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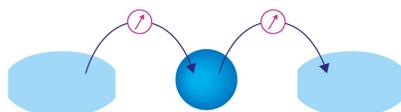
On the Well-posedness in Lorentz Spaces for the Inhomogeneous Heat Equation

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Abstract. We present new Strichartz estimates in Lorentz spaces for the solutions to the heat equation with inhomogeneous nonlinearity in the mass subcritical framework and space dimension $d \geq 1$. As an application we prove local and global well-posedness in the Strichartz-Lorentz space $L^q((0, T); L^{r,2}(\mathbb{R}^d))$, both in the focusing and the defocusing case, assuming the initial data are in the $L^2(\mathbb{R}^d)$ space.

INTRODUCTION

We consider the following Cauchy problem associated to the inhomogeneous nonlinear heat equation (INLH) for $d \geq 1$,

$$\begin{cases} \partial_t u - \Delta u + k|x|^{-b}|u|^{\alpha}u = 0, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d, \\ u(0, x) = f(x) \in L^2(\mathbb{R}^d), \end{cases} \tag{1.1}$$

where $k \in \mathbb{R}$, u is a real-valued function and Δ is the d-dimensional Laplace operator. Moreover $0 < b < \min\{2, d\}$ and α satisfies:

$$0 < \alpha \leq \alpha^*(d), \quad \alpha^*(d) = \frac{4 - 2b}{d}. \tag{1.2}$$

The solution u to (1.1) can be expressed with the following integral equation:

$$u(t, x) = e^{t\Delta}f(x) - k \int_0^t e^{(t-\tau)\Delta}|x|^{-b}|u(\tau)|^{\alpha}u(\tau)d\tau. \tag{1.3}$$

It satisfies two conservation laws (see for instance [31]),

$$\|u(t)\|_{L^2(\mathbb{R}^d)}^2 + \int_0^t E(u(\tau))\, d\tau \leq \|f\|_{L^2(\mathbb{R}^d)}^2, \quad k \geq 0, \tag{1.4}$$

$$\frac{d}{dt}E(u(t)) = -\|\partial_t u(t)\|_{L^2(\mathbb{R}^d)}^2, \tag{1.5}$$

with the energy defined as

$$E(u(t)) = \frac{1}{2} \int_{\mathbb{R}^d} |\nabla u(t, x)|^2 \, dx + \frac{k}{\alpha + 2} \int_{\mathbb{R}^d} |x|^{-b}|u(t, x)|^{\alpha+2}dx. \tag{1.6}$$

Because of these conservation laws, we can distinguish the defocusing case ($k > 0$), in which we can control the norm of the solution through the size of initial data, and the focusing case ($k < 0$), where we lack such control. In view of that, it is natural to investigate local (and global) existence of the solution to (1.1) in the $L^2(\mathbb{R}^d)$-based Strichartz-Lorentz spaces, a refined version of the classical Strichartz spaces.

In order to state our results we will need the following basic definition.

Definition 1. *We say that the pair (q, r) is* **admissible***, in short $(q, r) \in S_2$, if*

$$2 \leq q, r \leq \infty, \quad \frac{2}{q} + \frac{d}{r} = \frac{d}{2}, \quad (q, r, d) \neq (2, \infty, 2). \tag{1.7}$$

The subject of our main outstanding are the Strichartz estimates in Lorentz spaces, namely:

Theorem 1. *Let be $d \geq 1$. Then the following homogeneous estimate holds*

$$\left\| e^{t\Delta} f \right\|_{L^q((0,T);L^{r,2}(\mathbb{R}^d))} \leq C \left\| f \right\|_{L^2(\mathbb{R}^d)}, \tag{1.8}$$

when $(q, r) \in S_2$. Moreover, for any $(q, r), (\tilde{q}, \tilde{r}) \in S_2$, the following inhomogeneous estimate holds

$$\left\| \int_0^t e^{(t-\tau)\Delta} F(\tau, x) d\tau \right\|_{L^q((0,T);L^{r,2}(\mathbb{R}^d))} \leq C \left\| F \right\|_{L^{\tilde{q}'}((0,T);L^{\tilde{r}',2}(\mathbb{R}^d))}, \tag{1.9}$$

under the condition that

$$\frac{1}{q} + \frac{1}{\tilde{q}} = \frac{d}{2}\left(\frac{1}{\tilde{r}'} - \frac{1}{r}\right). \tag{1.10}$$

The Lorentz spaces and their properties will be displayed in a detailed way in the next section. By a straightforward application of the previous theorem, we get:

Theorem 2. *Assume $d \geq 1$, $0 < \alpha \leq \alpha^*(d)$. Then the problem (1.1) has a unique solution*

$$u(t, x) \in L^q([0, T); L^{r,2}(\mathbb{R}^d)) \tag{1.11}$$

for any $(q, r) \in S_2$ and initial data $f \in L^2(\mathbb{R}^d)$, with $T = T(\|f\|_{L^2(\mathbb{R}^d)}) > 0$. The solution can be extended globally if $\alpha = \alpha^*(d)$ and the initial data $f \in L^2(\mathbb{R}^d)$ is so that $\|f\|_{L_x^2} < \varepsilon$, for $\varepsilon > 0$ small enough or if $k > 0$ and $f \in \dot{H}^1(\mathbb{R}^d)$.

