More on the Inscribed Quadrilateral

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Different kinds quadrilaterals: parallelogram, trapezium, kite, pseudo square, threelateral, inscribed (cyclic), circumscribed. The latter two kinds are studied by Ptolemy, Brocard, Newton, etc. Soon were discovered multitude interesting points in arbitrary quadrilateral and through their properties – two important additions to Brocard’s theorem for the inscribed quadrilaterals.

Regional and global mathematical competitions include (geometric) problems of high difficulty and projects, including such about quadrilaterals studied.

Students should be more active: not only solve math problems but formulate and explore them, which is rare even in the educational programs for the exercises for mathematics competitors. In Bulgaria are rare didactical works and experiments in this direction.

Real exploration may rise too complex questions, require co-work of teachers and students, simplifying and modeling the problem, collecting and analyzing data, linking the conveyed conclusions with available facts and knowledge, interpret and share them with others.

The teachers in the classroom frequently tell the students what they have to observe, give them ready questions, show them the necessary methods and examine the results. And the students have only follow the teacher’s instructions.

This approach is challenges and motivates the students to search for the links between the facts and the laws, developing key skills as ***critical*** thinking, self reflection, planning experiment, analyze and presenting the results.
Huge part of the problems and the properties of the inscribed and circumscribed quadrilaterals given here are obtained by students in explorations during co-working with them.

Inscribed quadrilateral in simple constructions.

Task 1. ABC is an isosceles triangle of base AB. The intersection points of the chords CD and CE of its' circum-circle with it's side AB are M and N respectively. Prove that the quadrilateral $EDMN$ is inscribed.

Solution. (Fig.1) As the arcs AC = BC, then
the arcs DAC = AD + CB, hence <CMN = <CED;
analogically <CNM = <CDE. Then the opposite angles of $EDMN$ make sums of 180°.

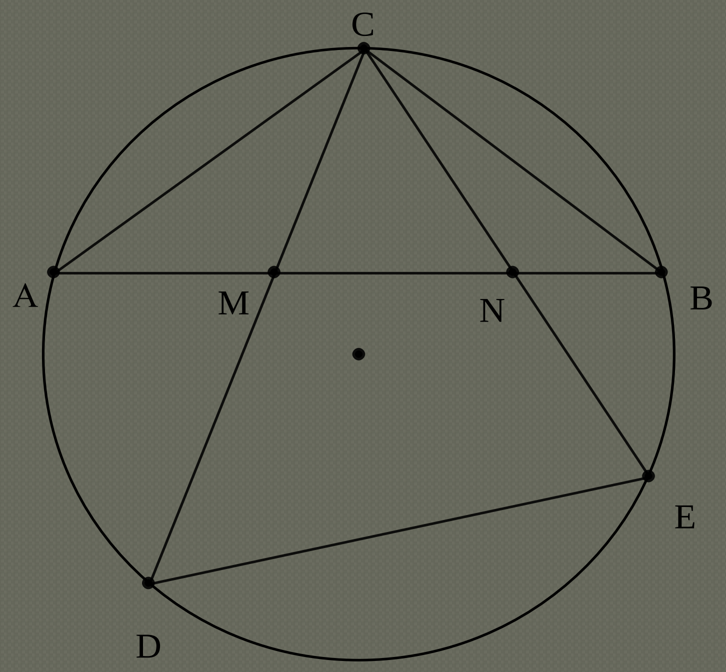


Fig. 1

Task 2. The bisectors of the interior angles of the quadrilateral $ABCD$ intersect at the points K,L,M and N, as it is shown in Fig.2. Prove, that the quadrilateral $KLMN$ is inscribed.

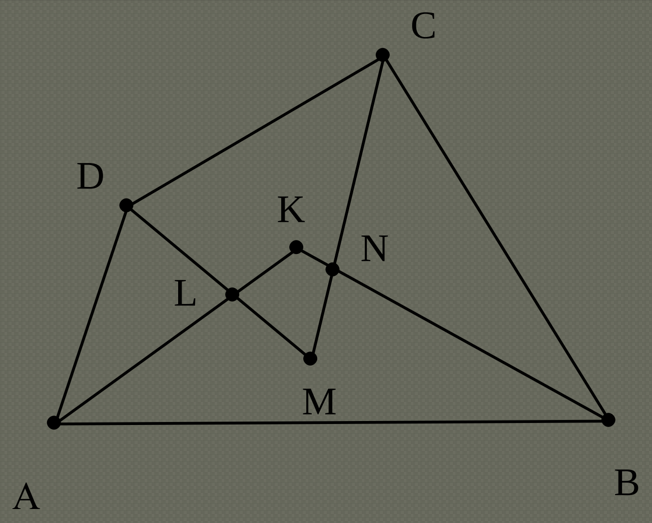


Fig. 2

Solution2. $\measuredangle AKB = 180° - \frac{1}{2} \cdot (\alpha + \beta)$, $\measuredangle DMC = 180° - \frac{1}{2} \cdot (\gamma + \delta)$.

Hence

$$\measuredangle LKN + \measuredangle LMN = 360° - \frac{1}{2} \cdot (\alpha + \beta + \gamma + \delta) = 360° - \frac{1}{2} \cdot 360° = 180°$$

and $KLMN$ is inscribed.

Task *3.* Each of the four given circles is tangent to two other, as it is shown in Fig. 3. Prove that the quadrilateral with vertices the common points A, B, C and D of the couples of circles is inscribed.

Solution 3. S - point on the mutual inner tangent of the circles of centers O_1 and O_2 (Fig. 3).

$\measuredangle DAS$ and $\measuredangle BAS$ as peripheral angles:

$$\measuredangle DAB = \measuredangle DAS + \measuredangle BAS = \frac{1}{2}\cdot DA + \frac{1}{2}\cdot BA = \frac{1}{2}\cdot(\measuredangle DO_1A + \measuredangle BO_2A)$$

from the other two circles:

$$\measuredangle DCB = \frac{1}{2}\cdot\left(\measuredangle DO_4C + \measuredangle BO_3C\right)$$

Adding the two equalities, we obtain:

$$\measuredangle DAB + \measuredangle DCB = \frac{1}{2}\cdot\left(\measuredangle DO_1A + \measuredangle BO_2A\right) + \frac{1}{2}\cdot\left(\measuredangle DO_4C + \measuredangle BO_3C\right) = \frac{1}{2}\cdot 360° = 180°$$

Hence $ABCD$ is inscribed.

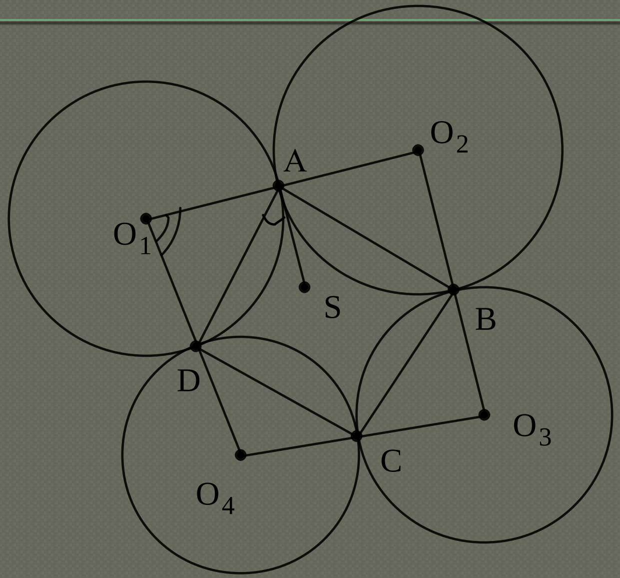


Fig. 3

Task 4. The sides AC and BC of the triangle ABC are tangent to it's in-circle at the points D and F respectively. The bisectors of the internal angles A and B intersect the line DF at the points N and M respectively. Prove that the points A, B, M and N lie on the same circle.

Solution 4. $\measuredangle CFD$ is external for $\triangle BFM$ and $\measuredangle BMF = \measuredangle CFD - \measuredangle FBM$.

As $\measuredangle FBM = \dfrac{\beta}{2}$ and $\measuredangle CFD$ is internal for the isosceles $\triangle DFC$, therefore $\measuredangle CFD = 90° - \dfrac{\gamma}{2}$. Hence

$$\measuredangle BMN = \measuredangle BMF = \left(90° - \frac{\gamma}{2}\right) - \frac{\beta}{2} = \frac{\alpha}{2} = \measuredangle BAN$$

and BN is seen at the same angle from M and A, hence A, B, M and N are concyclic.

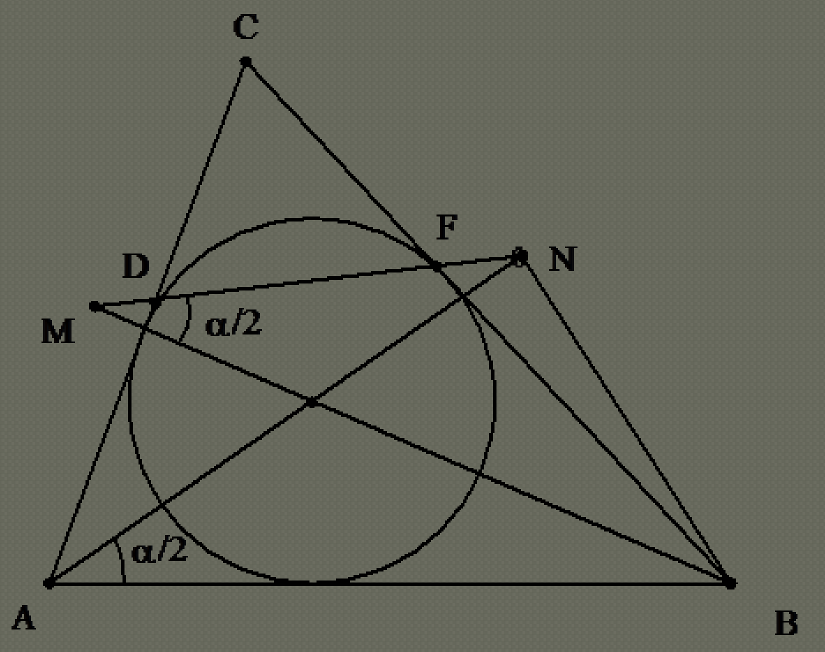


Fig. 4

Task 5. *N* is point on the side *AB* of the quadrilateral *ABCD*, for which $AN = AD$ and $BN = BC$. The bisectors of the angles ADC and BCD intersect at point *M*. Prove, that the points *C*, *D*, *M* and *N* lie on the same circle.

