COMPUTER-AIDED VISUALIZATION OF GRAPHICAL SOLVING OF EQUATIONS OF THE FORM $f(x) = g(x)$

Asen Velchev, Jordan Tabov
Institute of mathematics and informatics, Bulgarian Academy of Science,
“Acad. G. Bonchev” Str. Block 8, 1113 Sofia, Bulgaria
asen_v@abv.bg, tabov@math.bas.bg

Резюме

Важен недостатък на повечето графични методи за визуализация на решаване на уравнения с компютър е, че екранът показва част от координатната равнина. Затова в общия случай получаваме непълна информация за броя и разпределението на корените на уравнението. Тук предлагаме подход, при който този недостатък се избягва за ограничено (но достатъчно широко за училищното ниво) множество от функции $f(x)$ и $g(x)$. Той дава възможност за решаване и изследване и на параметрични уравнения, чрез дидактически софтуер за динамична геометрия като GeoGebra. Тук правим преглед на теоретични и практически разработки от други в областта на графичното решаване и изследване на уравнения с компютър, или по-скоро визуализация на такова, която неправилно е наречена решаване. Предлагаме също последователност от три примера, третиращи взаимосвързани нестандартни уравнения, които обобщаваме чрез параметрично уравнение. Най-интересните, стимулиращи изследователското мислене, са именно параметричните задачи.

Ключови думи: графично решаване на уравнение, визуализация, дидактически софтуер.

Example for illustration: In order to illustrate our idea (which will be discussed in details below), we start with an example.

Problem 1. Find, with accuracy to tenths, approximations of the roots of the equation $\sqrt{4-x^2}-\dfrac{1}{x+1}-2=0$.

Solution 1: This problem cannot be solved by the students analytically. On the other side, the graph of the function $y=\sqrt{4-x^2}-\dfrac{1}{x+1}-2$ is unknown and its shape outside the screen cannot be predicted. We rearrange the equation: $\sqrt{4-x^2}=\dfrac{1}{x+1}+2$. Now the graphs of the functions $f(x) = \sqrt{4-x^2}$ and $g(x) = \dfrac{1}{x+1}+2$ in the new equation are known: a semicircle and hyperbola, which can be obtained by translation of the graph of the **studied** in 8th grade function $h(x) = \frac{1}{x}$. The graphs of $f(x)$ and $g(x)$ are plotted in the same Cartesian plane in ***Fig. 1***. The abscissa of their intersection point visualizes the solution of the equation. We are sure that this is the only solution of the equation, because we know the shapes of the two graphs and therefore we can predict their mutual disposal outside the screen. It is easy to obtain (using the software) the solution $x = -1,8$ (with the required accuracy). We talk about visualization instead of solving, because the graphs, the intersection points and their abscissas are visualized on the screen **with some approximation**. Every approximation of a root of an equation, regardless of how close it approaches the root, **does not have** the property of the root(s) to turn the equation into an identity. Is the graphical solving with a computer worse in accuracy than the graphical solving on paper? As paper and pen cannot solve equations graphically and they cannot give the man enough data and tools for finding the root(s) of an equation (**not their approximations!**), we speak only for computer-aided/machine-aided graphical solving of equations as **the only** possible method. Of course, when the roots are integer, we may suspect this looking at a handmade sketch and check **arithmetically** if these values satisfy the equation. But the solving method then will be grapho-arithmetical, not graphical and furthermore we have the same opportunity when we use computers. So what was called "graphical solving" at school is also just a (worse) visualization. So what we can find is some approximations of the solutions and therefore we set condition for their accuracy in these examples.