Remark 1. *We underline that the previous Theorems 1 and 2 remain valid once one replaces the free heat propagator $e^{t\Delta}$ by $e^{(a+ib)t\Delta}$, with a and b real parameters. This fact can open a discussion devoted to more general nonlinear equations based on the heat and Schrödinger equations. We mention here the work [23] for more details. We emphasize also that our theory is general and could be extended to other possible scenarios, such as general operators $e^{(a+ib)tH}$ with $H = -\Delta + V$, for suitable potential functions $V = V(x)$ and $H = (-\Delta)^{\frac{\beta}{2}}$, with $\beta > 0$. We postpone such analysis for forthcoming works for the aim of simplicity.*

The nonlinear heat equation (NLH), that is (1.1) with $b = 0$, has been deeply investigated in the past decades by several authors. For example, in [16] [32], [33] and [3] were studied local well-posedness in the Lebesgue spaces $L^q(\mathbb{R}^d)$ and in the energy space $H^1(\mathbb{R}^d)$. In this framework, we want to recall [14] (see also references therein), where existence of global $C(I; H^1(\mathbb{R}^4))$-solutions in the focusing, energy-critical setting is treated by exploiting the *concentration compactness* method established in [18] and [19]. On the other hand, there is a wide literature on the nonlinear Schrödinger equation (NLS) with inhomogeneous nonlinearities. We mention here [15], [9], [6] for the well-posedness and blow up phenomena, [7] for scattering in the defocusing case, [4], [8], [5] (and references therein) in the focusing one. One of the main tools utilized to shed light on the properties of the nonlinear evolution equations and often used in the context of both NLH and NLS are the Strichartz estimates. We cite [17] [12],[13] for the general theory and [21] and [34] for Strichartz-type estimates for the fractional nonlinear heat equation (see also references therein). In [22] and [24] we introduced for the first time in the literature the extended version of the Strichartz estimates for the NLH as well as illustrated its application for proving local and global well-posedness in

$H^\sigma(\mathbb{R}^d)$ for the pure-power NLH (both in focusing and defocusing case). This technique was applied also in [23] in order to treat the Ginzburg-Landau equation with complex pure-power nonlinearity in $H^\sigma(\mathbb{R}^d)$ and in [26] for the heat equation, perturbed with a time dependent potential. Motivated by the previous papers and [26], we exhibited in [25] these extended Strichartz estimates for the solution to (1.1), assuming the initial data are in $H^\sigma(\mathbb{R}^d)$. The main target of the current paper is to manage the existence and uniqueness in $L^2(\mathbb{R}^d)$ for the solution to the INLH (1.1) by new Strichartz estimates settled in Lorentz spaces. In such way we not only improve the linear theory contained in [34], but also simplify significantly the fixed point step of the already cited work [25]: the Lorentz spaces are perfectly fitted to deal with singular function and then to control the nonlinear term $|x|^{-b}|u|^\alpha u$ avoiding any perturbation argument.

NOTATIONS AND PRELIMINARIES

We indicate by $f \in L^2(\mathbb{R}^d) = L_x^r$, for $1 \le r < \infty$, if

$$\|f\|_{L_x^r}^r = \int_{\mathbb{R}^d} |f(x)|^r \, dx < \infty,$$

with obvious modification for $r = \infty$ and introduce

$$\dot{H}^1(\mathbb{R}^d) = \dot{H}_x^1 = (-\Delta)^{-\frac{1}{2}} L_x^2.$$

Given any Banach space X we define, for any $1 \le q < \infty$, the mixed space-time norms

$$\|f\|_{L_t^q X} = \left(\int_{\mathbb{R}^+} \|f(x)\|_X^q \, dt \right)^{1/q},$$

with obvious modification for $q = \infty$. We embrace the notation $L_I^q X$, when one restricts $t \in I$, with I being a general time interval. Conjugate indices will be denoted with a prime throughout this work. For any two positive real numbers a, b, we write $a \lesssim b$ to express $a \le Cb$, with $C > 0$ and we unfold the constant only when it is essential.

Lorentz Spaces

In this section we recall some properties of the Lorentz spaces. For any measurable function ϕ in $\mathbb{R}^d$, we denote by $\mu_\phi = \mu_\phi(\lambda)$ the distribution function of ϕ, that is,

$$\mu_\phi(\lambda) = |\{x : |\phi(x)| > \lambda\}|,$$

for $\lambda > 0$. We define the decreasing rearrangement ϕ^* of ϕ by

$$\phi^*(t) = \inf\{\lambda > 0 : \mu_\phi(\lambda) \le t\},$$

for $t > 0$. Then, for any $1 \le r \le \infty$ and $1 \le \sigma \le \infty$, we introduce the Lorentz space $L^{r,\sigma}(\mathbb{R}^d)$ by

$$L^{r,\sigma}(\mathbb{R}^d) = L_x^{r,\sigma} = \left\{ \phi : \phi \text{ is measurable on } \mathbb{R}^d, \|\phi\|_{L^{r,\sigma}} < \infty \right\},$$

where

$$\|\phi\|_{L_x^{r,\sigma}} = \begin{cases} \left(\int_0^\infty \left(t^{1/r} \phi^*(t) \right)^\sigma \frac{dt}{t} \right)^{1/\sigma}, & \text{if } 1 \le \sigma < \infty, \\ \sup_{t>0} t^{1/r} \phi^*(t), & \text{if } \sigma = \infty. \end{cases}$$

