GRAPHS, GRAPHICAL REPRESENTATION AND GRAPHICAL SOLVING IN BULGARIAN MATHEMATICS TEXTBOOKS

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Abstract

The Computer Algebra Systems (CAS) and the Dynamic Geometry Software (DGS) enable graphical solving of equations. Information, which is presented graphically, is easy for grasping and it gives suitable overall picture of a number of data and facts. Graphical solving has similar properties. The school students and anyone else cannot solve equations just graphically by paper and pan only. When students draw hand-made graphs, they first put in the coordinate plane some points, through which the graph passes, for the coordinates of which students make **calculations using the function's formula**, i.e. they use arithmetical methods, not graphical. The computer makes numerical calculations, but shows graphs and the students use them. New technologies are integrating in school curriculum in many ways. The respective traditions in education should be analyzed carefully and then be combined, enhanced, developed and/or partially replaced with new elements. In this paper is analyzed the content of several Bulgarian maths textbooks about several interrelated topics, namely: graphs of right and inverse proportionality, graph of a function, graphical solving of equations, inequalities and systems of any kind, etc. These analyses include the links between these topics with the perspective that they form and should form a **system** of knowledge, which should be most easily and fully integrable with the whole maths content, with other disciplines and with real life problems. Suggestions for enriching the tradition are made.

Key words: graphical solving, coordinate methods, word problems, propaedeutics.

1 Introduction

Graphics and graphical representation appear for first time in Bulgarian maths curriculum in primary school, when pupils work with sets (Euler/Venn diagrams) [4], [6], [13]; when they do arithmetic operations with numbers [3], [4] by visualizing them with trees and other graphs (**Fig. 1**), and when they solve word problems [2], [8], [11], [12] (5^{th}, 6^{th} and 7^{th} grade) mainly those about distances, velocities and time (**Example** and **Fig. 2**). Thus pupils use directed segments unconsciously. Word problems for movement are currently also proposed to 8^{th} graders [21], [22] and to 9^{th} graders [23], [25], [26] with the same graphs – i.e. they meet this visualization **several years**. Only the equations involved in the mathematical models are different – in lower grades they are linear and later - more complex kinds.

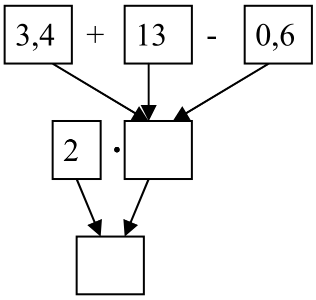

Fig. 1: Visualization with graph of arithmetic operations with numbers

Example: The distance between the settlements *A* and *B* is 220 km. From *A* for *B* set off a truck with velocity 50 km/h and at the same moment from *B* for *A* departed a car with velocity by 20% higher than the truck's one. Find how much time they will travel until they meet. [1]

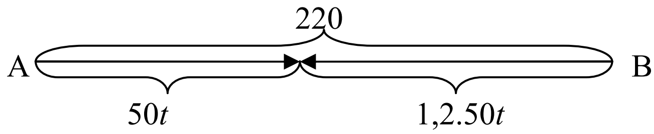

Fig. 2: Graph of the problem in Example 1

Very good propaedeutics of Cartesian coordinate system in the plane, matrices and electronic tables we see in 1st grade in [5], where pupils have to find the numbers of the row and the column of given elements of a table. As early propaedeutics of the concept "number half-line" appears for first time in 1st grade [5], where pupils use it without definition. 2nd grade students are introduced with the terms unit segment, half-line of the positive numbers, image of a number on the half-line of the positive numbers [6]. In [6], [19] (3rd grade) and [20] (4th grade) are introduced elements of graph theory and combinatorics. The same content is given in the three books, because they are intended for competitors too and the authors provide their customers the opportunity to have this content if they buy any of the books. The graphs also appear in forms of magic stars, pyramids and others [13], binary tree, schemes [13], [18], etc.

