METHODS FOR SOLVING SYSTEMS TRIGONOMETRIC EQUATIONS WITH TWO UNKNOWNS WITH AND WITHOUT A COMPUTER

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ABSTRACT

We consider several Systems Trigonometric Equations with Two Unknowns (STETU), solve them graphically by PC and analytically "by pen" and compare the solving methods and comment them. At this stage we don't aim to evaluate or judge them, but we rather share our experience.

1. METHODS FOR ANALYTICAL SOLVING (MAS) OF SYSTEMS TRIGONOMETRIC EQUATIONS WITH TWO UNKNOWNS (STETU).

Here we provide several "mini-types" of STETU:

Type 1 (reducible to $\left| \begin{array}{l} f_1(x) = g(y) \\ f_2(x) = g(y) \end{array} \right.$ **):**

Problem 1: $\left| \begin{array}{l} Cos^2\left(3x+\dfrac{\pi}{4}\right) = 2Cos^2 y \\ Cos2y + 2Sin2x + \dfrac{3}{4} = 2Sin^3 2x \end{array} \right.$. To obtain the same function of *y* in both equations, we transform the first one: $Cos^2\left(3x+\dfrac{\pi}{4}\right) - 1 = Cos2y$ and rearrange the second: $Cos2y = 2Sin^3 2x - 2Sin2x - \dfrac{3}{4}$. The two initial equations are "separable", i.e. they may be written as $f(x) = g(y)$. Here the STETU is reducible to $\begin{vmatrix} f_1(x) = g(y) \\ f_2(x) = g(y) \end{vmatrix}$. By equating the sides containing x from the two equations of such STETU we obtain an equation of one variable and we combine it with one of the given ones to solve the system.

Problem 2: $\begin{vmatrix} 9^{2Tanx+Cosy} = 3 \\ 9^{Cosy} - 81^{Tanx} = 2 \end{vmatrix}$. We may rearrange the STETU: $\begin{vmatrix} 9^{Cosy} = 3.9^{-2Tanx} \\ 9^{Cosy} = 9^{2Tanx} + 2 \end{vmatrix}$.

Problem 3: $\begin{vmatrix} |Sin3x| = -\sqrt{2}Siny \\ Cos2y + 2Cos2x.Sin^2 2x = \dfrac{3}{4} \end{vmatrix}$. By rising the modulo equation to second power we obtain $2Sin^2 y$ and transform it to $1 - Cos2y$.

Problem 4: $\begin{vmatrix} |Cos3x| = Siny + Cosy \\ 2Sin^2 2x.Cos2x + \dfrac{3}{4} = -Sin2y \end{vmatrix}$.

Problem 5: $\begin{vmatrix} \left|Sin\left(3x + \dfrac{\pi}{4}\right)\right| = Siny - Cosy \\ Sin2y + 2Sin2x = \dfrac{3}{4} + 2Sin^3 2x \end{vmatrix}$. By rising the modulo equations in these two systems to second power we obtain $1 \pm Sin2y$. Some STETU of $\begin{vmatrix} f_1(x) = g(y) \\ f_2(x) = g(y) \end{vmatrix}$-type are easily solvable also by other concrete methods having in mind the rich set of trigonometric formulas.

Type 2: $\begin{vmatrix} f_1(x)=g_1(y) \\ f_2(x)=g_2(y) \end{vmatrix}$ **not (easy) reducible to** $\begin{vmatrix} f_1(x)=g(y) \\ f_2(x)=g(y) \end{vmatrix}$**:**

Problem 1: $\begin{vmatrix} x-y=-\dfrac{1}{3} \\ Cos^2\pi x-Sin^2\pi y=\dfrac{1}{2} \end{vmatrix}$. Here $x=y-\dfrac{1}{3}$ (explicit function) we substitute in the second equation and decrease the powers: $1+Cos\pi\left(y-\dfrac{1}{3}\right)-1+Cos2\pi y=\dfrac{1}{2}\Leftrightarrow Cos\pi\left(2y-\dfrac{1}{3}\right)=\dfrac{1}{2}\ldots$

Problem 2: $\begin{vmatrix} x+y=\dfrac{\pi}{4} \\ tgx.tgy=\dfrac{1}{6} \end{vmatrix}$. We substitute *y* (inverse function, explicit function) from the first equation in the second one, simplify $Tan\left(\dfrac{\pi}{4}-x\right)$ and solve the obtained quadratic equation with respect to $Tanx$.