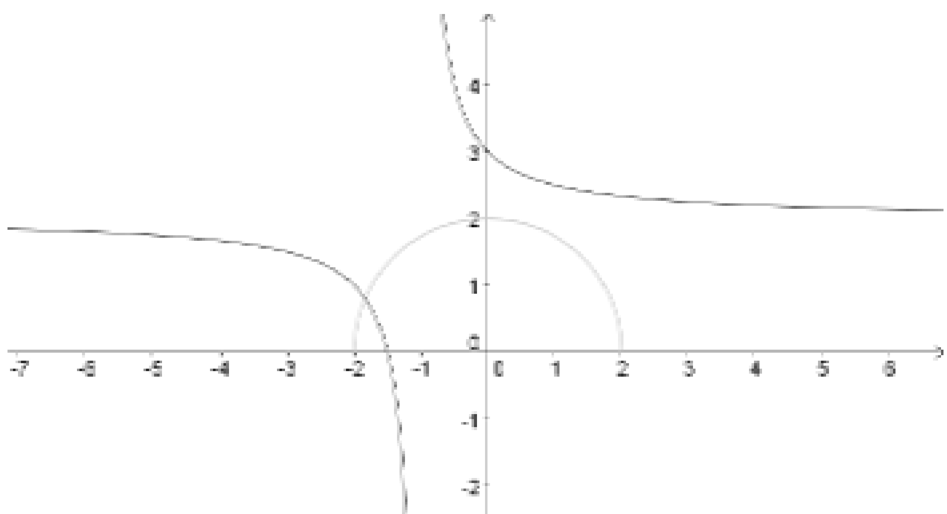

Fig. 1: Graphical presentation of the equation $\sqrt{4-x^2} = \frac{1}{x+1} + 2$ through the functions $f(x) = \sqrt{4-x^2}$ and $g(x) = \frac{1}{x+1} + 2$

We suggest a method, whose essential ideas can be described as follows: To work only with equations, which can be transformed as $f(x) = g(x)$ with functions $y = f(x)$ and $y = g(x)$, whose graphs the students know well. These are the

functions, which the students study at school ($y = ax + b$, $y = ax^2 + bx + c$, $y = \sin x$, etc.). Some other functions that are not too complicated (e.g. $y = \sqrt{1 - x^2}$, etc.) could be added gradually with suitable explanations (about their graphs).
When a student is familiar with the shapes of the graphs of the functions $y = f(x)$ and $y = g(x)$ in the equation $f(x) = g(x)$ and if he looks carefully at the computer screen, where these graphs are simultaneously plotted, then this student will be able to predict the behavior of the graphs out of the screen. If it is clear that these graphs could not intersect outside the screen, then:
(*) the solutions are the *x*-coordinates of the intersection points of the graphs within the screen.
Otherwise, i.e. if the graphs could intersect outside the screen, then by gradual zooming we put "within" the screen more and more of the co-ordinate plain until it is clear that the intersection points of the graphs of $y = f(x)$ and $y = g(x)$ are inside the screen. Then holds (*).
It is clear that almost all or at least 99% of the school equations plus new ones as the given in the example equation are solvable by other method for school students. A disadvantage of the **g**raphical **s**olving of **e**quations through **v**isualization (GSEV) is that we find the solutions with a certain approximation. However, this is not as essential as seems at a first glance. Actually, even in the case of "exact" solving of algebraic equations for a comparison and practical use of solutions of the type $x = \sqrt{3}$ one needs its "approximate" as well. In the same time GeoGebra provides the solutions with accuracy to the 6^{th} decimal place. In some special cases the approximations we use may significantly hamper the solving of the problem; such problems should not be given to the students. It should be noted that peculiarities of this kind appear also in the analytical "exact" form of solutions.

The analytical solving of equations perplexes many students by the following reasons:
1. The algorithms for solving equations algebraically are different for the different kinds of equations. We have one algorithm for linear equations, another for quadratic, third one for absolute value equations, etc.
2. When the equations are parametric, there is an additional difficulty: thorough consideration of all the possible cases.
3. The computational procedures involve risks of making errors.
4. Third- and higher-degree equations or polynomial-transcendent ones like some in [4] and [5] cannot be solved at school.
This black list could be extended. For the motivated students the overcoming of every difficulty leads to enriching in knowledge, skills and habits. For the others, which are majority, the difficulties usually are counter-indicative.

