# A GEONExT–based didactical technology for graphical solving of parametric equations of the type $f(x) = g(x)$

Asen Velchev,
Institute of mathematics and informatics, Sofia 1113, "Acad. G. Bounchev" blvd, block 8.

**ABSTRACT**

Our goal is to create didactical technologies for graphical solving of equations of type $f(x) = g(x)$ in school with a convenient computer program. The graphical solution of equation of this type is done by finding the abscissas of the points of intersection of the graphs of the functions $f(x)$ and $g(x)$. Here $f(x)$ and $g(x)$ are functions of type, studied by the school students. Why only such functions? Theoretically there are no limits for the kind of the functions. But we should be able to find together with school students the points of intersection of the graphs of $f(x)$ and $g(x)$. Therefore we should be able to see them on the computer screen using the technology we suggest here. To recognize that there are no other intersection points, except those on the screen, which is only a part of the infinite plane, we should know the behavior of the graphs of the functions $f(x)$ and $g(x)$ outside the screen, i.e. over the whole plane. For students it is possible only for functions of studied type, which they know well. We have to add, that the knowledge of the functions and their graphs' properties are necessary not only when we make conclusion that outside the screen there are no other generic points, but also when we know or suppose the opposite: that there are other generic points outside the screen area. In this case we can say "where in the plane" to expect more solutions and to drag the plane with the objects through the screen with the mouse until the suspected points of intersection enter the screen or prove to be non-existent. In particular, for equations of type $f(x) = 0$, i.e. when $g(x) \equiv 0$, the solutions are the abscissas of the points of intersection of the graph of the function $f(x)$ with the abscissa.

Here we present an attempt to create concrete didactic technology of the described type for students from 11th and 12th class of the Bulgarian secondary school **<u>on level two</u>**, which means students profiled in studying mathematics. We have experimented the didactic technology with students from 11th class, profiled in mathematics and science. They study mathematics five hours weekly. They had undergone the experiment successfully.

## 1. Basic goals in our work.

We aim to create exemplary GEONExT-based didactic technology for graphical solving of parametric equations of the type $f(x) = g(x)$ for the secondary school, for students of 11-th class of second level (profiled in mathematics). We also aim to investigate the effectiveness of its practical implementation. Here $f(x)$ and $g(x)$ are functions of type, studied by the school students. Why only such functions? To be able to find the generic points of the graphs of the functions $f(x)$ and $g(x)$, it's necessary to see them on the monitor screen, which shows only a very small part of the infinite plain. To be able to judge that there are no other generic points except these on the screen, or to be able to suspect the existence of such points and to know where to search for, we need to know the shape of the graphs of the functions $f(x)$ and $g(x)$ out of the screen, i.e. over the whole plain. For students this is possible only for functions of studied type.

2. **Particular didactic technology of the described type.**

I have created and experimented it with 11th grade school students at 23 school in Sofia, under the guidance of Prof. Jordan Tabov. This experiment is preliminary, because a number of elements of the didactic technology we have developed "on the move". But our results are optimistic and therefore we aim to continue our work on developing GEONExT-based didactic technologies and we think that it is necessary to reiterate the experiment, on purpose to corroborate the validity of these optimistic results.

I have started the experiment with two groups of students – experimental and marker, each one of 12 students. To study the level of the mathematical skills of the students to solve parametric equations before the carrying out the experiment, I have made the following test:

1. For which values of the real parameter *a* the equation $|a-x|=2$ has exactly 2 roots?

A) only if $a > 2$ B) only if $a \le 2$
C) never D) for every value of *a*.

2. For which values of the real parameter *a* the equation $\left(a^2-3\right)x^2=2$ has exactly 2 roots?

A) only if $a > 2$ B) for $a \in (-3; 3)$
C) for $a \in \left(-\infty; -\sqrt{3}\right) \cup \left(\sqrt{3}; \infty\right)$ D) for every value of *a*.

3. For which values of the real parameter *a* the equation $\left(a^2-3\right)x^2=6a-3$ has exactly 2 roots?

A) for $a \in \left(-\infty; -\sqrt{3}\right)$ B) for $a \in \left(-\sqrt{3}; \sqrt{3}\right)$
C) never D) for $a \in \left(-\sqrt{3}; \frac{1}{2}\right) \cup \left(\sqrt{3}; \infty\right)$.