Problem 3: $\begin{vmatrix} tg\left(\dfrac{x}{2}\right)+tg\left(\dfrac{y}{2}\right)=\dfrac{2}{\sqrt{3}} \\ tgx+tgy=2\sqrt{3} \end{vmatrix}$. Let $Tan\dfrac{x}{2}=a, Tan\dfrac{y}{2}=b$. Then we have $\begin{vmatrix} a+b=\dfrac{2}{\sqrt{3}} \\ \dfrac{2a}{1-a^2}+\dfrac{2b}{1-b^2}=2\sqrt{3} \end{vmatrix}\ldots$

As the all functions $f_i(x), g_i(x), i=1;2$ are invertible, these 3 systems might be transformed to $\begin{vmatrix} y=F(x) \\ y=G(x) \end{vmatrix}$ (sub-type of type 1), but via more transformations and the obtained equations will "look strange".

Type 3 (exactly one separable equation):

Problem 1: $\left|\begin{array}{l} x+y=\dfrac{5\pi}{6} \\ 5(Sin2x+Sin2y)=2(1+Cos^2(x-y)) \end{array}\right.$. The first equation is separable and its functions are invertible. Other different ways for solving it are based on different trigonometric formulas. **But they are not easier**.

Problem 2: $\left|\begin{array}{l} |x|+|y|=\dfrac{\pi}{4} \\ Cosx-Cosy=Sin(x+y) \end{array}\right.$. We may transform the sides of the second equation in products, simplify and solve it by considering the zeroes of the factors. The first equation is separable and its functions are invertible with pairs of inverse functions. Therefore the solution via using inverse function isn't easier, **but not so complicated**.

Type 4 (two non-separable equations):

Problem 1: $\left|\begin{array}{l} 6Sinx.Cosy+2Cosx.Siny=-3 \\ 5Sinx.Cosy-3Cosx.Siny=1 \end{array}\right.$. If $u=Sinx.Cosy$, $v=Cosx.Siny$, then $\left|\begin{array}{l} 6u+2v=-3 \\ 5u-3v=1 \end{array}\right. \Leftrightarrow u=-\dfrac{1}{4}, v=-\dfrac{3}{4}$; $\left|\begin{array}{l} Sin(x+y)=u+v=-1 \\ Sin(x-y)=u-v=0,5 \end{array}\right. \dots$

2. METHODS FOR GRAPHICAL SOLVING WITH A COMPUTER (MGSC) OF STETU.

Type 1: $\left|\begin{array}{l} f_1(x)=g_1(y) \\ f_2(x)=g_2(y) \end{array}\right.$**:**

How to present graphically a solution of one equation? We have a solution (*a*,*b*), when $f_1(a) = g_1(b)$ (along *Oz*). Their graphs are surfaces and their intersection is a skew curve in most cases. Solution of a system is intersection of two skew curves. Solution with 3D-software requires knowledge and skills, which the common school students do not have. It is rather for talented ones.

The graph of an equation with two unknowns is some locus (curve). Dynamics didactical software (and not only it) plots the curve, only when it is straight line or conic section. And even if some software is able to do so, since the curve's shape is unknown in the general case and therefore **unpredictable** out of the screen, we cannot use common method to solve graphically any system. Therefore we suggest here two special methods and they hold only for predictable functions: linear, quadratic, periodic … (trigonometrical here).

We have the system $\left| \begin{array}{l} f_1(x) = g_1(y) \\ f_2(x) = g_2(y) \end{array} \right.$. $f_1(x)$ takes its values along the *y*-axis in the co-ordinate plane. If we construct the graph of $g_1(x)$ instead of $g_1(y)$, we will have the $g_1(y)$-values **along the *y*-axis too**. As the second function is of **another** variable, the points of intersection of the two graphs present symmetric solutions of the equation (*a*,*a*). The ordered pair of the *x*-coordinates of **any** two points **on the same height** (the first point on $f_1(x)$ and the second - on $g_1(x)$) is solution of the equation. Let consider problem 5 from Type 1. The equations we rearrange thus: $\left| \begin{array}{l} \left| Sin\left(3x+\dfrac{\pi}{4}\right) \right| = Siny - Cosy \\ \dfrac{3}{4} + 2Sin^3 2x - 2Sin2x = Sin2y \end{array} \right.$. The period of $g_1(x)$ is 2π and the period of $f_1(x)$ is π/3. Then a vertical strip on the screen with width 2π (between the lines *Oy* and *e* in **Fig. 1**) includes enough from the graphs.

