Computer Methods for Graphical Presentation of Solutions of Systems of Trigonometric Equations for Advanced Students

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The mathematics activities of the students and the people in the real life will be more and more characterized by implementation of computer methods, mathematical software, more heuristics and "computer" proofs. The mathematics curriculum for schools needs gradual modernization of its content and teaching methods in this direction. We consider several Systems of Trigonometric Equations of Two Variables (STETV), solve them graphically by a PC and comment them. There is a lack of graphical solving of equations and graphical presentation of data and Boolean statements in our curriculum. But our work is also applicable to the current curriculum in Bulgaria and can enrich it. The idea to develop methods for graphical presentation of statements as "these two STETV are equivalent" or "the solutions of the STETV … are …" came to us during a didactics experiment on graphical solving of trigonometric equations, when we realized that such problem is not a standard one. Y. Papazov had solved a STETV during the experiment and we (V. Danova and A. Velchev) had a bit difficulty until we understood his solution. The whole class was totally unresponsive regardless of our detailed explanations and other didactic methods. The suggested here three methods for solving STETV appeared to be more "didactic". When we've introduced them to advanced students from Sofia High-school of Mathematics, they understood them, applauded us and solved themselves stated by us similar STETV. But we also consider here Y. Papazov's method and its plusses.

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1. Introduction

The computer software and hardware evolution make graphical solving of equations and inequalities not only possible but more and more easy and precise. Here we present methods for solving systems of two variables by software designed for work with one variable. This work is extension of [4]. During a didactical experiment at

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Sofia high school of mathematics we challenged students to solve some STETV-s and to prove STETV-s' equivalency. Methods described here are convenient for advanced students, not for the basic level.

2. Methods for graphical solving of STETV with a computer.

How to present graphically a solution of one equation? We have a solution (*a*,*b*), when $f_1(a)$ = $g_1(b)$ (along *Oz*). Their graphs are surfaces and their intersection is a skew curve in most cases. Solution of a system is intersection of two skew curves. Solution with 3D-software requires knowledge and skills, which the school students do not have. It is rather for talented ones. And even the software, which is able to plot skew curves, since the curve's shape is unknown in the general case and therefore unpredictable out of the screen, we cannot use common method to solve graphically any system. A curve has predictable shape outside the screen, only when it is a straight line, elementary function's graph or some conic section. Therefore we suggest here four special methods and they hold only for predictable functions: linear, quadratic, periodic … (trigonometrical here).

Method A: Let we have a system of type $\left|\begin{array}{l} f_1(x)=g_1(y) \\ f_2(x)=g_2(y) \end{array}\right.$. $f_1(x)$ takes values along the *y*-axis in the co-ordinate plane. If we construct the graph of $g_1(x)$ instead of $g_1(y)$, we will have the $g_1(y)$-values **along the *y*-axis too**. As the second function is of another variable, the points of intersection of the two graphs present symmetric solutions of type (*a,a*) of the equation. The ordered pair of the *x*-coordinates of **any two points on the same height** (the first point on $f_1(x)$ and the second - on $g_1(x)$) is solution of the equation. Let consider the system $\left|\begin{array}{l} \left|Sin\left(3x+\dfrac{\pi}{4}\right)\right| = Siny - Cosy \\ Sin2y + 2Sin2x = \dfrac{3}{4} + 2Sin^3 2x \end{array}\right.$. $\left|\begin{array}{l} \left|Sin\left(3x+\dfrac{\pi}{4}\right)\right| = Siny - Cosy \\ \dfrac{3}{4} + 2Sin^3 2x - 2Sin2x = Sin2y \end{array}\right.$ is an equivalent system. The periods of $f_1(x)$ and $g_1(x)$ are π/3 and 2π respectively. Then a vertical strip of width 2π (between the lines *Oy* and *e* in **Fig. 1**) includes enough from the graphs.