The Lorentz spaces have the following properties:

$$L_x^{r,r} = L_x^r, \quad \text{if } 1 \le r \le \infty, \tag{2.1}$$

$$L_x^{r,\sigma_1} \subset L_x^{r,\sigma_2}, \quad \text{if } 1 \le r < \infty \text{ and } 1 \le \sigma_1 \le \sigma_2 \le \infty, \tag{2.2}$$

$$(L_x^{r,\sigma})' = L_x^{r',\sigma'}, \quad \text{if } 1 \le r, \sigma \le \infty.$$

Here $(L_x^{r,\sigma})'$ is the associated dual space of $L_x^{r,\sigma}$ (see for example [1], Chapter 4). We have also that for $0 < m < \infty$, the space weak $L^m(\mathbb{R}^d)$, in short $L^{(m,\infty)}(\mathbb{R}^d)$, is defined as the set of all measurable functions ϕ on $\mathbb{R}^d$ such that

$$\|f\|_{L^{(m,\infty)}(\mathbb{R}^d)} = \inf\left\{ C > 0 : \mu_\phi(\lambda) \leq \frac{C^m}{\lambda^m}, \quad \text{for all} \quad \lambda > 0 \right\}$$

is finite. The space weak $L^\infty(\mathbb{R}^d)$ is by definition $L^\infty(\mathbb{R}^d)$. Moreover $L^{(m,\infty)}(\mathbb{R}^d) = L^{m,\infty}(\mathbb{R}^d)$ for $1 \leq m \leq \infty$. Operators that map $L^m(\mathbb{R}^{d_1})$ to $L^n(\mathbb{R}^{d_2})$, with $d_1, d_2 \geq 1$, are called of strong type (m,n) and operators that map $L^m(\mathbb{R}^{d_1})$ to $L^{n,\infty}(\mathbb{R}^{d_2})$ are called of weak type (m,n).

Let us state also a fundamental Marcinkiewicz type interpolation result in the Lorentz spaces (see [1], Theorem 1.12, Chapter 5, [2], Theorem 5.3.1 and also [20]). We set

$$(m,n,\sigma,\theta) \in \Lambda := \left\{\ 1 \leq m \leq n \leq \infty,\ \sigma, \theta \in [1, \infty]\ \right\}.$$

Notice that the structure of the set Λ is strongly connected with the Lorentz Spaces formerly introduced. This forces to some extra constraints on the quadruples (m,n,σ,θ), that is: $\sigma = 1$, if $m = 1$; $\sigma = \infty$, if $m = \infty$; $\theta = 1$, if $n = 1$; $\theta = \infty$, if $n = \infty$. Moreover $\sigma \leq \theta$, if $m = n$. Then we have:

Proposition 1. *Let $(m_0, n_0, \sigma_0, \theta_0), (m_1, n_1, \sigma_1, \theta_1) \in \Lambda$. For $0 < \eta < 1$, set*

$$\frac{1}{m} = \frac{1-\eta}{m_0} + \frac{\eta}{m_1}, \quad \frac{1}{n} = \frac{1-\eta}{n_0} + \frac{\eta}{n_1}.$$

Let T be a bounded linear operator from $L_x^{m_0,\sigma_0}$ to $L_x^{n_0,\theta_0}$ and from $L_x^{m_1,\sigma_1}$ to $L_x^{n_1,\theta_1}$, and define

$$M_0 = \sup_{f \in L_x^{m_0,\sigma_0} \setminus \{0\}} \frac{\|Tf\|_{L_x^{n_0,\theta_0}}}{\|f\|_{L_x^{m_0,\sigma_0}}}, \quad M_1 = \sup_{f \in L_x^{m_1,\sigma_1} \setminus \{0\}} \frac{\|Tf\|_{L_x^{n_1,\theta_1}}}{\|f\|_{L_x^{m_1,\sigma_1}}}.$$

Then the following holds:

i) *If $m_0 \neq m_1$ and $n_0 \neq n_1$, then*

$$\|Tf\|_{L_x^{n,\sigma}} \leq M_0^{1-\eta} M_1^\eta \|f\|_{L_x^{m,\sigma}}, \quad \text{for every } 1 \leq \sigma \leq \infty. \tag{2.3}$$

ii) *If $m_0 \neq m_1, n_0 = n_1$, and $\theta_0 = \theta_1 = \theta$, then*

$$\|Tf\|_{L_x^{n,\theta}} \leq M_0^{1-\eta} M_1^\eta \|f\|_{L_x^{m,\sigma}}, \quad \text{for every } 1 \leq \sigma \leq \infty.$$

iii) *If $m_0 = m_1, n_0 = n_1$, and $\theta_0 = \theta_1 = \theta$, then*

$$\|Tf\|_{L_x^{n,\theta}} \leq M_0^{1-\eta} M_1^\eta \|f\|_{L_x^{m,\sigma}}, \quad \text{for } \frac{1}{\sigma} = \frac{1-\eta}{\sigma_0} + \frac{\eta}{\sigma_1}.$$

One can see that the function $|x|^{-b}$ belongs to $L^{\frac{d}{b},\infty}$ and $\left\||x|^{-b}\right\|_{L^{\frac{d}{b},\infty}} = |B_1(0)|^{\frac{b}{d}}$ where $|B_1(0)|$ is the volume of the unit ball in $\mathbb{R}^d$, but it does not belong to any Lebesgue space. Furthermore, we have the following Hölder's inequality in Lorentz spaces (see [27]).