There is **one common algorithm for graphical solving of equations of the type** $f(x) = g(x)$: plotting the graphs of the functions $y = f(x)$ and $y = g(x)$, finding their intersection points and their abscissas. To judge that the graphs have no common points out of the screen, or to know where in the plane to search for such points, if we suspect their existence, we should only be familiar with the shapes of the graphs of $y = f(x)$ and $y = g(x)$ in the whole plain. Such knowledge the students have **only for studied functions.** The teachers should explain them this peculiarity.

The work with **dynamic geometry software (DGS)** has the following good sides:
1. The principle of visualization in teaching is kept. Furthermore, DGS offers opportunities that the chalk and the blackboard are not able to offer for parametric equations.
2. DGS clears the path for the **new technologies in math's teaching**. Students get familiar with software applications that can be used not only to do their homework but also after finishing school (**lifelong learning and professional usage**). [6]
3. The learners are more motivated to work.
4. The likelihood for delusion or error during the process of problem solving is smaller.
5. The time for work becomes shorter. This is very important, because the ministry of Education in Bulgaria minimized the amount of maths lessons in high school.

In our opinion, the inclusion of GSEV for equations of the form $f(x) = g(x)$ (and parametric equations $f(x,a) = g(x,a)$) via DGS as GEONExT, GeoGebra in the secondary school maths curriculum should have the following features:
1) replacement of the traditional analytical solving **of a part** of the equations studied at school by graphical solving,
2) assisting, by different means, the analytical solution **of another part** of equations through visual aids,
3) a **third part** of equations should still have to be solved like before – without using graphs,
4) **extending** the set of equations studied at school by including new types of equations (such as $\sin x = ax + b$) etc.

Revision and preliminary preparation

To teach GSEV by DGS to students, they should be familiar in advance with:
a) all kinds of functions $y = f(x)$ and $y = g(x)$ involved in the considered equations, and their graphs;
b) the "GSE" containing absolute value from the algebra course in 8th grade;
c) the potential of DGS for plotting graphs of functions, zooming, drawing objects in and out of the screen area etc.
For this purpose we suggest revision and preliminary preparation as follows:
1.1. Short revision of the "GSE" containing absolute value from 8th grade (for linear equations).
1.2. Brief introduction to GEONExT (GeoGebra).

Work with graphs of elementary functions in GEONExT (GeoGebra)

These programs allow plotting graphs in Cartesian co-ordinate system. On the axes tick marks can be put up to thousandths **manually** in GEONExT and thus the roots of an equation of the type $f(x) = 0$, if they exist, can be found with accuracy to the third sign, because on the screen we can see which is the closest tick mark to the solution (use maximum magnification). In GeoGebra tick marks appear **automatically** when zooming. This program has capacity to show the sixth decimal place. If we put in GEONExT the cursor on the point of intersection of the graphs of the functions $y = f(x)$ and $y = g(x)$, its co-ordinates appear at the upper right corner of the screen, with accuracy up to the 5th decimal place, when the magnification is maximal. GEONExT, run on different machines, provided results, which differed with hundredths and (!) even with tenths. GeoGebra provided the same results up to the 6th decimal place correctly. GeoGebra has a bit simpler syntax then GEONExT and therefore we prefer and recommend it.

Overview of some similar publications on computer-aided GSEV

We do not consider sources on GSEV without computer (or graphic calculator). We also omit here sources, where computer is involved, but the given equations are linear. We visited many free Bulgarian and English-language Internet sites by Google and Bulgarian articles in various periodicals from the period 2000 – 2007 and give below brief comments.

On the Internet site "Trigonometry – Equations, Identities, and Modeling Lesson 3: Equations which require a Graphical Solution" [4] there are more equations of the above type, but they are solved (graphically) by graphical calculator. The authors **discuss** the problem about predictability of the graphs of functions and therefore of the possibility of existence of solutions outside the computer screen.

A graphical solution of the equation $x = 2\sin x$ by MAPLE is presented on the Internet site of the Department of Mathematics, University of South Carolina [5] in an article entitled "What is a Project Report?". It has polynomial and transcendental terms. This paper is not focused on GSEV; it is only an illustration how certain empirical data can be expressed mathematically.