Solution 5. *DM* and *CM* are bisectors, $\triangle NDA$ и $\triangle NCB$ are isosceles, then:

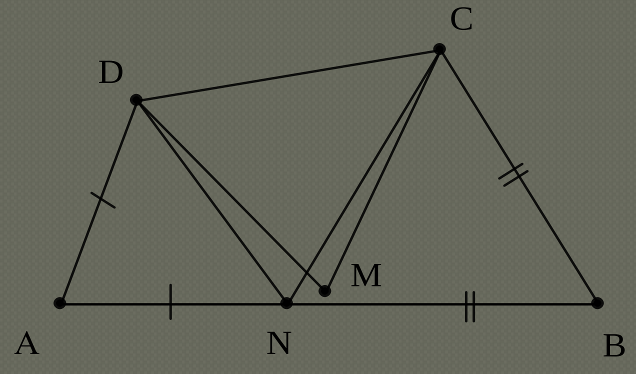
$$\measuredangle DMC = 180° - \frac{1}{2}\cdot(\measuredangle ADC + \measuredangle BCD) = 180° - \frac{1}{2}\cdot(360° - \measuredangle DAB - \measuredangle ABC) = \frac{1}{2}\cdot(\measuredangle DAB + \measuredangle ABC)$$

$$\measuredangle AND = 90° - \frac{1}{2}\cdot\measuredangle DAB, \ \measuredangle BNC = 90° - \frac{1}{2}\cdot\measuredangle ABC$$

and

$$\measuredangle DNC = 180° - (\measuredangle AND + \measuredangle BNC) = \frac{1}{2}\cdot(\measuredangle DAB + \measuredangle ABC),$$

and then $\measuredangle DMC = \measuredangle DNC$, and *C*, *D*, *M*, *N* are concyclic.



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Task 6. The bisector of the acute angle A of the paralellogram $ABCD$ crosses its side CD at point L and the continuation of BC at point K. Let O be the circum-center of $\triangle LCK$. Prove, that the points D, B, C and O are cocircular (Spring mathematics tournament, Kazanlak, 1993).

Solution 6. AK is bisector (Fig. 6), $AB \parallel CD$, $AD \parallel BC$, then

$\measuredangle CLK = \measuredangle BAL = \measuredangle LAD = \measuredangle KAD = \measuredangle AKC = \measuredangle CKL$.

Hence $CL = CK$ and analogically $AD = DL$. Then $DC = DL + LC = AD + CK = BC + CK = BK$, i.e. $DC = BK$.

O is the circum-center of $\triangle LCK$, then $OL = OC = OK$. From this and $LC = CK$ we obtain $\triangle LCO \cong \triangle CKO$.

Therefore $\measuredangle LCO = \measuredangle OCK = \measuredangle CKO$, i.e. $\measuredangle DCO = \measuredangle BKO$, then $\triangle DCO \cong \triangle BKO$; then $\measuredangle CBO = \measuredangle CDO$.

CO is seen at the same angle from B and D, which lie in the same semiplane. Therefore D, B, C, O are cocircular.

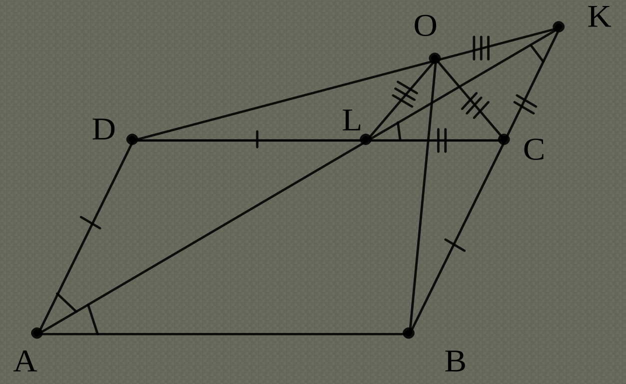

Fig. 6

2. Application of the properties of the inscribed quadrilateral in solving problems.

Interesting is when one needs to notice a cyclic tetragon and use its properties. Typical examples: proofs of two popular properties of triangle’s orthocenter:

Task 7. Let ABC be a triangle with orthocenter H. Prove that the points, symmetric to H with respect to:

a) triangles’s sides;

b) the mid-points of the sides:

lie on the triangle’s circum-circle.

Solution 7. Let ABC be an acute triangle and:

a) L be symmetric of H with respect to AC (Fig .7). Then $\measuredangle CLM = \measuredangle CHM$ and $\measuredangle CHM = \measuredangle CAB$ (acute angles, perpendicular arms). Then $\measuredangle CLM = \measuredangle CAB$, i.e. BC is seen at equal angles from A and L, which lie at the same side of BC, and the points A, B, C, L lie on the circumcircle of $\triangle ABC$;

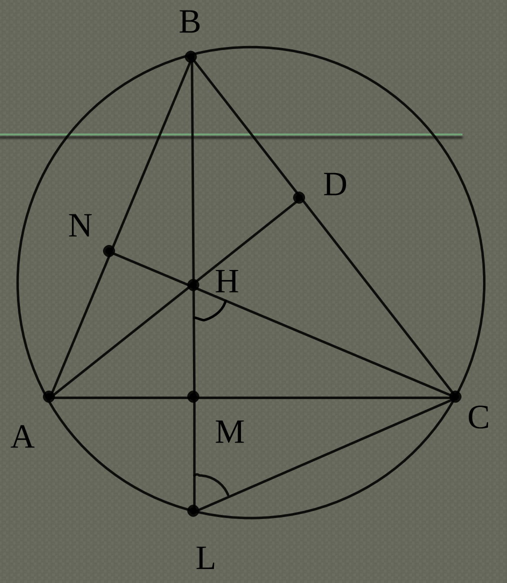

Fig. 7

b) A' - diametrically opposed of A on the circum-circle of $\triangle ABC$ (Fig. 8). We'll prove A' is symmetric of H with respect to midpoint A_1 of BC.

Angles ABA', ACA' subtend by diameter AA'=> they are right, $A'B \perp AB$ and as $CH \perp AB$ (H is orthocenter of $\triangle ABC$), then $A'B \| CH$.

Analogically $A'C \| BH$, then $BA'CH$ is a parallelogram, its diagonals BC and HA' has common midpoint A_1, then $A'A_1 = HA_1$, *which proves the statement*.

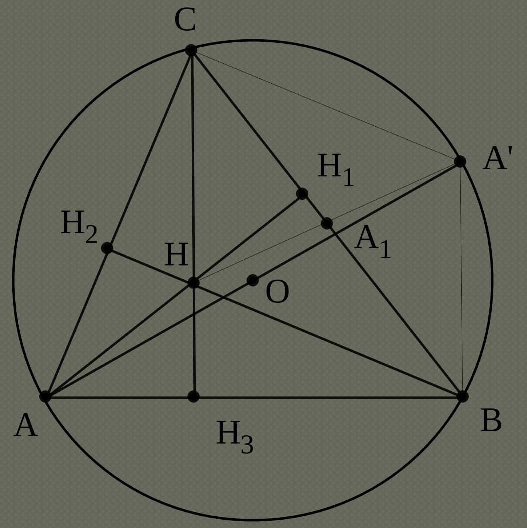

Fig. 8

Task 8. The triangles ABR, BCP, CAQ are constructed on the sides AB, BC and CA of the triangle ABC, out of it. If the angles ARB, BPC, CQA make sum $180°$, prove that the circumcircles of ABR, BCP, CAQ have common point.

Solution 8. Let F be the second mutual point of the circumcircles of the triangles BCP and CAQ (Fig. 9). Then $\measuredangle BFC = 180° - \measuredangle BPC$ and $\measuredangle CFA = 180° - \measuredangle AQC$, and therefore

$$< AFB = 360 - (< BFC + < CFA) =$$

$$= < BPC + < AQC = 180 - < ARB$$

Hence point F lies on the circumcircle of $\triangle ABR$.

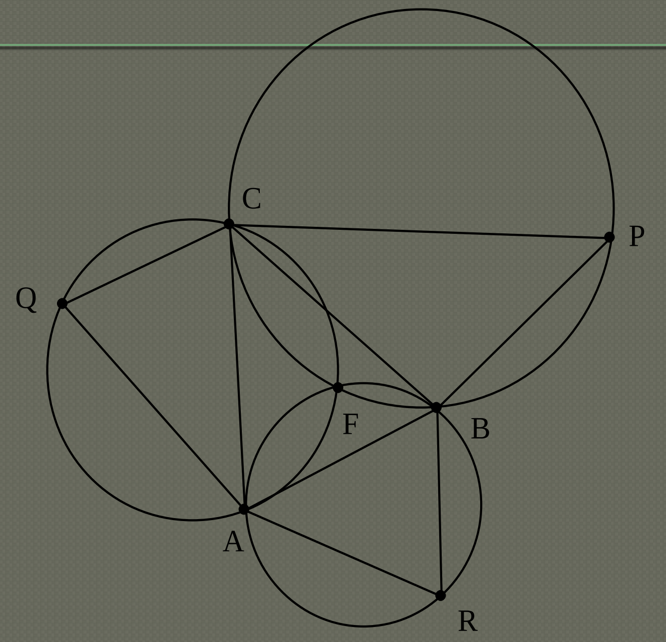

Fig. 9

Task ***9.*** (Mikel's theorem) *A*, *B* and *C* are arbitrary points on the sides *QR*, *PR*, *PQ* of the triangle *PQR*. Prove that the circumcircles of the triangles *ABR*, *BCP*, *CAQ* have common point.

Solution 9. The sum of the angles *ARB*, *BPC* and *CQA* is 180° as the triangles *ABR*, *BCP*, *CAQ* are on the sides *AB*, *BC*, *CA* of *ABC*, out of it. According to Task 8 the circumcirclses of the triangles *ABR*, *BCP*, *CAQ* have a common point.