2nd, 5th and 7th graders add and subtract [15] segments on one line, or find [11] their ratio, and similar ones [15], and they plot figures with given properties on a rectangular grid. Half-line of the positive numbers is used regularly during the first five years [19], [20] and pupils put on it integer numbers first, then decimals, then fractions [7] and learn the definition of it in 5th grade. Thus the propaedeutics of the concept "number half-line" is longitudinal, gradual, ladder-like and non-intrusive (because the number ray appears only several times per year in textbooks). 6th and 7th-graders use the number line to put on it concrete numbers or to present intervals with solutions [3], [14] of given equations and inequalities. Number axis is also used in the introduction of the concepts absolute value of a number, modulo equation and inequality [12], and systems. *The idea of the Euler/Venn diagrams is used, when pupils find intersections of number intervals on the number line*. This is the case, when they solve systems. They learn the concept Cartesian coordinate system in the plane in 6th grade [3] after the following propaedeutics: construction of images of numbers on the number line and vice versa, which starts from the beginning of 5th grade and searching the areas of figures, constructed on a grid [4].

In [14] after the lesson “Cartesian coordinate system” 6th graders are introduced with the concepts graphical presentation of data, which is propaedeutics of the concept “graph of a function”. We found earlier propaedeutics in [14] of the same concept. But this propaedeutics is not right before the introduction of the Cartesian coordinates. 5th graders become introduced with rectangular grid (1st quadrant) using the concepts half-line of the positive numbers, location of a point, horizontal/vertical half-line of the positive numbers; propaedeutics of the concept ordered pair is made by pointing that the point $x = 3, y = 4$ is not the same as $x = 4, y = 3$. There are tasks to determine the location of a figure on a chessboard [7] and location of points on the grid. The grid appears many times as a background of plane figures and solids in order to maintain this concept in the pupils’ minds.

6th-graders also use circle diagrams, which are introduced [14] to them on the basis of pieces of pie, which they had used to present ratios, fractions as parts of the whole thing. Thus studying ratios becomes propaedeutics of the circle diagrams. They also learn histograms, which are introduced in 5th grade by means of grid, or in 6th right after Cartesian coordinate system [3], [14]. **After the introduction of the Cartesian coordinates students do not use it for two years up to 8th grade.** They only use the number line or search areas of figures on a grid (regression to the propaedeutics level). In the exam of the candidates for elite schools after 7th grade there are almost no algebraic problems, which require graphical solving, neither such that require elementary knowledge on coordinate systems, except the number line, which is used when students present number intervals on it. There are **no** problems involving coordinate methods in the most appliances for these candidates (e.g. [9], [10]), and there are some in [11]. There are in [12] graphs of temperature and rent as functions of time, which is a propaedeutics of the concept graph of function. The tendency is graphing to abandon the above tests, teachers have less time to teach and many of them do **not** teach graphing, vectors and elements of the analytical geometry. Therefore 10th graders cannot understand graphical solving of quadratic inequality and what is x and what is y! The concepts function and graph of a function are introduced rigorously and studied well enough in an appliance [16] from … 1992. There are in a Book of mathematical problems for 6th grade [18] problems about finding coordinates of points, symmetrical to a given point with respect of the axes, the origin; such for proving that a given by the coordinates of its vertices figure is a parallelogram or other, for measuring angles, calculating areas, etc. But there in [18], in the themes right and inverse proportionality and their applications, is present no graphical interpretation; they are not considered as functions with respective graphs. All their properties are introduced, discovered and verified analytically. These themes are after the theme “Coordinate system” in [18]. The textbooks and appliances [21], [22] for 8th graders contain large amount of graphical visualizations. A revision on Cartesian coordinate system is done and then are introduced the concepts function and graph of function (at the beginning inductively by examples of functions and explanations) and then – rigorously. After that – right and inverse proportionality, and graphical presentations of both. Right-proportionality-graphing is based upon linear function (6th grade). Inverse-proportionality-graphing – upon calculation of the coordinates of several points – and then the graph is constructed with some approximation, which is made by human eye and hand, not mathematically. Then linear and modulo-linear functions are introduced and graphs of them – deductively, ***not*** dogmatically. There are introduced systems of linear equations with three variables as planes in 3D, ***only*** in order to visualize why the latter have exactly 1, 0 or infinitely many solutions. Graphical presentation of linear equations, inequalities and systems of equations with two variables serves as visualization to illustrate and support the analytical solving. What means to solve the equation $x+2y-3=0$ graphically? The intersection points of the line $x+2y-3=0$ with the coordinate axes are found analytically and the line is constructed thereafter. To know if a certain point is on or not on this line, you have to check it analytically again. Hand-made sketch could not help you recognize this in many cases. Or how can be solved graphically the system $\left| \begin{array}{l} x-3y=1 \\ x+y=5 \end{array} \right.$? Not only the two lines are constructed by analytical means, but the intersection point's coordinates have to be found analytically too. How to find them from a hand-made sketch? With what precision can be done this? The graphically solved systems in our textbooks have integer solutions, which might leave students with the impression that graphical method has the same accuracy as the analytical one. And even more: some students may **not** realize what exactly means to solve an equation or a system graphically! There are in [21] also introduced graphs of quadratic functions with different coefficients. Graphs are done by putting several points in the sketch first and parabolas plotting by approximations of the human eye and hand again. In [23], [24] 9^{th} graders are reminded that modulo equations could be solved graphically. Graphs of different quadratic and $y=\sqrt{x}$ functions and graphical solving of quadratic inequalities are introduced in [24] at length and lots of equations are solved (visualized in fact) graphically there. On the number line is shown in [24] that between any two rational numbers there are infinitely many other ones and inverse proportionality is considered. In [28], [29] the 10^{th} graders meet again many of the graphical methods of [23] and [24]. A parabola on rectangular grid in 1^{st} quadrant is given in [28] on p.25 and the respective quadratic function is sought. The method of intervals for solving inequalities of higher degree is introduced in [28], [29]. A physics problem is presented graphically on p.67, which provides understanding of the interpretation of problem's solution. Otherwise the solving would become rather mechanical then logical. Trigonometric functions of real argument are introduced [28], [29] by means of the unit circle and Cartesian coordinates and exponent with real argument. The 10^{th} graders become introduced with theorems for distribution of the roots of the quadratic equations in [29]. In [30] by convenient graphical presentation is introduced … **Fourier series**! There are also introduced the graphs of trigonometric functions by calculating the coordinates of several points, using periodicity, symmetries, boundedness of the sine and cosine functions and limits of tangent and cotangent when the argument tends to points of discontinuity. Graphs of inverse functions [30], of exponential and logarithmic function are introduced – again by plotting several concrete points and the curve through them using certain properties of the function. Graph of unit-step function of distribution is shown on p.103 histograms, circle diagrams in statistics. The equation $\log_2 x = 3 - x$ is very interesting [30] p.159. It is solved half analytically and half – graphically, but the reasoning is insufficient: are graphs predictable outside the screen, why?