Proposition 2 (Hölder's inequality). *Let $1 < r, r_1, r_2 < \infty$ and $1 \leq \sigma, \sigma_1, \sigma_2 \leq \infty$ be such that*

$$\frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{r}, \quad \frac{1}{\sigma_1} + \frac{1}{\sigma_2} \geq \frac{1}{\sigma}.$$

Then for any $f \in L^{r_1,\sigma_1}$ and $g \in L^{r_2,\sigma_2}$, it holds that

$$\|fg\|_{L^{q,\sigma}} \lesssim \|f\|_{L^{r_1,\sigma_1}} \|g\|_{L^{r_2,\sigma_2}}. \tag{2.4}$$

PROOF OF THEOREM 1

We start this section by proving an ancillary result associated to the heat semigroup $e^{t\Delta}$. That is:

Lemma 1. *Let* $(m, n, \sigma, \theta) \in \Lambda$. *Then, for any* $T > 0$,

$$\left\| e^{t\Delta} f \right\|_{L_x^{n,\sigma}} \lesssim t^{-\frac{d}{2}\left(\frac{1}{m}-\frac{1}{n}\right)} \|f\|_{L_x^{m,\sigma}} \tag{3.1}$$

for all $t \in (0, T)$. *Furthermore,*

$$\left\| e^{t\Delta} g \right\|_{L_x^{m',\sigma'}} \lesssim t^{-\frac{d}{2}\left(\frac{1}{m}-\frac{1}{n}\right)} \|g\|_{L_x^{n',\sigma}} \tag{3.2}$$

for all $t > 0$, *where* m', n' *and* σ' *are the Hölder conjugate numbers of* $m, n,$ *and* σ, *respectively.*

Proof. The operator $e^{t\Delta}$ obeys general decay estimates, since the heat kernel $e^{-t|\xi|^2}$, given here via Fourier transform in terms of its symbol, has a strong smoothing property for $t > 0$. More precisely, one has the following

$$\left\| e^{t\Delta} f \right\|_{L_x^p} \lesssim t^{-\frac{d}{2}\left(\frac{1}{r}-\frac{1}{p}\right)} \|f\|_{L_x^r}, \quad 1 \le r \le p \le \infty. \tag{3.3}$$

This inequality implies then, for $t > 0$,

$$\left\| e^{t\Delta} f \right\|_{L_x^r} \lesssim \|f\|_{L_x^r}, \tag{3.4}$$

with $r > 1$, and a decay estimate,

$$\left\| e^{t\Delta} f \right\|_{L_x^\infty} \lesssim t^{-\frac{d}{2}} \|f\|_{L_x^1}. \tag{3.5}$$

By the property (2.1), the above estimates become

$$\left\| e^{t\Delta} f \right\|_{L_x^{r,r}} \lesssim \|f\|_{L_x^{r,r}} \tag{3.6}$$

and

$$\left\| e^{t\Delta} f \right\|_{L_x^{\infty,\infty}} \lesssim t^{-\frac{d}{2}} \|f\|_{L_x^{1,1}}. \tag{3.7}$$

We observe at this point that $(1, \infty, 1, \infty)$ and (r, r, r, r) are both in Λ. Then we can use the Marcinkiewicz interpolation Proposition 1 with

$$\frac{1}{m} = 1 - \eta + \frac{\eta}{r}, \quad \frac{1}{n} = \frac{\eta}{r},$$

$$M_0 = t^{-\frac{d}{2}}, \quad M_1 = 1,$$

obtaining readily the bound (3.1). Finally, we get also

$$\left(e^{t\Delta} f, g \right)_{L_x^2} = \int_{\mathbb{R}^d} \left[e^{t\Delta} f \right](x) g(x)\, dx = \int_{\mathbb{R}^d} f(x) \left[e^{t\Delta} g \right](x)\, dx,$$

with $t > 0$, for all $f \in L_x^{m,\sigma}$ and $g \in L_x^{n',\sigma'}$. This fact, in combination with (3.1), implies (3.2) and the proof of the lemma is finished. $\qquad \square$

Proof of Theorem 1. We will focus first on showing (1.8). We apply the classical Marcinkiewicz interpolation theorem (see [28], Appendix). The idea of the proof is essentially the same as in [11] and [33]. Consider the map $\mathcal{U}$ defined by $\mathcal{U}f = \left\| e^{t\Delta} f \right\|_{L_x^{r,2}}$ from $L^{r,2}$ to functions on $[0, T)$. The inequality (3.1) in Lemma 1, with $\sigma = 2$, $m = 2$ and $n = r$, shows that $\mathcal{U}$ is of weak type (r, q), where $2/q = (1/2 - 1/r)d$ and with $1 < q$. Moreover $\mathcal{U}$ is sub-additive and of weak type (r, ∞). If $2 \le q$, the interpolation theorem can be applied. So $\mathcal{U}$ is of strong type $(2, q_1), 2 < q_1$ with $2/q_1 = (1/2 - 1/r)d$, which then gives the desired result. Notice that our proof does not include the endpoint $(2, 2d/(d-2))$, for $d \ge 3$.