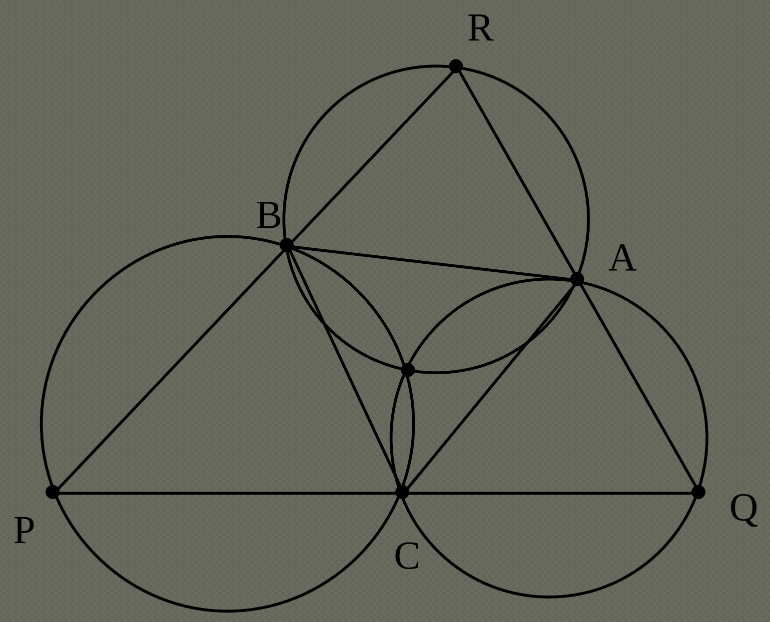

Fig. 10

Task ***10.*** The quadrilateral $ABCD$ is inscribed and its diagonal AC is its diameter. Prove that the projections of every two opposite sides on the diagonal BD are equal.

Solution 10. A_1 and C_1 - projections of A and C on BD (Fig.11). It is sufficient to prove that $BA_1 = DC_1$.

O_1 - projection of circumcentre O.

O is midpoint of AC; O_1 - midpoint of A_1C_1, then $O_1A_1 = O_1C_1$.

OO_1 lies on diameter and it is perpendicular of BD therefore OO_1 halves BD. From $O_1A_1 = O_1C_1$ and $O_1B = O_1D$ follows $BA_1 = DC_1$.

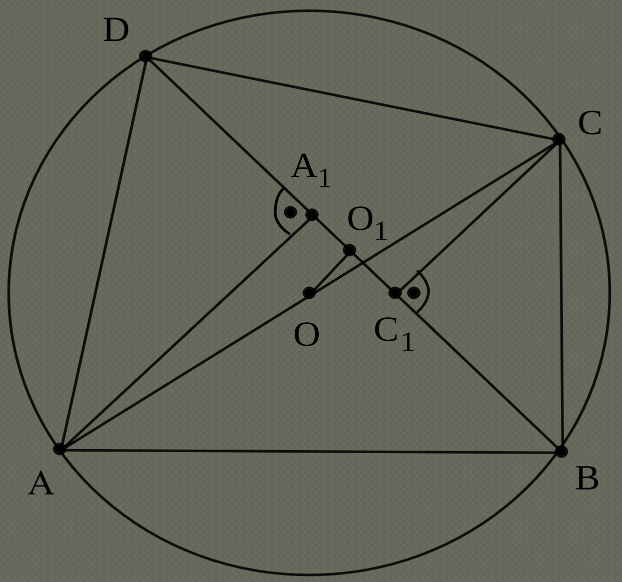

Fig. 11

Here we prove that four points are concyclic, proving first and using that two quadrilaterals are inscribed.

Task ***11.*** ABC is a right-angled triangle with acute angles $\measuredangle BAC = \alpha$ and $\measuredangle ABC = \beta$, and incenter J. The point D in <ACB is such, that $\measuredangle ADC = 90° - \frac{\beta}{2}$ and $\measuredangle BDC = 90° - \frac{\alpha}{2}$. The bisectors l_1 and l_2 of the catheti BC and AC cross the lines AD and BD respectively at the points P and Q. Prove that P, D, Q and J are conciclic points.

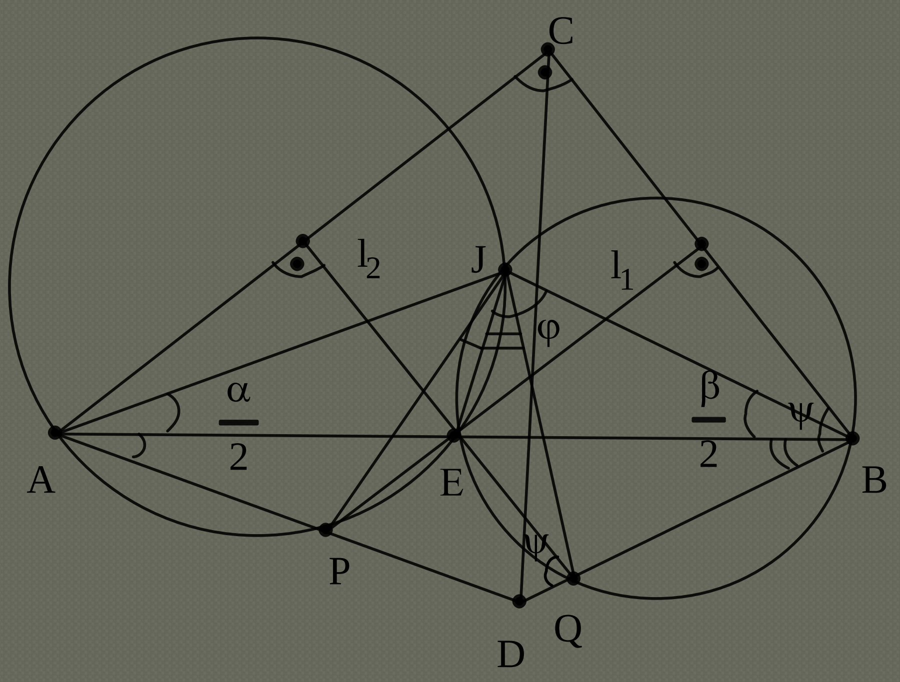

Fig. 12

Solution 11. l_1 and l_2 cut AB at midpoint E and $EQ \parallel BC$, $EP \parallel AC$ (Fig. 12). Wi first prove J, E, Q and B are concyclic.

Let $\measuredangle EJB = \varphi$, $\measuredangle EQD = \psi$.

As $\measuredangle AJB = 135°$, then $JE \cdot JB \cdot \sin\varphi = 2 \cdot S_{EJB} = 2 \cdot S_{EJA} = JE \cdot JA \cdot \sin(135° - \varphi)$, then

$$\frac{\sin(135° - \varphi)}{\sin\varphi} = \frac{JB}{JA} = \frac{\sin\frac{\alpha}{2}}{\sin\frac{\beta}{2}}, \text{ i.e.}$$

(1) $$\frac{\sin\frac{\alpha}{2}}{\sin\frac{\beta}{2}} = \frac{\sin(135° - \varphi)}{\sin\varphi}$$

As $EQ \parallel BC$ (from above), we obtain:

(2) $$\measuredangle DBC = \measuredangle DQE = \psi$$

We easy calculate $\measuredangle ADB = \measuredangle ADC + \measuredangle BDC = 135°$.

From $ADBC$: $\measuredangle DAC = 360° - 90° - 135° - \measuredangle DBC = 135° - \psi$, i.e.

(3) $$135° - \psi = \measuredangle DAC$$

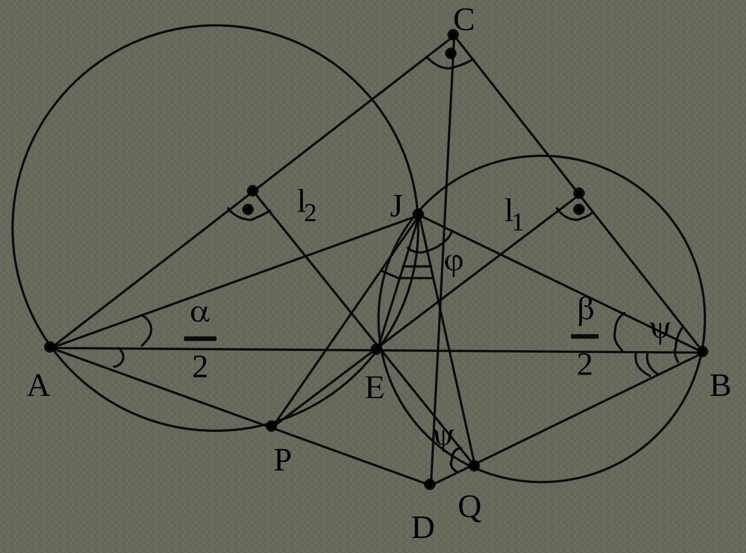

Fig. 12

Sine rule for ΔDBC provides $\sin\measuredangle DBC = \frac{CD}{BC} \cdot \sin\measuredangle BDC = \frac{CD}{BC} \cdot \sin(90° - \frac{\alpha}{2}) = \frac{CD}{BC} \cdot \cos\frac{\alpha}{2}$

Analogically $\sin\measuredangle DAC = \frac{CD}{AC}\cdot\cos\frac{\beta}{2}$; dividing these two equalities jointly, we obtain:

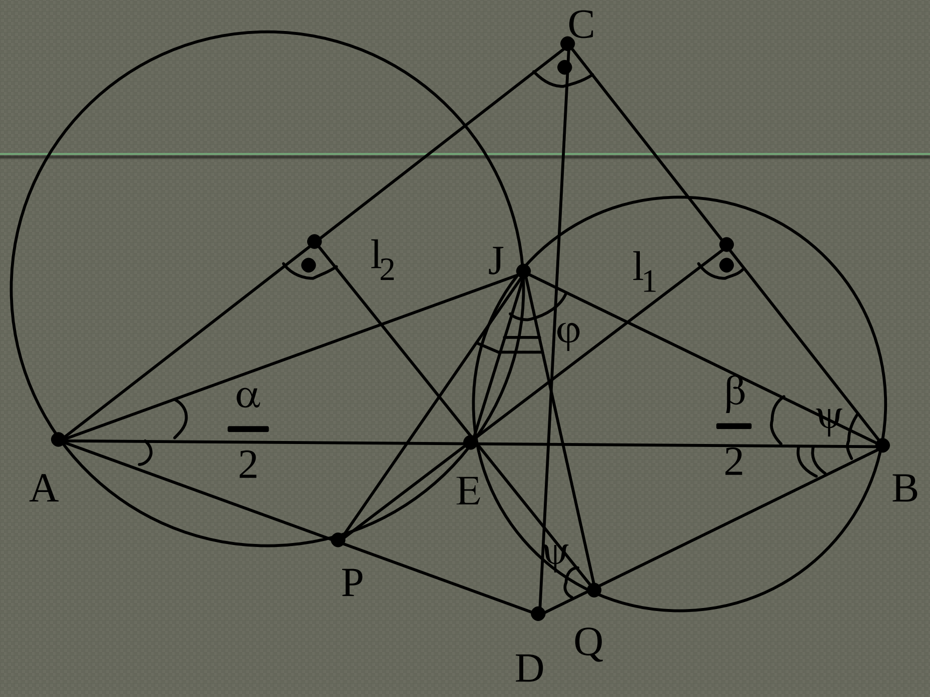

Fig. 12

(1) $$\frac{\sin\measuredangle DAC}{\sin\measuredangle DBC} = \frac{BC}{AC}\cdot\frac{\cos\frac{\beta}{2}}{\cos\frac{\alpha}{2}}$$

After consequtive applying of (3), (2), (4), (1) we find

$$\frac{\sin(135^\circ-\psi)}{\sin\psi} = \frac{\sin\measuredangle DAC}{\sin\measuredangle DBC} = \frac{BC}{AC}\cdot\frac{\cos\frac{\beta}{2}}{\cos\frac{\alpha}{2}} =$$

$$= \frac{\sin\alpha}{\sin\beta}\cdot\frac{\cos\frac{\beta}{2}}{\cos\frac{\alpha}{2}} = \frac{\sin\frac{\alpha}{2}}{\sin\frac{\beta}{2}} = \frac{\sin(135^\circ-\varphi)}{\sin\varphi}$$

After simple transfornations we get $\cot g\psi = \cot g\varphi$, i.e. $\psi=\varphi$, i.e. $\measuredangle EQD = \psi = \varphi = \measuredangle EJB$. Therefore $JEQB$ is cyclic. $JEPA$ is cyclic too (analog.) and $\measuredangle PJQ = \measuredangle PJE + \measuredangle QJE = \measuredangle PAE + \measuredangle QBE = 180^\circ - \measuredangle PDQ$ i.e. $\measuredangle PJQ + \measuredangle PDQ = 180^\circ$, i.e. the points P, D, Q, J are concyclic.

1.3. Some popular and interesting properties of the cyclic tetragon.

Property 1. The bisectors of the angles, formed by the opposite sides of a cyclic teragon, are parallel to the bisectors of the angles, between its'diagonals.

Proof 1**:** $ABCD$ - cyclic teragon; opposite sides $BC \cap AD = E$ (Fig.13). Let bisector of $\measuredangle CED$ cuts the circum-circle of $ABCD$ = $L\,;M$ (Fig .13), PQ is the line trough $S \| LM$. Enough to show SQ is bisector of $\measuredangle CSD$. As the arcs PL = QM and $\measuredangle MED = \measuredangle CEM$, we obtain consecutively:

$$\measuredangle CSQ = \frac{1}{2}\cdot\left(AP + CQ\right) = \frac{1}{2}\cdot\left(AL + LP + CM - QM\right) = \frac{1}{2}\cdot\left(AL + CM\right) =$$

$$= \frac{1}{2}\cdot\left[\left(MD - 2\cdot\measuredangle MED\right) + \left(BL + 2\cdot\measuredangle CEM\right)\right] = \frac{1}{2}\cdot\left(MD + BL\right) =$$

$$= \frac{1}{2}\cdot\left(MD + QM + BL - PL\right) = \frac{1}{2}\cdot\left(QD + BP\right) = \measuredangle QSD$$

Hence the so constructed line is the bisector of $\measuredangle CSD$.

Analogically: the line through $S \|$ bisector of < (AB, CD) halves $\measuredangle BSC$.

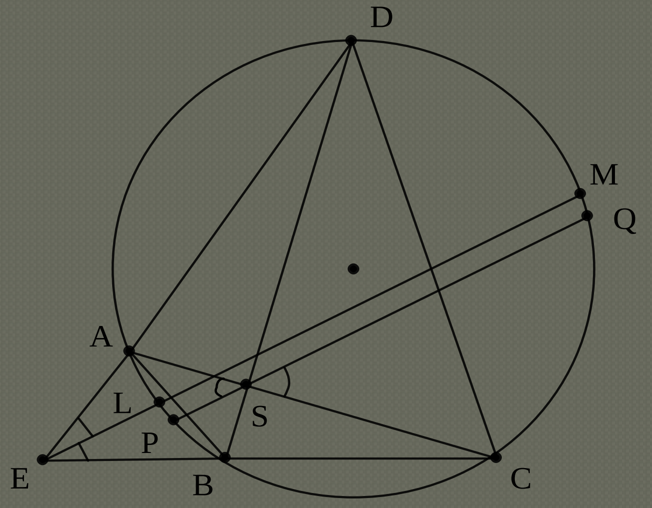

Fig. 13

Property 2. The bisectors of the angles between the opposite sides of a cyclic tetragon are perpendicular and cut it at points, which are vertices of a rhombus.

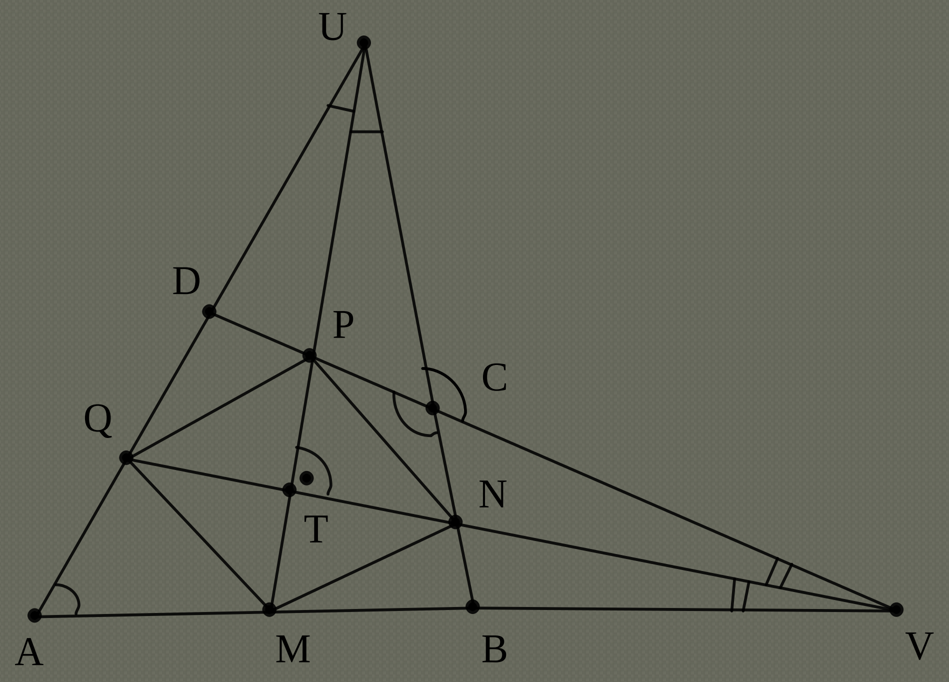

Fig. 14

Proof 2. Let $U=AD\cap BC$, $V=AB\cap CD$

M,N,P,Q - cuts of bisectors of $<AUB$, $<AVD$ with AB, BC, CD, DA resp. (Fig.14).

If $MP\cap NQ=T$, from $AVTU$ we obtain:

$\measuredangle UTV-\measuredangle UAV=\measuredangle AUT+\measuredangle AVT$, and (analog) from $TVCU$: $\measuredangle UCV-\measuredangle UTV=\measuredangle CUT+\measuredangle CVT$

As UT, VT are bisectors of <AUB and <AVD, then: $\measuredangle AUT+\measuredangle AVT=\measuredangle CUT+\measuredangle CVT$.

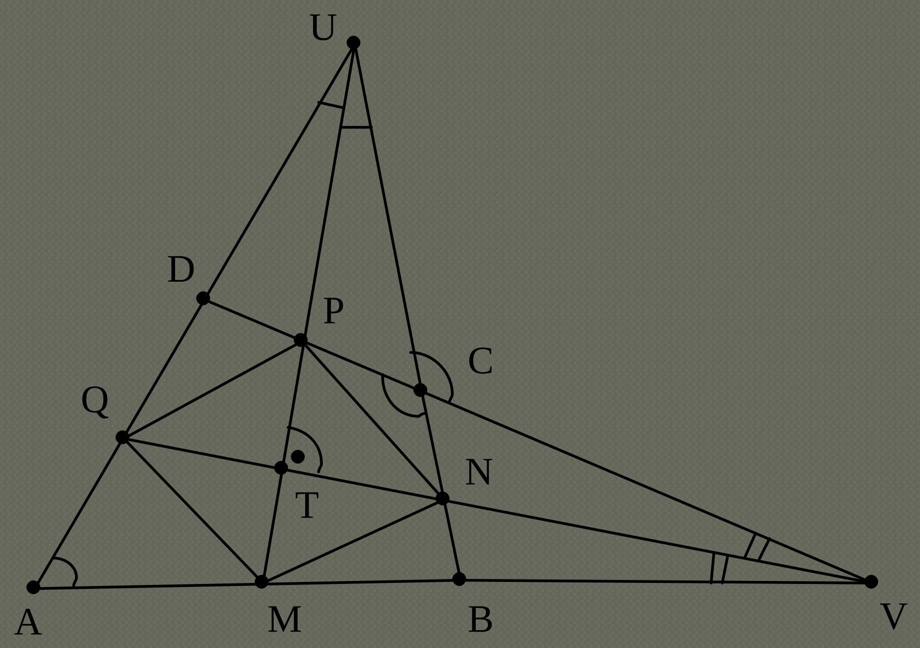

Fig. 14

The right sides of the upper equalities are equal, then are aqual their left sides, i.e. $\measuredangle UTV - \measuredangle UAV = \measuredangle UCV - \measuredangle UTV$, then $<UTV = (<UAV + <UCV)/2 =$

$= (<UAV + < DCB)/2 = 180/2 = 90^{\circ}$. So, the bisectors $UM \perp VQ$ of $<AUB$ and $<AVD$. Then UT: altitude and bisector in ΔCUN, and then median to QN, hence T is midpoint of QN. Analogically T is midpoint of PM in $MNPQ$ and it's a parallelogram (with perpendicular diagonals), so it's a rhombus.