The time for the students to do the test was 15 min. Each true answer I have estimated with 2 points; no answer with 1 point and wrong answer with 0 points. My aim was to avoid guess-work. The students were able to use and in fact they have used only standard school mathematical methods to solve these problems.

The mean-arithmetic test-result for a student from the marker group was 2,455 points and for the experimental group – 3,25 points. This *eventually* shows a slight advantage that the experimental group gains over the marker group before the beginning of the experiment. I have compared the test-results with the mean-arithmetic marks of the two groups for the first term of the school year. For the experimental group it was 4,54 and for the marker one – 4,33. This fact *eventually* confirms a slight higher level of mathematical skills and knowledge of the experimental group (in Bulgaria the marks vary from poor 2 to excellent 6).

The carrying out of the experiment consisted in this: I have reminded the students of the experimental group of the theme "Graphical solving of equations", which is studied in 8th class and I have introduced them with the tools of GEONExT for plotting graphs of functions. We have solved graphically several linear and quadratic non-parametric equations with GEONExT in order to get acquainted with the program and after that – some parametric equations, which we have solved algebraically (without GEONExT) and also graphically with GEONExT. The work in the marker group consisted *only* in algebraic solving of *the same* parametric equations, namely:

**Problem 1:** Find the number of the roots of the equations, depending on the values of the real parameters *a* and *b*:

a) $\left||x|-a\right|=3$;

b) $\left||x|-3\right|=b$;

c) $\left||x|-a\right|=b$.

**Problem 2:** Find the number of the roots of the equations, depending on the values of the real parameters *a, b* и *c*:

a) $|ax-1|-1=0$;

b) $|x+b|-1=0$;

c) $|x+1|+c=0$.

**Problem 3:** Find the values of the integer parameter $p\in[-10;10]$, for which the equation $|px+1|=|3x+p|$ has exactly one real root.

**Problem 4:** Find the values of the integer parameter $p\in[-10;10]$, for which the equation $|px-2|=|2x+p|$ has exactly one real root.

**Problem 5:** Find the number of the roots of the equation $|x-2|-1=p$ depending on the values of the real parameter *p*.

**Problem 6:** Find the number of the roots of the equation $|x^2-4x+3|=p$ depending on the values of the real parameter *p*.

**Problem 7:** Find the values of the real parameter *k*, for which the equation $x^2-kx+k-2=0$ has diverse real roots.

**Problem 8:** Find the values of the real parameter *k*, for which the equation $x^2-kx+k=0$ has diverse real roots.

**Problem 9:** Find the values of the real parameter *k*, for which the equation $x^2-kx+k=0$ has positive real roots.

**Problem 10:** Find the values of the real parameter *k*, for which the equation $kx^2-x+k-1=0$ has diverse real roots.

I have decided to solve these problems in the experimental group algebraically (without GEONExT) and also graphically with GEONExT with the purpose to see if the visualization of the problem solving with GEONExT helps the students to comprehend the sense of the algebraic procedures for solving of a parametric equation. The way to work with a parameter in GEONExT is to draw its value from the *x*-coordinate of a slider point on the *x*-axis. When we solved the most of the **given problems** we moved the slider point over a certain interval on the *x*-axis and we easily guessed what would be the behavior of the graphs of the functions for further movement of the slider to $\pm\infty$. So we solved the problems for the parameter varying from $-\infty$ to $+\infty$, but this is not a proof. This is a weakness of our method. Other weakness of GEONExT is that one cannot fix very precisely the endpoints of the intervals for the sought values of the parameter – it can be done only with some approximation. But when you solve the same problems algebraically you find them precisely and after that you only visualize the problem solving with GEONExT.

After teaching these problems to the students I have carried out the following test:

1. Find the values of the real parameter *k*, for which the equation $x^2+(k-1)x+k+2=0$ has diverse real roots.

A) there are no such values; B) $k\in(-\infty;-1)\cup(7;\infty)$;

C) $k\in(-\infty;-1)\cup(2;\infty)$; D) $k\in(-2;1)$.