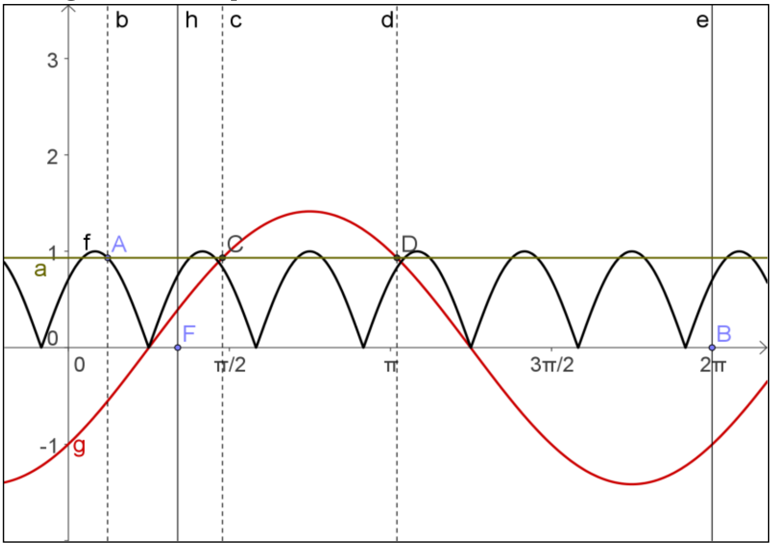


Fig. 1: Graphical presentation of the equation $\left|Sin\left(3x+\frac{\pi}{4}\right)\right| = Siny - Cosy$.

A is a slider point on $f_1(x)$, *a* is the horizontal line through *A*. *a* intersects $g_1(x)$ at several (or none) points (*C* and *D* in **Fig. 1**). Hence the ordered pairs of *x*-coordinates (x_A, x_C) and (x_A, x_D) are solutions of the equation. It is enough to move the slider along the sector of the curve between the vertical lines *Oy* and *h*, which is one period of $f_1(x)$, in order to show with a good approximation solutions of the equation. GeoGebra shows **the 15th decimal place** of the coordinates. The line *a* intersects the graph of $f_1(x)$ at two points on the sector between *Oy* and *h* in **Fig. 1**, but this does not provide new solutions, because the slider *A* will pass through the additional point and the intersection of *a* with $g_1(x)$ will be again $\{C, D\}$.

The system's solutions are **among** the solutions of the first equation. The graphical presentation of $\frac{3}{4}+2Sin^3 2x-2Sin2x=Sin2y$ is analogous. We plot the graphs of $f_2(x)$ and $g_2(x)$ (**Fig. 2**). If (a,b) is a solution of the system, then the points $A(a, f_1(a))$ and $C(b, g_1(b))$ lie on a horizontal line *a*) and the points $E(a, f_2(a))$ and $G(b, g_2(b))$ - on another horizontal line, i.e. *AEGC* is a rectangle. Else it is a trapezium. Hence the second equation will have solution if *EG* or *EH* is parallel to *Ox* (**Fig. 2**). To find the solutions easier we construct a horizontal line *i* through *E*, move the slider *A* and look **only** if **either** *G* or *H* will "step" on *i*. This will be a solution of the system.

