x'(0) = 0. \tag{17}$$

For the zero approximation:

$$\mathcal{E}_0 = \frac{{x_0}^2+{x_0}'^2}{2} = \frac{l^2}{2}. \tag{18}$$

The first approximation satisfies the conditions:

$$\mathcal{E}_1 = \frac{2x_0x_1+2\sigma_0\sigma_1{x_0}'^2+2{\sigma_0}^2{x_0}'{x_1}'}{2} - \frac{x^4}{4}. \tag{19}$$

Let the function x_1 satisfies the conditions:

$$x_1(0) = 0, {x_1}'(0) = 0. \tag{20}$$

In general, the function x_1 has the form:

$$x_1 = \frac{\lambda_1}{4}{x_0}^3 + \frac{\lambda_2}{4}{x_0}^2 l + \frac{\lambda_3}{4}x_0 l^2 + \frac{\lambda_4}{4}l^3. \tag{21}$$

In (21) $\lambda_i, i = 1,2,3,4$ are constants that will be determined later.

In order for the initial conditions (17) to be satisfied, the unknown constants $\lambda_i, i = 1,2,3,4$ must satisfy the condition:

$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 0. \tag{22}$$

From (18) it follows that:

$${x_0}'^2 = l^2 - {x_0}^2. \tag{23}$$

After substitution in the equation, for the first approximation, it gets:

$$\mathcal{E}_1 = \left(\frac{\lambda_1}{4} - \frac{3\lambda_1}{4} - \frac{1}{4}\right) {x_0}^4 + \left(\frac{\lambda_2}{4} l - \frac{\lambda_2}{4} l\right) {x_0}^3 + \left(\frac{3\lambda_1}{4} l^2 - \sigma_1\right) {x_0}^2 + \left(\frac{\lambda_4}{4} l^3 + \frac{\lambda_2}{4} l^3\right) x_0 + \sigma_1 l^2 + +\lambda_3 \frac{l^4}{4}. \tag{24}$$

For $\mathcal{E}_1$ to be a constant, all expressions in the brackets, which multiply $x_0$and its powers, must be equals to 0. Therefore:

$$\lambda_1 = -\frac{1}{2} \tag{25}$$

$$\sigma_1 = -\frac{3}{8} l^2 \tag{26}$$

For full determination of the desired feature, let:

$$\lambda_4 = \lambda_2 = 0, \tag{27}$$

$$\lambda_3 = \frac{1}{2}. \tag{28}$$

It can be immediately determined that:

$$\mathcal{E}_1 = -\frac{l^4}{4}, \tag{29}$$

as should be expected.

Referred to the physical side of the task, the oscillating motion in the first approximation is performed by law:

$$x = x_0 + \varepsilon \left[-\frac{{x_0}^3}{8} + \frac{l^2 x_0}{8}\right], \tag{30}$$

where $x_0 = l \cos \Omega t$ with period $\frac{2\pi}{\Omega}$, and:

$$\Omega = \omega \left(1 - \frac{3}{8} \varepsilon l^2\right). \tag{31}$$

USING THE METHOD FOR FINDING A SOLUTION IN CASE OF THIRD DEGREE DISTURBANCE WITH RESPECT TO THE VARIABLE

Let the perturbing potential be in the form:

$$V(x) = -\frac{x^3}{3}. \tag{32}$$

Let:

$$x(0) = l, x'(0) = 0. \tag{33}$$

The zero approximation has the form:

$$\mathcal{E}_0 = \frac{{x_0}^2 + {x_0'}^2}{2} = \frac{l^2}{2}. \tag{34}$$

The function x_1 has the form:

$$x_1 = \frac{\lambda_1}{3} {x_0}^2 + \frac{\lambda_2}{3} x_0 l + \frac{\lambda_3}{3} l^2. \tag{35}$$

Let again:

$$x_1(0) = 0, x_1{}'(0) = 0. \tag{36}$$

Condition (36) leads to the relation between the constants $\lambda_i, i = 1,2,3$ which must be determined:

$$\lambda_1 + \lambda_2 + \lambda_3 = 0. \tag{37}$$

Substituting the expression for x_1 in the equation for the first approximation (35), it turns out:

$$\mathcal{E}_1 = \left(\frac{\lambda_1}{3} - \frac{2\lambda_1}{3} - \frac{1}{3}\right) x_0{}^3 + \left(\frac{\lambda_3}{3} l^2 + \frac{2\lambda_1}{3} l^2\right) x_0 + \sigma_1 l^2 - \sigma_1 x_0{}^2 + \lambda_2 \frac{l^3}{3}. \tag{38}$$

All expressions in the brackets, which multiply $x_0$and its powers, must be equals to 0. Then:

$$\lambda_1 = -1, \tag{39}$$
$$\lambda_3 = 2, \tag{40}$$
$$\sigma_1 = 0. \tag{41}$$

From the relation (37), it gets that:

$$\lambda_2 = -1. \tag{42}$$

The correction in the first approximation to $\mathcal{E}$ is:

$$\mathcal{E}_1 = -\frac{l^3}{3}. \tag{43}$$

The solution of the problem in the first approximation is:

$$x_0 = l\cos \omega t, \tag{44}$$
$$x = x_0 + \varepsilon\left[-\frac{x_0{}^2}{3} - \frac{l x_0}{3} + \frac{2l^2}{3}\right]. \tag{45}$$

CONCLUSION

The technique used to find an approximate solution makes it possible to work with the energy integral, not with the differential equation, as in the classical approach. In sense, this complicates the task, but the application of a formal order decomposition technique simplifies the calculations, despite the complexity of the resulting expressions. The proposed way to finding higher approximations, through the power function of the solution in the zero approximation, leads to quick finding of the required functions. The hypothesis that this is appropriate in all cases where the disturbance is a power function of the variable can be said.

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