2. Find the values of the real parameter *k*, for which the inequality $x^2+2(k+1)x+9k-5>0$ is true for every *x*.

A) there are no such values; B) $k\in(-\infty;-5)\cup(9;\infty)$;

C) $k\in(-\infty;-9)\cup(2;\infty)$; D) $k\in(1;6)$.

3. Find the values of the real parameter *k*, for which the equation $k|k-x|=2$ has exactly 2 solutions?

A) $k\in[-1;0)$; B) $k\in(0;\infty)$;

C) $k\in(-\infty;-12)\cup(2;\infty)$; D) $k\in(-2;1]$.

4. Find the values of the real parameter *k*, for which the equation $||x|-k|=2$ has exactly 2 solutions.

A) $k\in[-1;5)$ ; B) $k\in(-2;2)$;

C) $k\in(-\infty;-9)\cup(2;\infty)$; D) $k\in[-1;0)$.

5. Find the values of the real parameter *a* for which the equation $(a^2-3)x^2=6a-3$ has exactly 2 solutions?

A) $a\in(-\infty;-\sqrt{3})$ B) $a\in(-\sqrt{3};\sqrt{3})$

C) never D) $a\in\left(-\sqrt{3};\frac{1}{2}\right)\cup(\sqrt{3};\infty)$.

The time for work was 25 min. Each true answer I have estimated with 2 points, no answer with 1 point and wrong answer with 0 points. My aim was to avoid guess-work. The students have used only standard algebraic methods to solve the problems (without GEONExT).

The mean-arithmetic result for the marker group was 6,545455 points and for the experimental group – 5,5 points. The experimental group handled the test worse than the marker group, regardless of the better starting level and the longer time for work (because we have solved each problem with and without GEONExT). But by my personal observation the marker group has done the test very quickly with a tendency of guessing-work. The slight difference in the results is *probably* of accidental character, but it is almost sure, that **in our didactic technology GEONExT does not help students to grasp better the concept parametric equation**. Both the groups have showed low results, regardless of the facts that they are from 11th class, they are profiled in study of mathematics (five hours per week regular work and two hours complementary work) and we have trained them additionally in extra time – in classes of informational technologies. The test results show that **the understanding of the concept parametric equation is very difficult didactic problem.** We have come to the idea to be satisfied at that moment if the students are able to understand the graphical method with visualization with GEONExT. We have decided to explore what good sides may have a didactic technology, which includes using of GEONExT.

We have solved the test problems algebraically in the two groups and also with GEONExT in the experimental group. Since GEONExT made a problem at this lesson on the most computers, I have demonstrated to the students with a beamer how to solve the problems. But it was necessary for the students to work on their computers and therefore we have had additional computer lesson with the experimental group. At this lesson we have solved 5 similar problems.

The end of the experiment was a test, which the students of the experimental group have been allowed to solve in traditional algebraic way or with GEONExT. All of them have chosen to work with GEONExT. This is a strong argument pro GEONExT. This is the test:

1. Find the values of the real parameter $k \in [-10;10]$, for which the equation $x^2 + (k-1)x + k + 2 = 0$ has no real roots.

A) $k \in (-1;\, 7)$; B) there are no such values;

C) $k \in \left(-1-\sqrt{10};\, -1+\sqrt{10}\right)$; D) $k \in (-2;\, 1)$.

2. Find the values of the real parameter $k \in [-10;10]$, for which the equation $x^2 + 2(k+1)x + 9k - 5 = 0$ has no real roots.

A) there are no such values; B) $k \in \left(\frac{7-\sqrt{73}}{2};\, \frac{7+\sqrt{73}}{2}\right)$;

C) $k \in [-10;\, 2) \cup (9;\, 10]$; D) $k \in (1;\, 6)$.

3. Find the values of the real parameter $k \in [-10;10]$, for which the equation $k|k-x| = 0$ has infinite number of solutions.

A) $k \in [-1;\, 0)$; B) $k \in (0;\, 10]$;

C) there are no such values; D) $k = 0$.

4. Find the values of the real parameter $k \in [-10;10]$, for which the equation $\big||x| - k\big| = 2$ has exactly one real root.