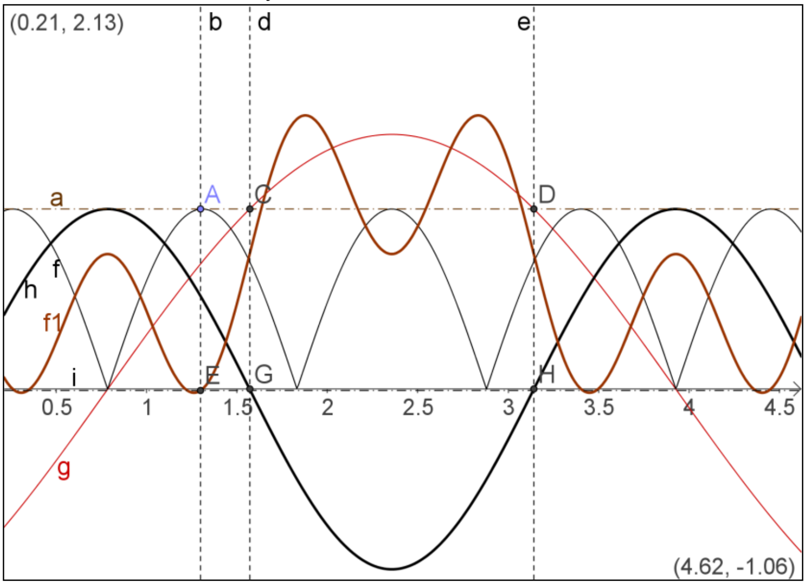


Fig. 2: Graphical presentation of the system.

Construction (we use the free didactics software GeoGebra):

1) The four graphs

2) The slider *A* (on the graph of $f_1(x)$)

3) The horizontal line *a* and the vertical line *b* through *A*

4) *E* – the intersection of *b* with the graph of $f_2(x)$

5) The intersection points *C*, *D* of *a* with the graph of $g_1(x)$

6) The vertical lines *d*, *e* through them

7) Their points of intersection *G*, *H* with the graph of $g_2(x)$

8) The horizontal line *i* through *E*.

Because the sketch is overladen, we may make the verticals through *A*, *C* and *D* invisible after constructing *E, G, H*. The graph of $f_2(x)$ may be hidden too after the construction of *E*. The graph of $g_2(x)$ is necessary as points of intersection of it with additional verticals may become necessary to find in case of more points of intersection of *a* with the graph of $g_1(x)$. The graph of $f_1(x)$ is not necessary too, because the slider “knows” its trajectory as we pull it from left to right. Another important thing is to check if the period of $f_2(x) \setminus g_2(x)$ is longer than π/3 \ 2π. If so, then we take the domain(s) broader until they include the periods of the four functions.

Method B (at least one separable equation with (easy) invertible function):

Let the system be of type $\begin{vmatrix} (f(x))^2 = (g(y))^2 \\ F(x,y) = 0 \end{vmatrix}$. Then $f(x) = \pm g(y)$ and $y_{1,2} = g^{-1}(\pm f(x))$. The curves through *B* and *C* in **Fig. 3** are the graphs of $y_{1,2} = g^{-1}(\pm f(x))$ and the other curve is the graph of $F\left(x, g^{-1}(f(x))\right)$. The graph of $F\left(x, g^{-1}(-f(x))\right)$ is the same in our case: $\begin{vmatrix} Cos^2\left(3x + \frac{\pi}{4}\right) = 2Cos^2 y \\ Cos2y + 2Sin2x + \frac{3}{4} = 2Sin^3 2x \end{vmatrix}$. Every solution (*a,b*) of such STETV has these properties: $F\left(a, g^{-1}(f(a))\right) = F\left(a, g^{-1}(-f(a))\right) = 0$ where $b = g^{-1}(\pm f(a))$. If $F\left(c, g^{-1}(f(c))\right) = 0$, then $\left(c, g^{-1}(f(c))\right)$ is solution. I.e. the co-ordinates of *B* and *C* in **Fig. 3** are solutions.