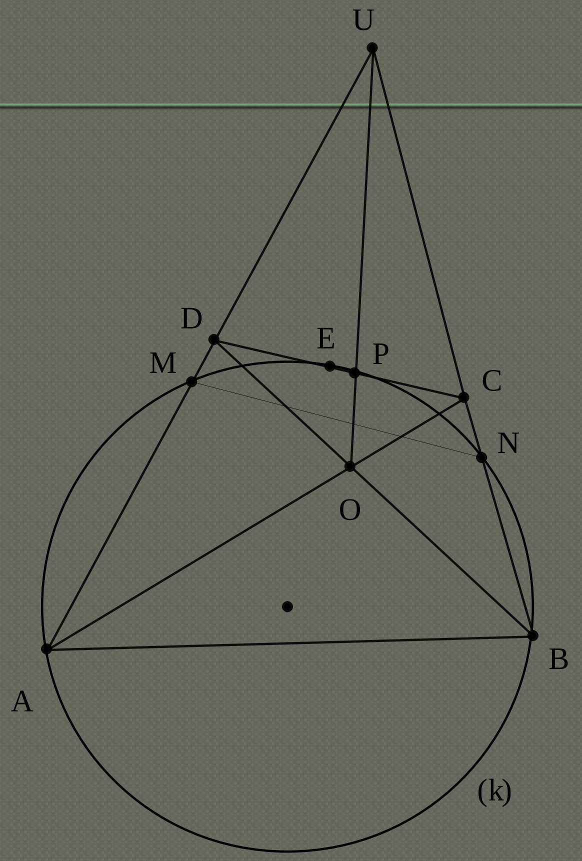

Fig. 15

Property ***3.*** The diagonals $AC \cap BD = O$ of the cyclic tegragon $ABCD$, and the continuatitons of sides AD$\cap BC = U$. If $OU \cap DC = P$, then the circle (k) through A, B and P passes through the midpoint of CD.

Proof 3: Let $(k) \cap DC = E$, $(k) \cap AU = M$, $(k) \cap BU = N$. We'll prove E is midpoint of DC. From secant lines property we have $DE \cdot DP = DM \cdot DA$ and $CE \cdot CP = CN \cdot CB$, and then, by jointly division we get $\frac{DE \cdot DP}{CE \cdot CP} = \frac{DM \cdot DA}{CN \cdot CB}$. Then, to prove $DE = CE$, we'll show that

(1) $$\frac{DP}{CP} = \frac{DM \cdot DA}{CN \cdot CB}$$

As $\measuredangle ADB = \measuredangle ACB$, then $\measuredangle ODU = \measuredangle OCU$, and therefore

$$\frac{DP}{CP} = \frac{S_{DOU}}{S_{COU}} = \frac{\frac{1}{2} \cdot DU \cdot DO \cdot \sin \measuredangle ODU}{\frac{1}{2} \cdot CU \cdot CO \cdot \sin \measuredangle OCU} = \frac{DU \cdot DO}{CU \cdot CO}$$

Thus (1) is equivalent to

(1) $$\frac{DU \cdot DO}{CU \cdot CO} = \frac{DM \cdot DA}{CN \cdot CB}$$

As $\measuredangle NMU = 180° - \measuredangle AMN = \measuredangle ABU$ and $\measuredangle ABU = 180° - \measuredangle ADC = \measuredangle CDU$, we have $\measuredangle NMU = \measuredangle ABU = \measuredangle CDU$. Then $MN \parallel DC$, therefore $\frac{DU}{CU} = \frac{DM}{CN}$ and (2) is thus equivalent to $\frac{DO}{CO} = \frac{DA}{CB}$. But $\triangle AOD \sim \triangle BOC$, hence the last equality *is true*.

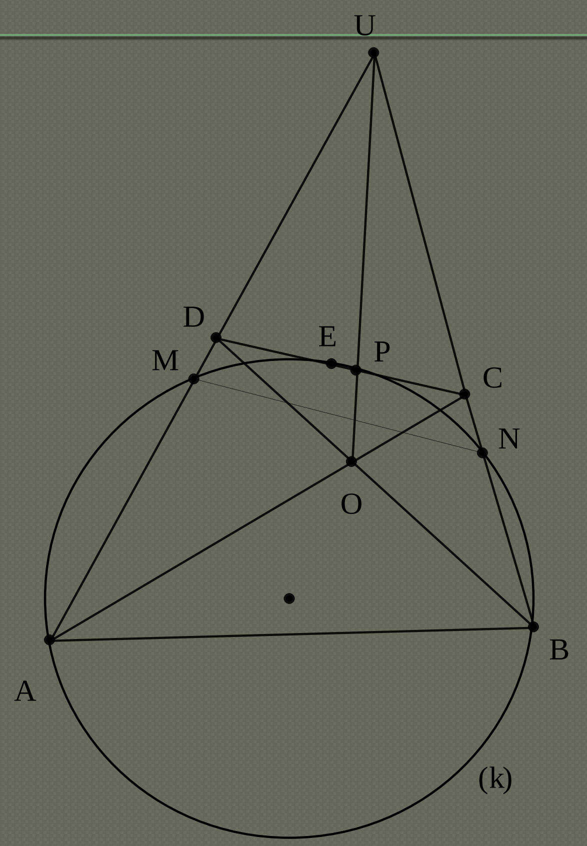

Fig. 15

1.4. Properties of some subkinds of cyclic tetragons and their application in solving problems.

Cyclic tetragons, satisfying more requirements, have more additional properties. We consider here cyclic tetragons with perpendicular diagonals and *harmonic* ones (the product of any two two their opposite sides equals the product of the other two).

Property 4. If $ABCD$ is a cyclic tetragon with circum-center O and perpendicular diagonals then $\measuredangle AOB + \measuredangle COD = 180°$.

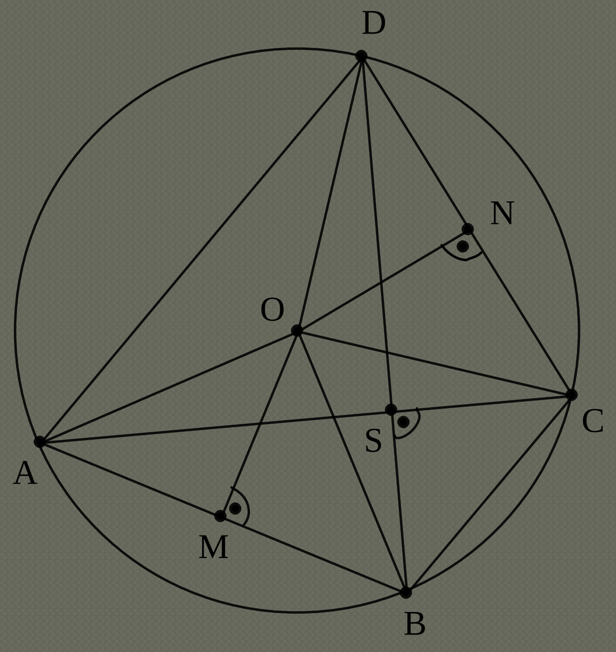

Fig. 16

Proof 4: Let S=$AC \cap BD$. We have $\measuredangle AOB = AB = 2 \cdot \measuredangle ADB$ and analogically $\measuredangle COD = CD = 2 \cdot \measuredangle CAD$.

Therefore $\measuredangle AOB + \measuredangle COD = 2 \cdot (\measuredangle ADB + \measuredangle CAD) = 2 \cdot (180° - \measuredangle ASD) = 2.90° = 180°$.

Property 5. If *ABCD* is cyclic tetragon with circum-center *O* and perpendicular diagonals, the distance from *O* to any of its sides is half the length of the opposite side.

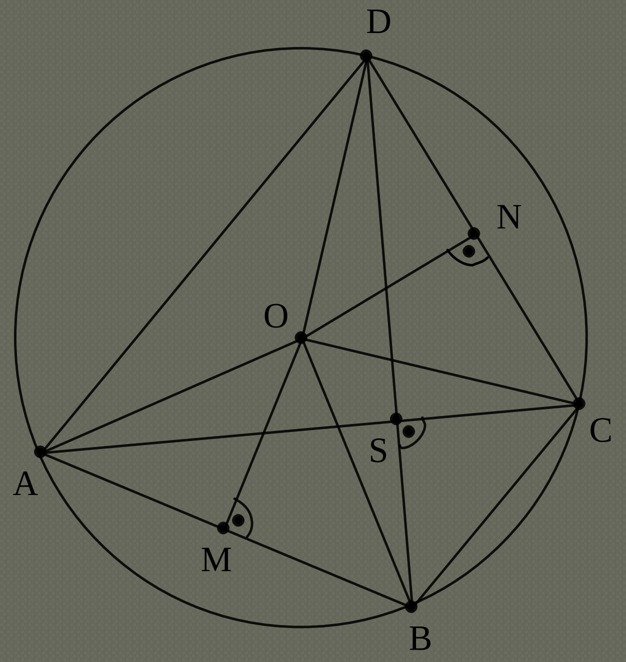

Fig. 16

Proof 5: If $OM \perp AB$, $ON \perp CD$, then *M*, *N* are midpoints of chords *AB* and *CD*. From Property 4 $\measuredangle AOB + \measuredangle COD = 180°$ and therefore $\measuredangle AOM + \measuredangle DON = 90°$. But $\measuredangle AOM + \measuredangle OAM = 90°$ (AOM is right triangle) and $\measuredangle DON = \measuredangle OAM$. From $DO = OA$, $\measuredangle DON = \measuredangle OAM$ and $\measuredangle DNO = \measuredangle OMA = 90°$ follows $\triangle DON \cong \triangle OAM$ and then $OM = DN = \frac{1}{2} \cdot CD$.