Let us consider now (1.8). It follows from Lemma 1 that

$$\left\| e^{(t-\tau)\Delta} F(\tau, x) \right\|_{L_x^{r,2}} \lesssim |t - \tau|^{-\frac{d}{2}\left(\frac{1}{\bar{r}}-\frac{1}{r}\right)} \|F(\tau, x)\|_{L_x^{\bar{r}',2}},$$

for $1 \le \tilde{r}' < r \le \infty$ and any $\tau < t$. Then the Hardy-Littlewood-Sobolev inequality enhances to

$$\left\| \int_0^t e^{(t-\tau)\Delta} F(\tau, x) d\tau \right\|_{L_t^q L_x^{r,2}} \lesssim \left\| \int_0^t \left\| e^{(t-\tau)\Delta} F(\tau, x) \right\|_{L_x^{r,2}} d\tau \right\|_{L_t^q} \tag{3.8}$$

$$\lesssim \left\| \int_0^t |t - \tau|^{-\frac{n}{2}\left(\frac{1}{\tilde{r}'} - \frac{1}{r}\right)} \left\| F(\tau, x) \right\|_{L_x^{\tilde{r}',2}} d\tau \right\|_{L_t^q} \tag{3.9}$$

$$\lesssim \left\| F(\tau, x) \right\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}',2}},$$

with $(q, r), (\tilde{q}, \tilde{r}) \in S_2$ satisfying identity (1.10). This completes the proof of the theorem . $\qquad\square$

We can summarize, then, the results achieved so far as follows.

Proposition 3. *Let $d \ge 1$. Then one has fulfilled the following estimate*

$$\left\| e^{t\Delta} f \right\|_{L_t^q L_x^{r,2}} + \left\| \int_0^t e^{(t-\tau)\Delta} F(\tau) d\tau \right\|_{L_t^q L_x^{r,2}} \lesssim \|f\|_{L_x^2} + \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}',2}}, \tag{3.10}$$

assuming that (q, r) and $(\tilde{q}, \tilde{r})$ are S_2 pairs. Moreover, by exploiting the properties (2.1) and (2.2) we get the classical Strichartz estimates for the free heat propagator

$$\left\| e^{t\Delta} f \right\|_{L_t^q L_x^r} + \left\| \int_0^t e^{(t-\tau)\Delta} F(\tau) d\tau \right\|_{L_t^q L_x^r} \lesssim \|f\|_{L_x^2} + \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}. \tag{3.11}$$

PROOF OF THEOREM 2

We begin this section with a key lemma, which is compulsory for the proof of Theorem 2. Inspired by [30] (see also [29]), we prove the existence of quadruples $(q, r, \tilde{q}, \tilde{r}) \in S_2 \times S_2$, fulfilling all conditions stated in Theorem 1. This result is contained in:

Lemma 2. *For any $d \ge 1, k \in \mathbb{R}$, $0 < b < min\{2, d\}$ and α satisfying the assumption (1.2), there exist S_2 pairs (q, r) and $(\tilde{q}, \tilde{r})$, such that*

$$\frac{1}{\tilde{r}'} = \frac{\alpha + 1}{r} + \frac{b}{d}, \quad \frac{1}{\tilde{q}'} \ge \frac{\alpha + 1}{q}. \tag{4.1}$$

Proof.   *Case $d \ge 3$:* The conditions on $(q, r, \tilde{q}, \tilde{r}) \in S_2 \times S_2$ can be summarized as follows:

$$0 < \frac{1}{r}, \frac{1}{\tilde{r}}, \frac{1}{q}, \frac{1}{\tilde{q}} < \frac{1}{2}, \tag{4.2a-d}$$

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2}, \quad \frac{2}{\tilde{q}} + \frac{d}{\tilde{r}} = \frac{d}{2}, \tag{4.2e,f}$$

$$\frac{1}{\tilde{r}'} = \frac{\alpha + 1}{r} + \frac{b}{d}, \quad \frac{1}{\tilde{q}'} \ge \frac{\alpha + 1}{q}. \tag{4.2g,h}$$

We start by rewriting them with help of the identities

$$\frac{1}{\tilde{r}} = 1 - \frac{\alpha + 1}{r} - \frac{b}{d},$$

$$\frac{1}{q} = \frac{d}{4} - \frac{d}{2r},$$

$$\frac{1}{\tilde{q}} = \frac{d}{4} - \frac{d}{2\tilde{r}} = \frac{d(\alpha + 1)}{2r} + \frac{b}{2} - \frac{d}{4}.$$

We can see that the inequality $1/\tilde{q}' \geq (\alpha + 1)/q$ is always satisfied, since it simplifies as:

$$1 - \frac{d(\alpha + 1)}{2r} - \frac{b}{2} + \frac{d}{4} \geq \frac{d(\alpha + 1)}{4} - \frac{d(\alpha + 1)}{2r} \Rightarrow \alpha \leq \frac{4 - 2b}{d}. \tag{4.3h}$$

Expressing the remaining inequalities in terms of $1/r$ yields

$$0 < \frac{1}{r} < \frac{1}{2}, \tag{4.4a}$$

$$0 < 1 - \frac{\alpha + 1}{r} - \frac{b}{d} < \frac{1}{2} \Rightarrow \frac{d - 2b}{2d(\alpha + 1)} < \frac{1}{r} < \frac{d - b}{d(\alpha + 1)}, \tag{4.4b}$$

$$0 < \frac{d}{4} - \frac{d}{2r} < \frac{1}{2} \Rightarrow \frac{d - 2}{2d} < \frac{1}{r} < \frac{1}{2}, \tag{4.4c}$$

$$0 < \frac{d(\alpha + 1)}{2r} + \frac{b}{2} - \frac{d}{4} < \frac{1}{2} \Rightarrow \frac{d - 2b}{2d(\alpha + 1)} < \frac{1}{r} < \frac{2 + d - 2b}{2d(\alpha + 1)}, \tag{4.4d}$$

which in turn leads to the following chain of bounds

$$\rho_{(d;\alpha;b)}^{(1)} = \max\left\{0, \frac{d - 2b}{2d(\alpha + 1)}, \frac{d - 2}{2d}\right\} < \frac{1}{r} < \min\left\{\frac{1}{2}, \frac{d - b}{d(\alpha + 1)}, \frac{2 + d - 2b}{2d(\alpha + 1)}\right\} = \rho_{(d;\alpha;b)}^{(2)}. \tag{4.5}$$