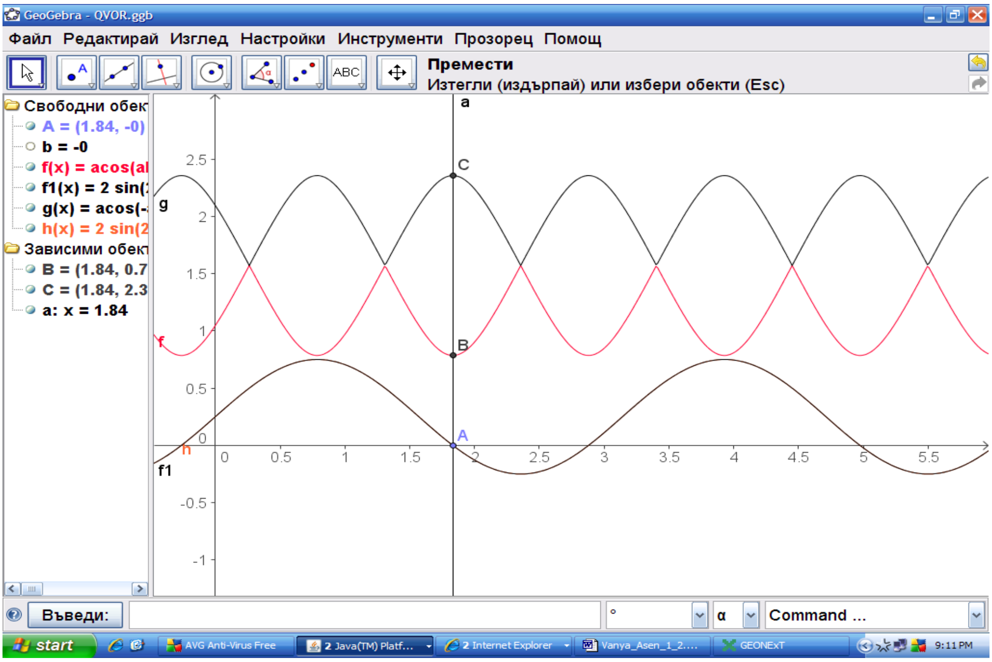


Fig. 3: Graphical solving of a STETV via inverse functions.

Method C (by considering the one variable as a parameter):

One may create a slider object (parameter), representing the value of one of the unknown variables (say y). Then draw the graphs of both equations and find the solutions changing the value of the slider. The true translation of the Boolean statement "($a;b$) is solution of the system" is "The two graphs intersect on the x-axis at the point a for y (slider) = b". Some students tend to use the wrong statement: "The STETV has solution ($a;b$) if and only if the two graphs intersect at the point ($a;b$)". It x is represented by a slider then we may set x on the place of y and the statement's "($a;b$) is solution of the system" true translation will be "The two graphs intersect on the x-axis at the point b for x (slider bar) = a". Why we speak about Boolean statements? We do this, because this term allows us to **generalize and upgrade** the concept system of equations to the concept conjunction of Boolean statements. The equations are kind of Boolean statements – for some values of the variables they become true and for other values - false. The same holds for systems of inequalities and combined systems – containing both equations and inequalities.

Method D:

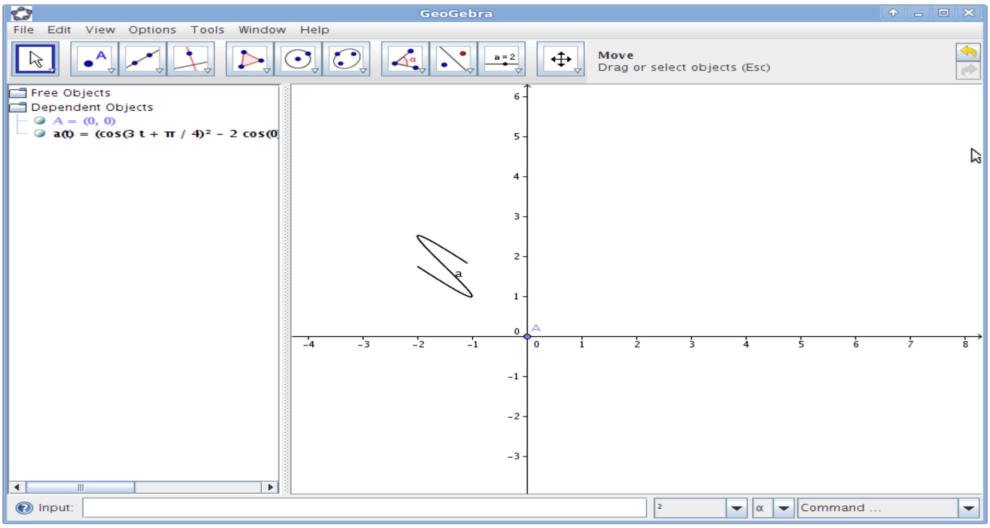
Example 1:

$$\begin{cases} \cos^2\left(3x+\dfrac{\pi}{4}\right)=2\cos^2 y \\ \cos 2y+2\sin 2x+\dfrac{3}{4}=2\sin^3 2x \end{cases}$$