We prove analogically $ON = \frac{1}{2} \cdot AB$.

Property 6 (discovered in 7th century from Indian mathematician Brahmagupta)**.** If the diagonals of cyclic tetragon are perpendicular, then the line through their common point, perpendicular to any of its sides, halves the opposite one.

Proof 6: Let $AC \perp BD$, $AC \cap BD = S$ and l is line throgh S, perpendicular to AD. Let it meets BC, AD at M, N. From the right $\triangle ADS$ we have $\measuredangle NSA = \measuredangle ADS$, but $\measuredangle NSA = \measuredangle CSM$ and $\measuredangle ADS = \measuredangle ADB = \measuredangle ACB = \measuredangle SCM$. Then $\measuredangle CSM = \measuredangle NSA = \measuredangle ADS = \measuredangle SCM$, i.e. $\measuredangle CSM = \measuredangle SCM$, hence $CM = SM$. Analogically $BM = SM$, hence M is midpoint of BC.

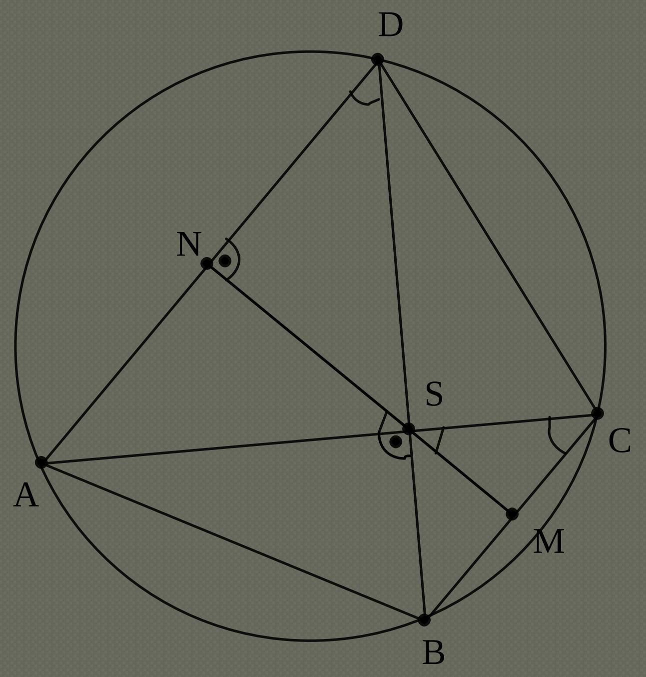

Fig. 17

Property 7. In circle of center O is inscribed tetragon $ABCD$ with perpendicular diagonals. From their intersection point S are constructed perpendiculars to AB, BC, CD and DA, whose other ends are L, M, P and Q respectively. Then the tetragon $LMPQ$ is simultaneously cyclic and circumscribed. It's circumcircle passes through the midpoints of the sides of $ABCD$ and its center is the midpoint of SO.

Proof 7: Tetragons *QALS*, *LBMS* are cyclic (each has two opposite right angles) and hence $\measuredangle QAS = \measuredangle QLS$, $\measuredangle SLM = \measuredangle SBM$.

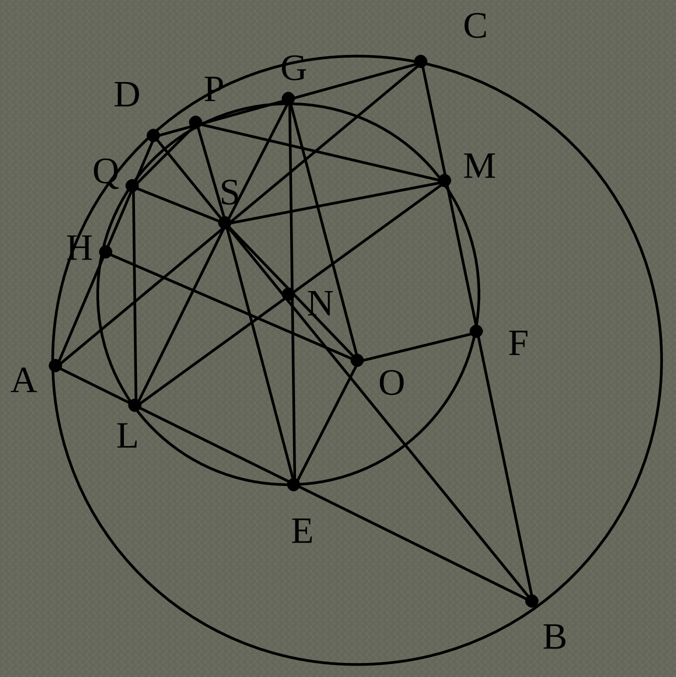

Fig. 18

As $ABCD$ is also cyclic, then $\measuredangle DAC = \measuredangle DBC$, i.e. $\measuredangle QAS = \measuredangle SBM$. Thus $\measuredangle QLS = \measuredangle SLM$ and therefore SL is bisector of $\measuredangle QLM$. Analogically SM, SP, SQ are bisectors of the rest angles of $LMPQ$. The four bisectors meet at S => $LMPQ$ has an incircle of center S.

E, F, G, H - midpoints of AB, BC, CD, DA; $EFGH$ is parallelogram of sides $\|$ diagonals of $ABCD$. As $AC \perp BD$, $EFGH$ is rectangle. Then it has circumcircle k of diameters EG, FH. We'll prove L, M, P, Q Є k.

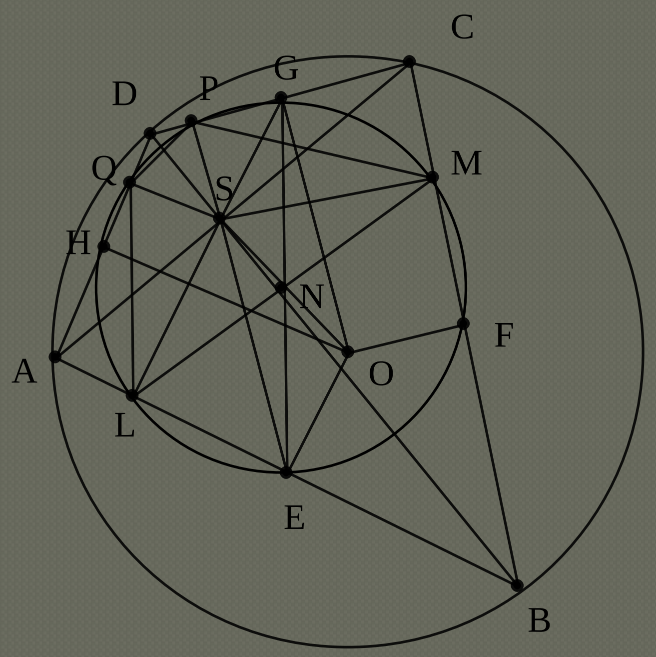

Fig. 18

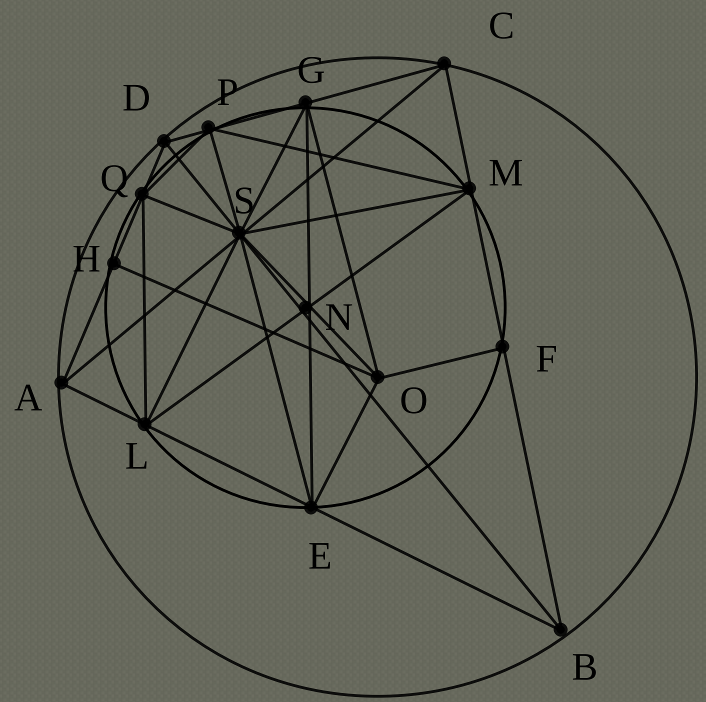

Fig. 18

From Property 6 => the perpendicular SL from S to AB cuts CD at its midpoint G and therefore $\measuredangle GLE = \measuredangle SLE = 90°$, i.e. EG is seen at right angle from L => L is on k and analogically M, P, Q are on k.

We'll show the center N of k is midpoint of SO. As $OG \| ES$ and $OE \| GS$ (again Property 6) => $OGSE$ is parallelogram => its diagonals EG, SO halve each other at the center of k = midpoint of diameter EG = midpoint of SO.

Another cyclic tetragon is the harmonic one. The products of lengths of its couples opposite sides are equal. It has additional properties, one of which we will apply for a hexagon.

Property 8. $ABCD$ is harmonic tetragon with midpoint U of its diagonal AC. Prove that these equalities hold:

1) $\measuredangle AUB = \measuredangle AUD$;

2) $BU \cdot DU = \frac{1}{4} \cdot AC^2$.