Since we are assuming $d \geq 3$, we have $0 < b < 2$, so that the following,

$$0 < \frac{d - 2}{2d},$$

$$\frac{d - b}{d(\alpha + 1)} > \frac{2 + d - 2b}{2d(\alpha + 1)},$$

are trivial. We further simplify (4.5) and it is equivalent to

$$\rho_{(d;\alpha;b)}^{(1)} = \max\left\{\frac{d - 2b}{2d(\alpha + 1)}, \frac{d - 2}{2d}\right\} < \frac{1}{r} < \min\left\{\frac{1}{2}, \frac{2 + d - 2b}{2d(\alpha + 1)}\right\} = \rho_{(d;\alpha;b)}^{(2)}.$$

The last condition will be fulfilled once the following relations are satisfied

$$\frac{d - 2b}{2d(\alpha + 1)} < \frac{1}{2}, \tag{4.6}$$

$$\frac{d - 2b}{2d(\alpha + 1)} < \frac{2 + d - 2b}{2d(\alpha + 1)}, \tag{4.7}$$

$$\frac{d - 2}{2d} < \frac{1}{2}, \tag{4.8}$$

$$\frac{d - 2}{2d} < \frac{2 + d - 2b}{2d(\alpha + 1)}. \tag{4.9}$$

The second and the third relations are self-evident. The first inequality simplifies to

$$-\frac{2b}{d} < 0 < \alpha,$$

which is always true, according to the choice of α. Lastly, inequality (4.9) can be rewritten like

$$(d - 2)\alpha + d - 2 < 2 - 2b + d,$$

or equivalently

$$\alpha < \frac{4 - 2b}{d - 2},$$

which is again always fulfilled.

Case d = 2: It is important to remind that for $d = 2$ we have the following restriction $0 < b < 2$ and therefore (4.5) can be narrowed down to:

$$\rho_{(2;\alpha;b)}^{(1)} = \max\left\{0, \frac{1-b}{2(\alpha+1)}\right\} < \frac{1}{r} < \min\left\{\frac{1}{2}, \frac{2-b}{2(\alpha+1)}\right\} = \rho_{(2;\alpha;b)}^{(2)}.$$

Then we note that all four inequalities can be easily verified as above.

Case d = 1: We underline that for $d = 1$ alongside with the condition $0 < b < 1$, we get as a consequence of Definition 1, that $0 < 1/r < 1/2$ and $0 < 1/q < 1/4$. Therefore (4.5) turns into:

$$\rho_{(1;\alpha;b)}^{(1)} = \max\left\{0, \frac{1-2b}{2(\alpha+1)}\right\} < \frac{1}{r} < \min\left\{\frac{1}{2}, \frac{2-2b}{2(\alpha+1)}\right\} = \rho_{(1;\alpha;b)}^{(2)}.$$

Proceeding as before, one can check again that the previous set is non-empty for any $\alpha > 0$. This completes the proof of Lemma 2. □

Consequently, we present the proof of the main Theorem 2.

Proof of Theorem 2. We set $I = (0, T)$ and introduce the spaces

$$\|w\|_{X_I} = \sup_{(q,r)\in S_2}\left\{\|w\|_{L_I^q L_x^{r,2}}\right\} \tag{4.10}$$

and

$$\|w\|_{X_I'} = \inf_{(\tilde{q},\tilde{r})\in S_2}\left\{\|w\|_{L_I^{\tilde{q}'} L_x^{\tilde{r}',2}}\right\}. \tag{4.11}$$

We will perform now a contraction argument. Namely, let be defined the integral operator associated to (1.1), i.e. to (1.3),

$$\mathcal{T}_f(u) = e^{t\Delta}f - k\int_0^t e^{(t-\tau)\Delta}(|x|^{-b}|u|^\alpha u)(\tau, x)\, d\tau.$$

One needs to show that for any (q, r) as in Lemma 2 and any $t \in I$, if $f \in L_x^2$, there exist a $T = T(\|f\|_{L_x^2}) > 0$ and a (unique) $u(t, x) \in X_I$, satisfying the property

$$\mathcal{T}_f(u(t)) = u(t). \tag{4.12}$$

For simplicity, we split the proof in three different steps.

Step One: For any $f \in L_x^2$, there exist $T = T(\|f\|_{L_x^2}) > 0$ and $R = R(\|f\|_{L_x^2}) > 0$, such that $\mathcal{T}_f(B_{X_I}(0, R)) \subset B_{X_I}(0, R)$, for any $T' < T$.