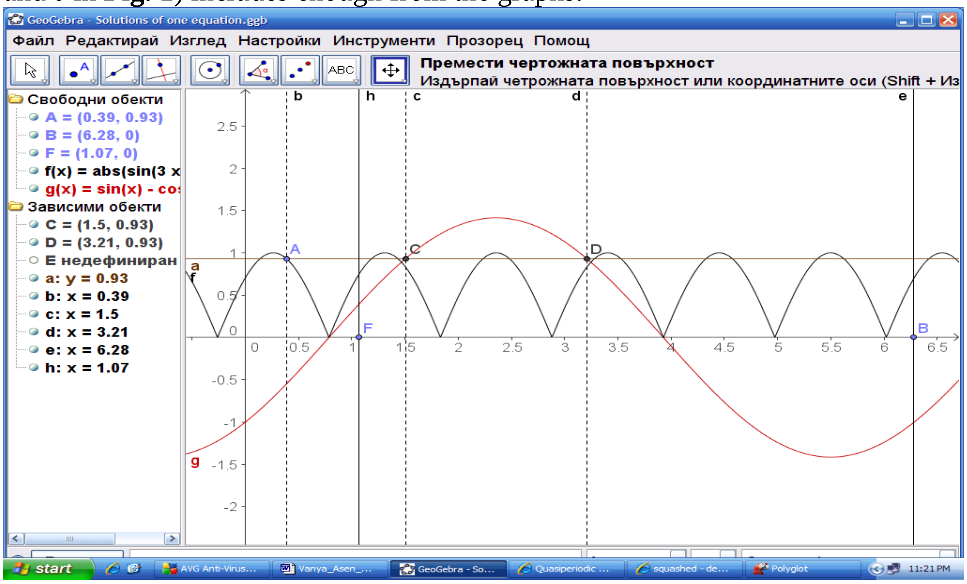

Fig. 1: Graphical presentation of the equation $\left| Sin\left(3x+\dfrac{\pi}{4}\right) \right| = Siny - Cosy$.

A is a slider point on $f_1(x)$, *a* is the horizontal line through *A*. *a* intersects $g_1(x)$ at several (or none) points (two on the screenshot – *C* and *D*). Hence the ordered pairs (x_A, x_C) and (x_A, x_D) of the *x*-coordinates of the points are solutions of the equation. It is enough to move the slider along the sector of the curve between the vertical lines *Oy* and *h*, which is one period of $f_1(x)$, in order to show with some approximation different solutions of the equation. The line *a* intersects the graph of $f_1(x)$ at two points on the sector between *Oy* and *h* in **Fig. 1**, but this does not provide new solutions, because the slider *A* will pass through the additional point and the intersection of *a* with $g_1(x)$ will be again {*C*, *D*}.

The system's solutions are **among** the solutions of the first equation. How to find them? The graphical presentation of $\frac{3}{4}+2Sin^3 2x-2Sin2x=Sin2y$ is analogous. We plot the graphs of $f_2(x)$ and $g_2(x)$ (**Fig. 2**). If (*a*,*b*) is a solution of the system, then the points $A(a, f_1(a))$ and $C(b, g_1(b))$ have the same level (lie on the parallel to *Ox* line *a*) and the points $E(a, f_2(a))$ and $G(b, g_2(b))$ lie on another parallel to *Ox* line, i.e. *AEGC* is a rectangle. Else *AEGC* is a trapezium. (See **Fig. 2**). Hence the second equation will have solution if *EG* or *EH* is parallel to *Ox*. To find the solutions <u>**easier**</u> we construct a horizontal line *i* through *E*, move the slider *A* and look <u>**only**</u> if <u>**either**</u> *G* or *H* will "step" on *i*. This will be a solution of the system.