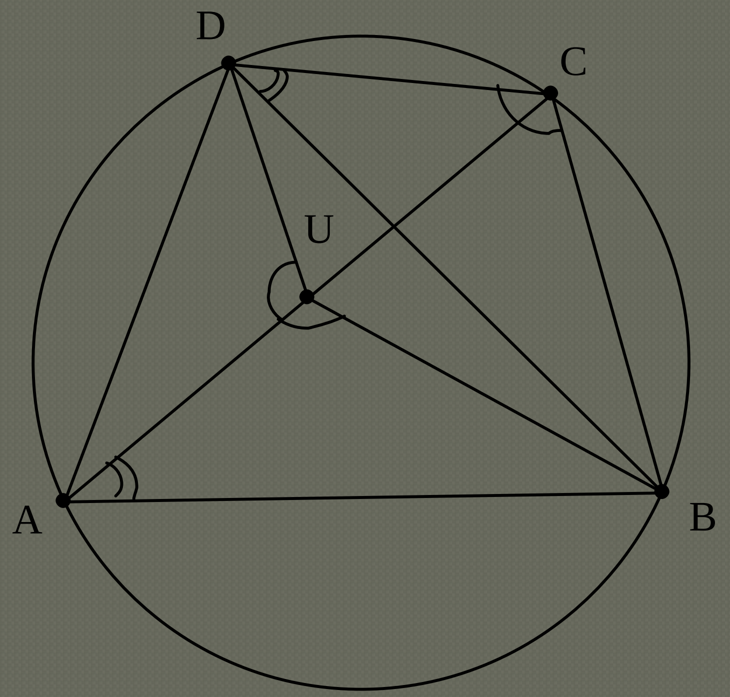

Fig. 19

Proof 8: $AB \cdot CD + BC \cdot DA = AC \cdot BD$ from Ptolemy theorem for cyclic tetragon and from harmonic tetragon definition $AB \cdot CD = BC \cdot AD$ => $2 \cdot AB \cdot CD = AC \cdot BD \Leftrightarrow \dfrac{AB}{AU} = \dfrac{BD}{CD}$ (Fig. 19). Further $\measuredangle BAU = \measuredangle BDC$ as inscribed angles => $\triangle ABU \sim \triangle DBC$ => $\measuredangle AUB = \measuredangle DCB$ and $\dfrac{AU}{BU} = \dfrac{CD}{BC}$.

Similarly $\triangle UAD \sim \triangle CBD$ => $\measuredangle AUD = \measuredangle DCB$ and $\dfrac{AU}{DU} = \dfrac{BC}{CD}$.

From the equalities of angles => $\measuredangle AUB = \measuredangle AUD$ and from the equalities of proportions $BU \cdot DU = AU^2 = \dfrac{1}{4} \cdot AC^2$.

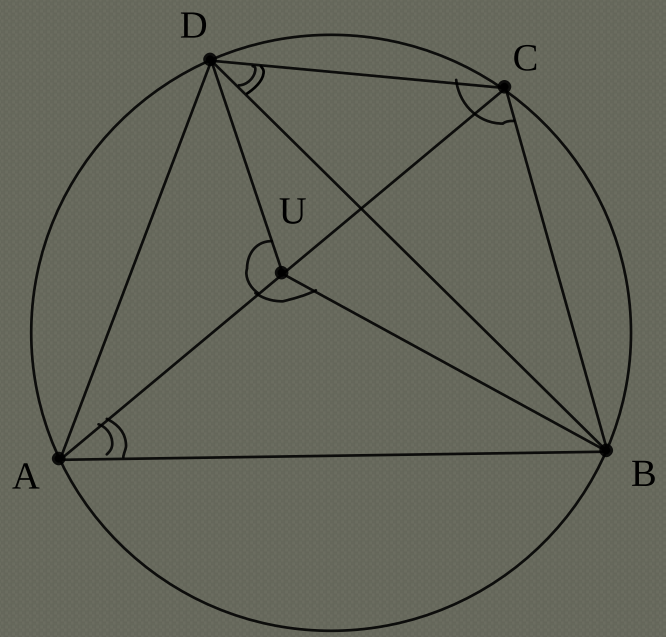

Fig. 19

Task 12. $ABCDEF$ is a convex hexagon, where the tetragons $ABDF$, $ACDE$ are harmonic. Prove that the midpoints M, N, P of the diagonals AD, BE, CF respectively and the common point Q of BE and CF are concyclic.

Solution 12: Equality 1) of Property 8 for $ABDF$, $ACDE$ provides $\measuredangle AMB = \measuredangle AMF$ and $\measuredangle CMD = \measuredangle EMD$ (Fig. 20). Adding jointly these we obtain $\measuredangle AMB + \measuredangle CMD = \measuredangle AMF + \measuredangle EMD$ => $\measuredangle BMC = \measuredangle FME$ => $\measuredangle BME = \measuredangle BMF + \measuredangle FME = \measuredangle BMF + \measuredangle BMC = \measuredangle FMC$, i.e. $\measuredangle BME = \measuredangle FMC$. Equality 2) of Property 8 for $ABDF$, $ACDE$ provides $BM \cdot FM = \frac{1}{4} \cdot AD^2$ and $EM \cdot CM = \frac{1}{4} \cdot AD^2$. Hence $\frac{BM}{CM} = \frac{EM}{FM}$. From this and the last equation between angles => $\triangle BME \sim \triangle CMF$. As MN, MP are respective medians in similar triangles => $\measuredangle MNE = \measuredangle MPF$, i.e. $\measuredangle MNE = \measuredangle MPQ$ => $\measuredangle MNQ + \measuredangle MPQ = 180°$ => M, N, P, Q are concyclic.

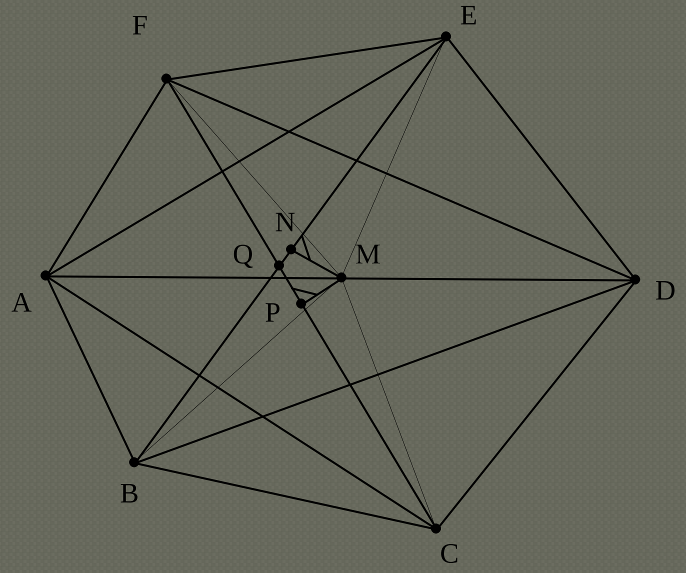

Fig. 20

1.5. Orthocenter of an inscribed quadrilateral.

The cyclic tetragon has all remarkable points of a convex tetragons, considered in the publications below. It has, in particular, such point, known from long time, which is a generalization of triangles' orthocenter and therefore is also called orthocenter. Let $ABCD$ is inscribed in circle Γ of center O and Ha, Hb, Hc, Hd be respectively the orthocenters of BCD, CDA, DAB, ABC. We'll prove:

Theorem 1. The lines AHa, BHb, CHc, DHd meet at one point and the tetragons $HaHbHcHd$ and $ABCD$ are symmetric with respect to this point (Fig. 21).

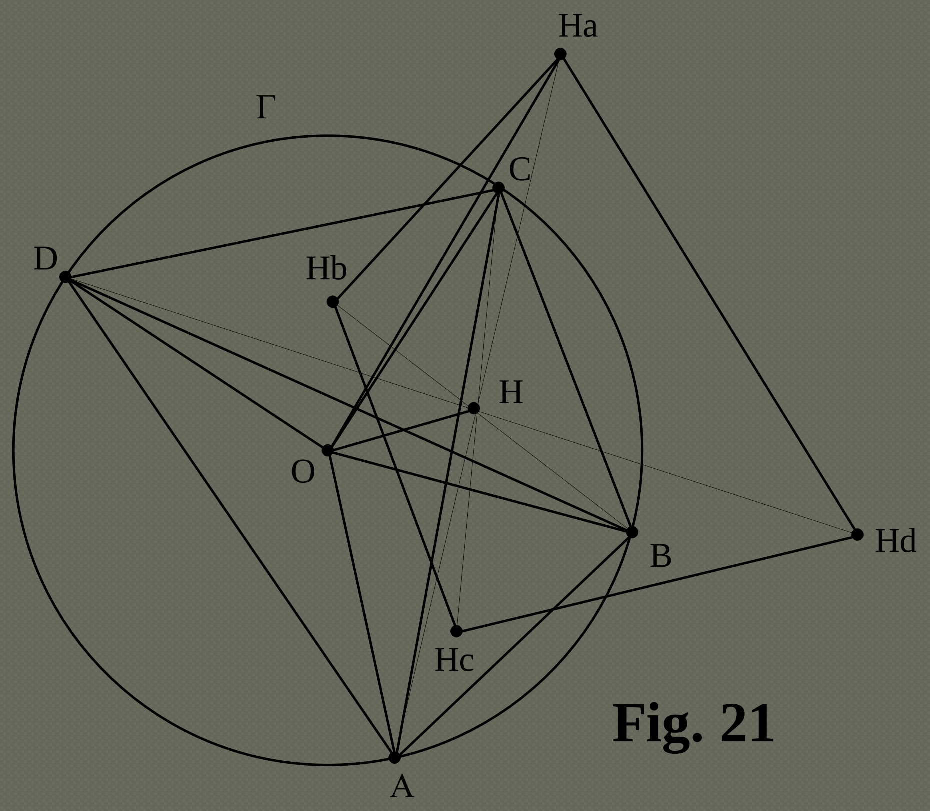

Fig. 21

Proof Th 1: As Ha is orthocenter of $\triangle BCD$, then $\overrightarrow{OHa} = \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD}$ [11]. If H is midpoint of AHa, then $\overrightarrow{OH} = \frac{1}{2} \cdot \left(\overrightarrow{OA} + \overrightarrow{OHa} \right)$ =>

(*) $\overrightarrow{OH} = \frac{1}{2} \cdot \left(\overrightarrow{OA} + \overrightarrow{OHa} \right) = \frac{1}{2} \cdot \left(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD} \right)$

Analogous (*) holds also for midpoints of BHb, CHc, DHd => these midpoints coincide => AHa, BHb, CHc, DHd meet at H, defined by (*), which halves them => AHa, BHb, CHc, DHd pass through one point and $ABCD$ and $HaHbHcHd$ are symmetric with respect to it.