By (3.10) in Proposition 3, combined with (4.10) and (4.11) we have

$$\left\|\mathcal{T}_f u\right\|_{X_I} \lesssim \|f\|_{L_x^2} + \left\||x|^{-b}|u|^\alpha u\right\|_{X_I'}. \tag{4.13}$$

Thus, by Hölder's inequality (2.4) and picking up (q, r), $(\tilde{q}, \tilde{r})$ as in Lemma 2 so that (4.1) is satisfied with the strict inequality, the nonlinear term on the r.h.s. of (4.13) can be bounded like,

$$\left\||x|^{-b}|u|^\alpha u\right\|_{X_I'} \lesssim \left\||x|^{-b}|u|^\alpha u\right\|_{L_I^{\tilde{q}'} L_x^{\tilde{r}',2}} \lesssim \left\||x|^{-b}\right\|_{L_x^{\frac{d}{b},\infty}} \left\||u|^{\alpha+1}\right\|_{L_x^{r,2}}\bigg\|_{L_I^{\tilde{q}'}} \lesssim T^{\kappa(\alpha)} \|u\|_{L_I^q L_x^{r,2}}^{\alpha+1} \lesssim T^{\kappa(\alpha)} \|u\|_{X_I}^{\alpha+1}, \tag{4.14}$$

for some $\kappa(\alpha) > 0$. By (4.13) and (4.14) we attain then

$$\left\|\mathcal{T}_f u\right\|_{X_I} \lesssim \|f\|_{L_x^2} + T^{\kappa(\alpha)} \|u\|_{X_I}^{\alpha+1}. \tag{4.15}$$

Step Two: Let $T, R > 0$ be as in the previous step, then, there exists $\overline{T} = \overline{T}(\|f\|_{L_x^2}) < T$, such that $\mathcal{T}_f$ is a contraction on $B_{X_{\overline{T}}}(0, R)$, equipped with the norm $\|\cdot\|_{X_{\overline{T}}}$.

Given any $v_1, v_2 \in B_{X_I}(0, R)$ we achieve, by an use of (3.10), the following:

$$\left\| \mathcal{T}_f v_1 - \mathcal{T}_f v_2 \right\|_{X_I} \lesssim \left\| |x|^{-b} \left(v_1 |v_1|^\alpha - v_2 |v_2|^\alpha \right) \right\|_{L_I^{\tilde{q}'} L_x^{\tilde{r}',2}} .$$

Then, proceeding as above, one gets finally

$$\left\| \mathcal{T}_f v_1 - \mathcal{T}_f v_2 \right\|_{X_{\bar{I}}} \lesssim \overline{T}^{\kappa(\alpha)} \left(\|v_1\|_{X_{\bar{I}}}^\alpha + \|v_2\|_{X_{\bar{I}}}^\alpha \right) \|v_1 - v_2\|_{X_{\bar{I}}},$$

with $\kappa(\alpha)$ as above.

Step Three: The solution exists and is unique in $X_{\bar{I}}$, where $\bar{I}$ is as in the previous step.
We are in position to show existence and uniqueness of the solution by applying the contraction principle to the map $\mathcal{T}_f$ defined on the complete metric space $B_{X_{\bar{I}}}(0, R)$, endowed with the topology induced by $\|\cdot\|_{X_{\bar{I}}}$.

L_x^2-*critical case.* The previous lemma remains valid also in the L_x^2−critical case, that is if we select $\alpha = \alpha^*(d) = (4 - 2b)/d$. In fact, arguing as in the previous steps we infer that the chain of inequalities (4.14) can be replaced by

$$\left\| |x|^{-b} |u|^\alpha u \right\|_{X_I'} \lesssim \left\| |x|^{-b} |u|^\alpha u \right\|_{L_I^{\tilde{q}'} L_x^{\tilde{r}',2}} \lesssim \left\| |x|^{-b} \right\|_{L_x^{\frac{d}{b},\infty}} \left\| |u|^{\alpha+1} \right\|_{L_x^{r,2}} \Big\|_{L_I^{\tilde{q}'}} \lesssim \|u\|_{L_I^q L_x^{r,2}}^{\alpha+1} \lesssim \|u\|_{X_I}^{\alpha+1}, \tag{4.16}$$

with (q, r), $(\tilde{q}, \tilde{r})$ given as in Lemma 2, so that (4.1) is provided now with the equality. The remaining part of the proof can be deduced similarly to the one we carried out above. Thus, the details will be omitted.

Global well-posedness. One can pick $I = [0, \infty)$ in the following frames:

i)   If $\alpha = \alpha^*(d)$. In this critical case, by (4.13) and (4.16), one needs to take $\|f\|_{L_x^2} < \varepsilon$, with $\varepsilon > 0$ suitable small.

ii) If we impose the extra conditions $k > 0$ and $f \in \dot{H}_x^1$. In fact we have that the conservation laws (1.4) and (1.5), which read as

$$E(u(t)) + \int_0^t \int_{\mathbb{R}^d} |\partial_t u|^2 \, dx dt = E(f),$$

are both satisfied with $E(u(t)) \geq 0$ for all $t \geq 0$. This fact allows to extend the solution globally w.r.t. the time variable.

The proof of the theorem is now accomplished. $\qquad\qquad\square$

REFERENCES

[1]   C. Bennett, R. Sharpley, "Interpolation of Operators", Pure and Applied Mathematics, vol. 129, Academic Press, Inc., Boston, MA, 1988.

[2]   J. Bergh, J. Löfström, "Interpolation Spaces", Springer, Berlin–Heidelberg–New York, 1976.

[3]   H. Brezis and T. Cazenave, "A nonlinear heat equation with singular initial data", J. d'Anal. Math. **68**, 277–304 (1996).

[4]   L. Campos, "Scattering of radial solutions to the inhomogeneous nonlinear Schrödinger equation", Nonlinear Anal. **202**, 112–118 (2021).

[5]   M. Cardoso, L. G. Farah, C. M. Guzmán, J. Murphy, "Scattering below the ground state for the intercritical non-radial inhomogeneous NLS", e-print arXiv:2007.06165 [math.AP] (2020).