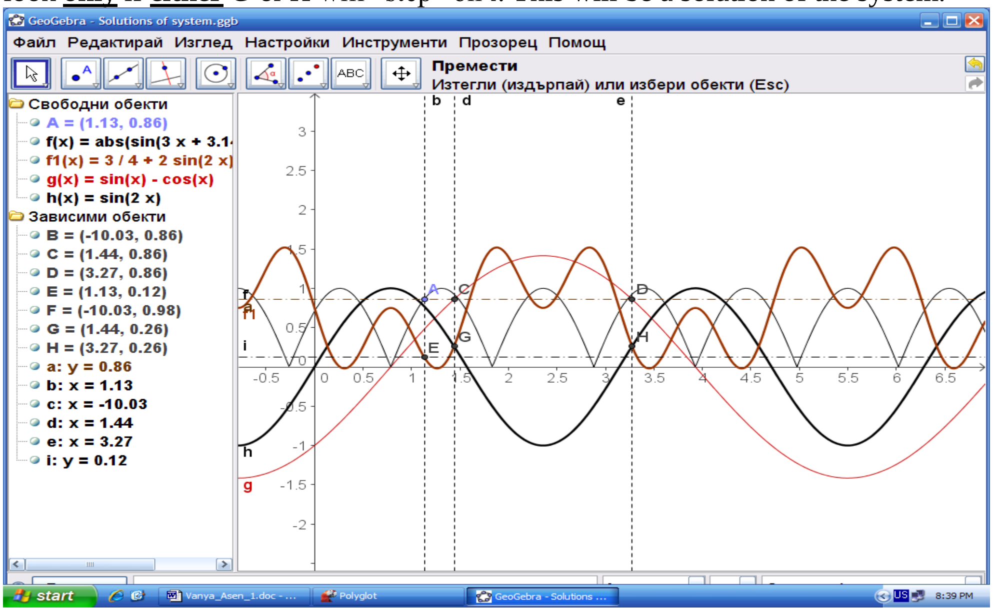

Fig. 2: Graphical presentation of the system.

We use GeoGebra – free software with good precision tested by us at school. Construction:

1) The four graphs

2) The slider *A* (on the graph of $f_1(x)$)

3) The horizontal line *a* and the vertical line *b* through *A*

4) *E* – the intersection of *b* with the graph of $f_2(x)$

5) The intersection points of *a* with the graph of $g_1(x)$

6) Vertical lines through them

7) Their points of intersection with the graph of $g_2(x)$

8) The horizontal line *i* through *E*.

Because the screen is overladen, we may make the verticals through *A*, *C* and *D* invisible after constructing *E, G, H*. The graph of $f_2(x)$ may be hidden too after the construction of *E*. The graph of $g_2(x)$ is necessary as points of intersection of it with additional verticals may become necessary to find in case of more points of intersection of *a* with the graph of $g_1(x)$. The graph of $f_1(x)$ is not necessary too, because the slider "knows" its trajectory as we pull it from left to right. Another important thing is to check if the period of $f_2(x) \setminus g_2(x)$ is longer than π/3 \ 2π. If so, then we take the domain(s) of *x* or/and *y* broader (until they include the periods of all functions).

Type 2 (at least one separable equation with (easy) invertible function):

Let the system be of type $\left| \begin{array}{l} \left(f(x)\right)^2 = \left(g(x)\right)^2 \\ F(x,y) = 0 \end{array} \right.$. Then $f(x) = \pm g(y)$ and $y_{1,2} = g^{-1}\left(\pm f(x)\right)$. The curves through *B* and *C* in **Fig. 3** are the graphs of $y_{1,2} = g^{-1}\left(\pm f(x)\right)$ and the other curve is the graph of $F\left(x, g^{-1}\left(f(x)\right)\right)$. The graph of $F\left(x, g^{-1}\left(-f(x)\right)\right)$ is the same in our case: $\left| \begin{array}{l} Cos^2\left(3x + \dfrac{\pi}{4}\right) = 2Cos^2 y \\ Cos2y + 2Sin2x + \dfrac{3}{4} = 2Sin^3 2x \end{array} \right.$. Every solution (*a*,*b*) of such STETU has these properties: $F\left(a, g^{-1}\left(f(a)\right)\right)$ = $F\left(a, g^{-1}\left(-f(a)\right)\right)$ = 0 and $b = g^{-1}\left(\pm f(a)\right)$.