In [31] (12th grade, math profile) we see graphical presentation of derivative of a function, graphical presentation of complex numbers; elements of statistics – histograms, polygon of a discrete and normal distribution of stochastic quantities; elements of analytical plane geometry – coordinates of vector, equation of a straight line and conic sections. In [32] (appliance for candidate-students) graphical presentation is used for assisting students to systematize the problem and to build up a plan how to solve it analytically. We see in [32] problems with modulo in modulo and complicated ones. In [33] we see the number axis used for introduction of the concept monotone number sequence and for propaedeutics of the concept definite integral (there is required to find the area between the graph of $y = x^2$ and the abscissa in the interval [0,1]). In [34] we see problems, which include graphs of functions and require graphical methods to be integrated in the solution. [35] includes graphical solving of equations as $\lg x = x^2$ and plotting graphs of exponential, logarithmic and inverse functions. Plenty of graphs and notes on graphical solving we see in the very beginning of [36] (matures and candidate-students, printed in 2009). There are plenty of graphical presentation, graphs of functions and graphical solving in it (on almost all pages!), which improves understanding significantly.

2 Instead of conclusion

The tradition in our education is rich and steady. A colleague asked me if the textbooks from 1998 for example are out-dated. The answer is “no”. Even if there are new versions of them, the content, the didactic technologies and approach and the most mathematics problems are the same or nearly the same – i.e. slightly reduced, but not enriched. Every concept becomes finally shaped on its place after long development. New technologies have radically changed and continue changing the maths activities of people – merely of the users, but also those of the mathematicians. But the new technologies still cannot find their place in teaching because many mathematics activities cannot be taught well by them. There is a strong tendency the computer software to be used to provide us results, not to help us develop critical thinking, heuristic methods and Boolean-rules-cognition. The new technologies do not help us to learn how to prove theorems. There are many open questions and therefore I prefer to leave the end of this talk open.

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