Solution:

Let we plot the parametric Curve[cos(3 t + π / 4)² - 2 cos(x(A))², cos(2 x(A)) + 2 sin(2 t) + 3 / 4 - 2 sin(2 t)³, t, 0, π]. We will seek for solutions of the equation by a slider point. There exists a solution of the system for a given x(A), which is the *x* coordinate of the slider point *A*, only when the curve passes through the point (0,0), i.e. when there exists such *t* that for x(A) and *t* hold the statements:

Cos(3*t+Pi/4)^2-2*Cos(x(A))^2=0, Cos(2*x(A))+2*Sin(2*t)+3/4-2*Sin(2t)^3=0,



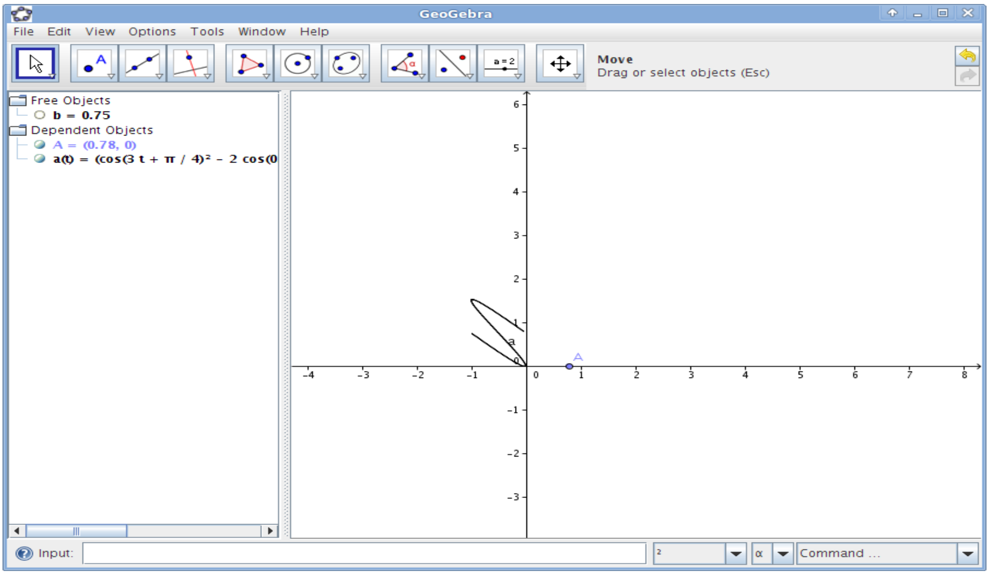
but in this case x(A) and *t* are solutions of the initial system and vice versa. Let we demonstrate the method step by step:

1) By moving the slider *A* along *Ox* we obtain the roots of the equation of the parametric curve (no solution in Fig. 4 and solution in Fig. 5):

The solutions here are: 0.78 and 2.36.

2) After finding the values of x(A), for which the equation has roots, we find the respective values of *t*, which are solutions of the system by constructing the graph of function of *t*, in which x(A) is a constant. This value here is approximately 1.82.

3) The approximate values in the interval [0,π], for which the system has solutions, are: (0.78,1.82), (2.36,1.82), i.e. (π /4, 7*π /12) and (3*π /4, 7*π /12).



Methods for graphical solving of a STETV hold for many STETV and some of them are really challenging even for excellent students. [1] and [2] contain solutions of systems with the professional package MATHEMATIKA. Our methods are different and we use didactics software, because it's free, with easier syntax and translated in Bulgarian. We put here screenshots to enable readers to see not only graphics window, but also the algebraic one, the toolbar and the menus. The considered here STETV are extracted from [3].

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