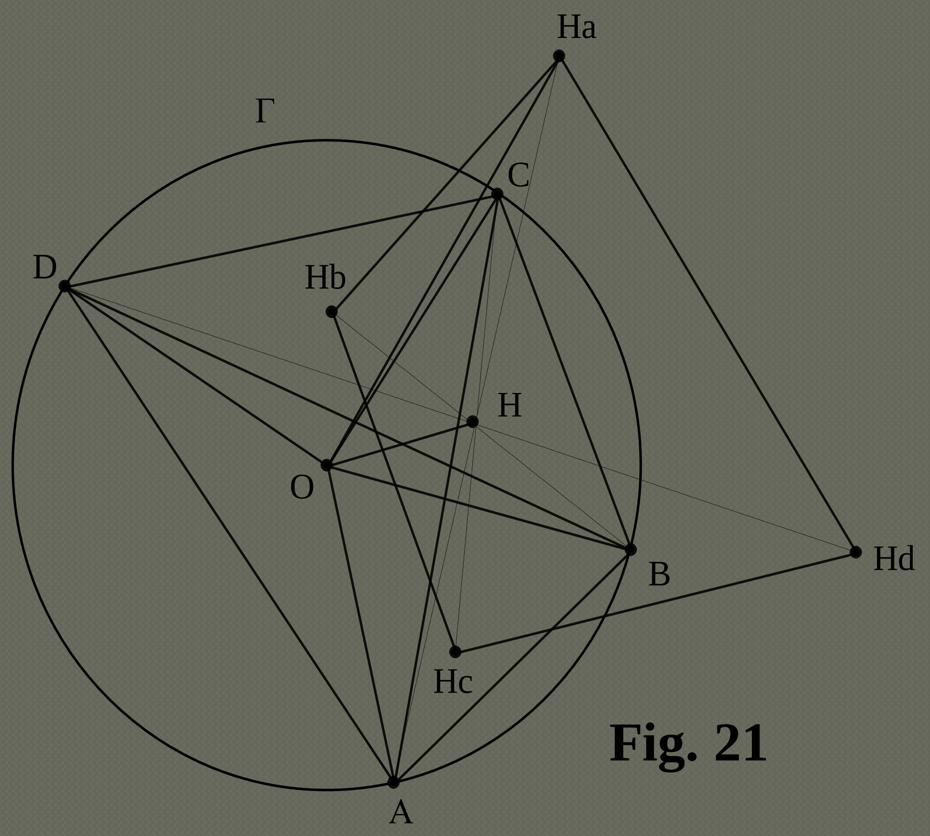

Fig. 21

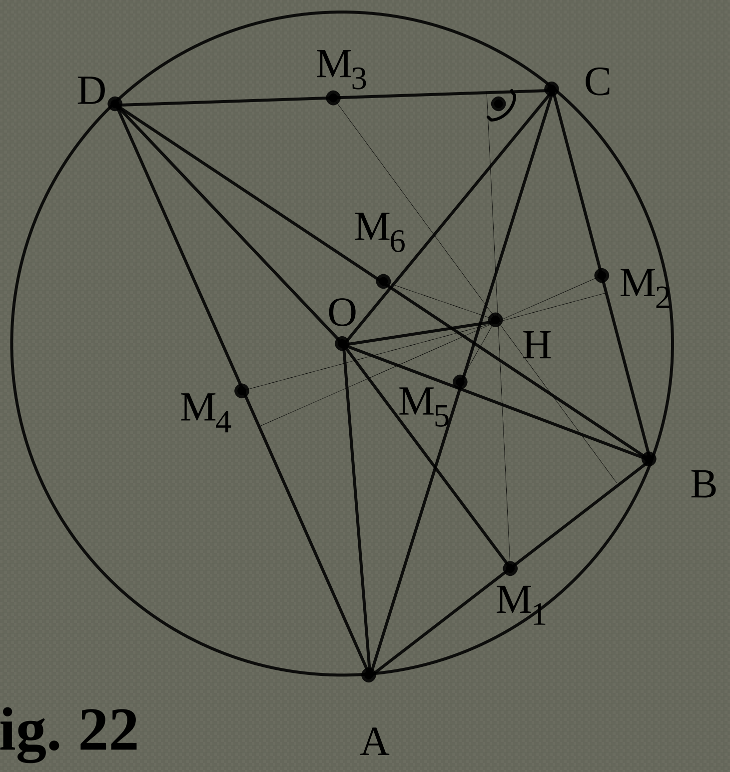

Fig. 22

Definition. The point H in the cyclic tetragon, defined by (*) is called tetragon's orthocenter. H characterizes by these properties:

Theorem 2. a) The straight lines through the midpoints of the sides of the tetragon $ABCD$, perpendicular to their opposite sides, intersect at the orthocenter H of $ABCD$ (Fig. 22);

b) The straight lines through the midpoints of the diagonals AC and BD, perpendicular to BD and AC respectively, intersect at the orthocenter H of $ABCD$.

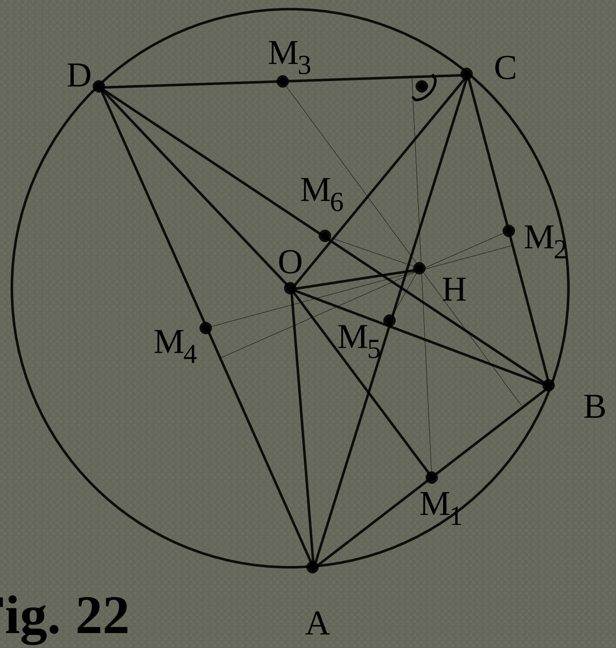

Fig. 22

Proof: M_1, M_2, M_3, M_4 - midpoints of AB, BC, CD, DA and M_5, M_6 - midpoints of diagonals AC, BD. Then

$$\overrightarrow{OM_1} = \frac{1}{2}\cdot\left(\overrightarrow{OA}+\overrightarrow{OB}\right) \Rightarrow$$

$$\overrightarrow{M_1H} = \overrightarrow{OH} - \overrightarrow{OM_1} = \frac{1}{2}\cdot\left(\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}+\overrightarrow{OD}\right) - \frac{1}{2}\cdot\left(\overrightarrow{OA}+\overrightarrow{OB}\right) = \frac{1}{2}\cdot\left(\overrightarrow{OC}+\overrightarrow{OD}\right).$$

Therefore $C, D \in \Gamma$ and

$$\left(\overrightarrow{M_1H}, \overrightarrow{DC}\right) = \frac{1}{2}\cdot\left(\left(\overrightarrow{OC}+\overrightarrow{OD}\right),\left(\overrightarrow{OC}-\overrightarrow{OD}\right)\right) = \frac{1}{2}\cdot\left(\overrightarrow{OC}^2-\overrightarrow{OD}^2\right) = \frac{1}{2}\cdot\left(\left|\overrightarrow{OC}\right|^2-\left|\overrightarrow{OD}\right|^2\right) = 0 \Rightarrow$$

$M_1H \perp DC$. We analogously prove similar things for M_i, $i = 2,3,4,5,6$.

Theorem 3. The orthocenter H, median point G and the circum-center O of a cyclic tetragon are collinear and $\overrightarrow{HG} = \overrightarrow{GO}$ (HGO is called Euler's line of the tetragon).

Theorem 4. The orthocenter H of a cyclic tetragon $ABCD$ with perpendicular diagonals AC, BD coinsides with the common point of these diagonals. ([2])

Theorem 5. The orthocenter of a cyclic tetragon is the common point of Euler's circles of the four triangles, formed by its sides and diagonals. ([17])

Theorem 6. The orthocenter of a cyclic tetragon is consyclic with the intersection points of its diagonals and those of the continuations of its opposite sides (This property si soon discovered). ([2])

1.6. Classic Theorems about cyclic tetragons

Cyclic tetragons were stydied by solid matematitians, who discovered their classic properties and formulated them in classical theorems:

Theorem 7. (Ptolemy) If $ABCD$ is a cyclic tetragon, then holds the equality $AC \cdot BD = AB \cdot DC + AD \cdot BC$.

Theorem 8. (Brocard) If $ABCD$ is inscribed in circle of center O, $AD \cap BC = P$, $AB \cap DC = Q$ and $AC \cap BD = T$, then O is the orthocenter of the triangle PQT.

Theorem 9. (*sypplement to Theorem 8*) Let $ABCD$ is inscribed in circle of center O and $AD \cap BC = P$, $AB \cap DC = Q$. The triangle POQ has orthocenter the common point T of the diagonals AC, BD and K_1K_2M for orthocentric triangle, where M is Mikel's point, and K_1, K_2 - the Brocardians of the tetragon. ([16])

Theorem 10. Let $ABCD$ is inscribed in circle of center O and $AD \cap BC = P$, $AB \cap CD = Q$. The median point G of $ABCD$ lies on the Euler's circle of $\triangle PQO$. ([19])