And if $F\left(c, g^{-1}\left(f\left(c\right)\right)\right)=0$, then $\left(c, g^{-1}\left(f\left(c\right)\right)\right)$ is solution of the system. I.e. the co-ordinates of B and C in **Fig. 3** are solutions.

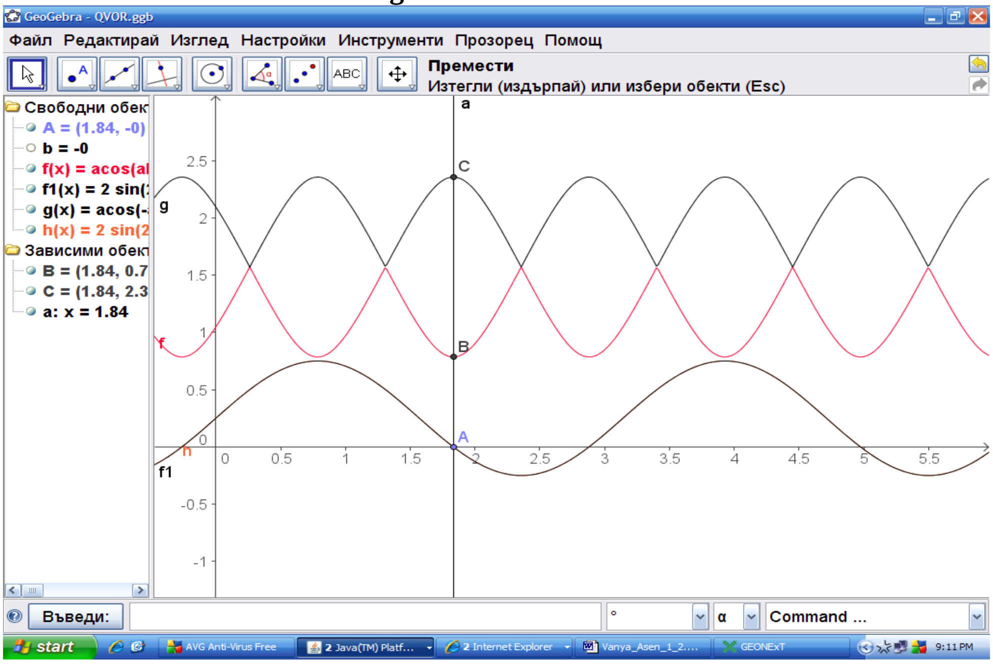

Fig. 3: MGSC of a STETU via using of inverse function.

3. COMPARISON OF THE METHODS.

There are different MAS for the different types of STETU. We do not classify them. To deal with an arbitrary STETU successfully a student has to have a very solid mathematical training – to handle with the section trigonometry, to know a set of MAS of algebraic and trigonometrical equations, systems of equations; to judge which transformation is equivalent etc; to construct mentally several-steps strategy for solution of the problem and so on. We do not analyze these skills here.

MGSC of a STETU are **less** in number and each of them holds for **many types** of STETU. For success is not necessary the student to know a large set of tools/prioms. The same time MGSC are real difficulty even for the excellent student at the heuristic and construction steps of the solution. The concept MGSC of a STETU might be introduced to students via step-by-step didactic technology (DT), which we plan to create further. A system of one unknown (STEOU) might be the first step from such DT, two-three parametric STEOU as the parameter is like a middle step before introduction of STETU.

The articles [1], [2] contain solutions of systems with the professional package MATHEMATIKA. Our methods are different and we use didactics software, because it's free, with a more easy syntax and option for interface (menus) in

Bulgarian. Graphic files are easy exportable in different formats, but we put here screenshots to enable readers to see not only graphics window, but also the algebraic one, the toolbar and the menus.
The STETU that we've provided here are from [3].

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[2] Lazarov B., Mathematics and Informatics Quarterly, vol. 7, No. 4, 1997.

[3] Королев, С. В. "Тригонометрия на экзамену по математике", Москва, 2006.