[6]   M. Cardoso, L. G. Farah, C. M. Guzmán, "On well-posedness and concentration of blow-up solutions for the intercritical inhomogeneous NLS equation", e-print arXiv:2004.06706 [math.AP] (2020).

[7]   V. D. Dinh, "Energy scattering for a class of inhomogeneous nonlinear Schrödinger equation in two dimensions", Journal of Hyperbolic Differential Equations, **18(1)**, 1–28 (2021).

[8]   V. D. Dinh, "Energy scattering for a class of the defocusing inhomogeneous non- linear Schrödinger equation", Journal of Evolution Equations **19**, 411–434 (2019).

[9]   L. G. Farah, "Global well-posedness and blow-up on the energy space for the inhomogeneous nonlinear Schrödinger equation", Journal of Evolution Equations **16**, 193–208 (2016).

[10]   D. Foschi, "Inhomogeneous Strichartz estimates", J. Hyper. Differential Equations **02(1)**, 1–24 (2005).

[11]   Y. Giga, "Solutions for semilinear parabolic equations in L^p and regularity of weak solutions of the Navier-Stokes system", J. Differential Equations **62**, 186–212 (1986) .

[12]   J. Ginibre and G. Velo, "Smoothing properties and retarded estimates for some dispersive evolution equations", Comm. Math. Phys. **123**, 535–573 (1989).

[13]   J. Ginibre and G. Velo, "Generalized Strichartz inequalities for the wave equation", J. Funct. Anal. **133**, 50–68 (1995).

[14]   S. Gustafson and D. Roxanas, "Global, decaying solutions of a focusing energy-critical heat equation in $\mathbb{R}^4$", J. Differential Equations **264(9)**, 5894–5927 (2018).

[15]   C. M. Guzmán, "On well posedness for the inhomogeneous nonlinear Schrödinger equation", Nonlinear Anal. Real World Appl. **37**, 249–286 (2017).

[16]   A. Haraux and F. Weissler, "Non uniqueness for a semilinear initial value problem", Indiana Univ. Math. J. **31**, 167–189 (1982).

[17]   M. Keel and T. Tao, "Endpoint Strichartz estimates", Amer. J. Math. **120**, 955–980 (1998).

[18]   C.E. Kenig and F. Merle, "Global well-posedness, scattering and blow-up for the energy-critical, focusing, non-linear Schrödinger equation in the radial case", Invent. Math. **166(3)**, 645–675 (2006).

[19]   C.E. Kenig and F. Merle, "Global well-posedness, scattering and blow-up for the energy critical focusing non-linear wave equation", Acta Math. **201(2)**, 147–212 (2008).

[20]   N. Ioku, K. Ishige, E. Yanagida. "Sharp decay estimates in Lorentz spaces for nonnegative Schrödinger heat semigroups", J. Math. Pures Appl. **103**, 900–923 (2015).

[21]   C. Miao, B. Yuan, and B. Zhang, "Well-posedness of the Cauchy problem for the fractional power dissipative equations", Nonlinear Anal. **68**, 461–484 (2008).

[22]   E. Nikolova, M. Tarulli, and G, Venkov, "On the Cauchy Problem For The Nonlinear Heat Equation", AIP Conference Proceedings **2159**, 030025 (2019).

[23]   E. Nikolova, M. Tarulli, and G, Venkov, "Unconditional well-posedness in the energy space for the Ginzburg-Landau equation", AIP Conference Proceedings **2159**, 030024 (2019).

[24]   E. Nikolova, M. Tarulli, and G, Venkov, "On the extended Strichartz estimates for the nonlinear heat equation", AIP Conference Proceedings **2172**, 030015(2019).

[25]   E. Nikolova, M. Tarulli, and G, Venkov, "Local and global space-time integrability for the inhomogeneous heat equation", AIP Conference Proceedings **2321**, 030027 (2021).

[26]   E. Nikolova, M. Tarulli, and G, Venkov, "Extended Strichartz estimates for the heat equation with a Strichartz type potential", AIP Conference Proceedings **2333**, 120008 (2021).

[27]   R. O'Neil, "Convolution operators and $L(p, q)$ spaces". Duke Math. J. **30**, 129–142 (1963)

[28]   E. M. Stein, "Singular Integrals and Differentiability Properties of Functions", Princeton Univ. Press, Princeton, N.J., 1971.

[29]   M. Tarulli, "Well-posedness and scattering for the mass-energy NLS on $\mathbb{R}^n \times \mathcal{M}^k$", Analysis **37(3)**, 117–131 (2017).

[30]   N. Tzvetkov and N. Visciglia, "Well-posedness and scattering for NLS on $\mathbb{R}^d \times \mathbb{T}$ in the energy space", Rev. Mat. Iberoamericana **32(4)**, 1163–1188 (2016).

[31]   B. Wang, Z. Huo, C. Hao, and Z. Guo, "Harmonic analysis method for nonlinear evolution equations I", World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2011.

[32]   F. Weissler, "Local Existence and Nonexistence for Semilinear Parabolic Equations in L^p", Indiana Univ. Math. J. **29(1)**, 79–102 (1980).

[33]   F. Weissler, "Existence and nonexistence of global solutions for a semilinear heat equation", Israel J. Math. **38**, 29–40 (1981).

[34]   Z. Zhai, "Strichartz type estimates for fractional heat equations", J. Math. Anal. Appl. **356**, 642–658 (2009).