A) $k \in [-1;\, 5)$; B) $k = -2$;

C) $k = 2$; D) $k \in [-1;\, 0)$.

5. Find the values of the real parameter $a \in [-10;10]$ for which the equation $(a^2 - 3)x^2 = 6a - 3$ has exactly one real root?

A) $a \in [-10;\, -\sqrt{3})$; B) $a \in (-\sqrt{3};\, \sqrt{3})$;

C) $a = \frac{1}{2}$; D) there are no such values.

6. Find the values of the real parameter $a \in [-10;10]$ for which the equation $|a-x| = 2$ has exactly one real root.

A) only if *a > 2;* B) only if $a \leq 2$;

C) there are no such values*;* D) for every *a*.

7. Find the values of the real parameter $a \in [-10;10]$ for which the equation $(a^2-3)x^2 = 2$ has exactly one real root.

A) only if *a > 2;* B) there are no such values;

C) for $a \in (-10;-\sqrt{3}) \cup (\sqrt{3};10)$; D) for every *a*.

8. Find the sum of the integer values of the parameter $a \in [-4;4]$, for which the equation $|\mathrm{x}|\text{-ax} = \frac{1}{2}$ has exactly one real root.

A) $\frac{1}{2}$; B) there are no such values;

C) 0; D) for every *a*.

9. Find the sum of the integer values of the parameter $a \in [-10;10]$, for which the equation $||x|-a| = x$ has exactly one real root.

A) -10; B) 55;

C) there are no such values; D) 0.

10. Find the sum of the values of the parameter $a \in [-2;2]$, for which the equation $(a^2-3)x^2 = x$ has exactly one real root.

A) 3*;* B) there are no such values;

C) 0; D) for every *a* the statement is true.

11. Find the values of the real parameter $a \in [-10;10]$ for which the equation $(a^2-3)x^2 = a-1$ has exactly one real root.

A) $\mathrm{a} \in [\text{-}10;\text{-}\sqrt{3})$; B) $a \in (-\sqrt{3};\sqrt{3})$;

C) $a = 1$; D) there are no such values.

12. Find the values of the real parameter $a \in [-5;5]$ for which the equation $||x|-a| = |a-x|$ has infinite number of solutions.

A) for every $a \in [-5;5]$; B) there are no such values;

C) $a \in [-5;0]$; D) $a \in [0;5]$.

The parameter in these problems always varies in certain (short) intervals unlike in the problems in the previous two tests. This is because of the permission to the students in the experimental group to use GEONExT with the aim to avoid the weakness that we have earlier commented. For each true answer – again 2 points, but this time for wrong answer or no answer – 0 points. One reason for this decision is that I hadn't avoided the guess-work in the previous tests. Second reason is that I have decided to follow the practice for carrying out of diagnostic tests [1]. The previous observations have showed that the students quickly "solve" the problems and the time for a solving of a problem could be shortened. Especially with GEONExT can be achieved good speed of true problem-solving. To show this advantage of GEONExT I have decided to give 40 min for 12 problems. By private problems the true time for work was shortened to 30 min. The mean-arithmetic of the marker group was very poor - 5,8333333 points and for the experimental group – very high – 16,1666667.

3. **Conclusions.** Despite the preliminary leadership of the experimental group and the longer time to work with them, theirs high achievement probably could not be explained only with these factors. The more so as the same factors have been also at the second test, where the experimental group had worse result. This third test shows the following advantages of GEONExT:

1. The students are more interested and motivated to work.
2. The necessary time for solving of a problem is shorter.
3. The possibility for misunderstanding and to make mistakes is very low.
4. The new technologies of education and especially GEONExT-based are coming at school.

Because of different our mistakes, gaps, uncertain points during the work and our poor practice it is necessary for us to continue with betterment of our didactic technology and experiments. Further experiments are also necessary to prove the advantages of GEONExT that we diagnosed.

**References:**

[1] Tabov J., For the multiple choice tests, Mathematics and Informatics, issue 1/2007, pp. 4-10 (in Bulgarian: Табов Й., За тестовете от задачи с избираем отговор, Математика и информатика, кн. 1/2007, стр. 4-10).