A didactical approach for solving parametric equations (and inequalities) by GEONExT and its Internet version is presented by Tonova, Goushev & Kopeva in [7] and [8], respectively. They are based on online dynamic Java applets. But in this didactical module GEONExT is used only for visualization and to help the heuristics in searching for a way of analytical solving of the problems.

In the articles [9] and [10] Lazarov & Vassileva describe graphical solutions of systems of parametric equations for university students by MATHEMATICA.

Stanilov & Slavova explain in [11] how MAPLE can be used to solve irrational equations. Obtained by the procedure "solve" and verified by "eval", the solutions are **visualized** graphically.

GSEV by MATLAB is described (but called GSE) in the cited above e-book [4] on the Internet site of the Department of Mathematics at the college of Staten Island. The method presented here prescribes to use GSEV when the analytical algorithms are very complicated or if there is no such algorithm, since the graphical solving is not precise. The examples illustrating the method involve equations with both polynomial and transcendental terms. The authors sometimes transform them from the form $f(x)$ = $g(x)$ to the form $F(x) = 0$, which causes difficulties, because the solvers do not know the shape of the graph of the function $y = F(\mathrm{x})$.

On the Internet site "**Solve** quadratic equations graphically" [12] we find a tutorial for analytical and graphical solving of quadratic equations. The GSEV is presented as GSE practically by a dynamic applet too. The users can vary any of the three coefficients of the quadratic trinomial and to observe how its graph changes. If the equation has real roots, they appear on the screen with accuracy of 8 places after the decimal point.

On the Internet site "Simultaneous equations" [13] the concepts of linear equation, power of equation, system of equations are introduced formally and also graphical method for solving linear, quadratic and higher power systems of equations by dynamic Java applet and tutorial marks. Unfortunately they neither say that the graphical method is only approximate, nor provide systems with solutions, which are not integer, nor mention the method to be rather visualization than real solving.

There are given in [5] equations, which require graphical "solving" (like $x = \sin x$). The used device is graphical calculator.

A method for GSEV and systems of equations by MAPLE with several examples is described on the Internet site "Syntax and Hints for Selected Maple Commands" [14]. The authors of [14] sometimes transform equation of the type $f(x) = g(x)$ to $F(x) = 0$, which is unsuitable, if the solvers do not know the shape of the graph outside the computer screen. No mention the method to be visualization only.

On the Internet site "Trig Solutions Review" [15] an approach is present for "solving" easy trigonometric equations by a handheld device. The functions are periodic and hence their graphs are predictable, **as stated** on the site. No mention that the graphical method is only visualization, a picture on the screen and that the solving procedure is mere analytical and it is even done numerically by interpolation made by CAS (computer algebra system).

For all the papers cited above except [4] the common detail is that their authors **do not require the functions** $y = F(x)$**,** $y = f(x)$ and $y = g(x)$ **to have been studied by the students.** They do not discuss the predictability of the function graphs outside the screen. This is significant disadvantage because the students might think that they can **"solve" any equation** graphically. The authors do not describe didactic experiments or any impressions from classroom or results from online interviews, i.e. **they do not present approbated didactic technologies**.

In [16] Baycheva and Kirilova present a successful didactic experiment with software for graphical solving of tasks on quadratic function, quadratic equations and inequalities. The software works under DOS, its interface looks out-of-date and it deals only with quadratic trinomials. No mention the method to be visualization only.

Further examples for illustration

Problem 2. Find approximations of the roots of the equation $\sqrt{14-x^2} - \frac{1}{x+1} - 2 = 0$ with accuracy to tenths.

Solution 2: As in ***Solution 1***, we transform the given equation to the form $\sqrt{14-x^2} = \frac{1}{x+1} + 2$ (***Fig. 2***) and proceed similarly. Here $x_1 = -3,3;\ x_2 = -0,4;\ x_3 = 2,9$.

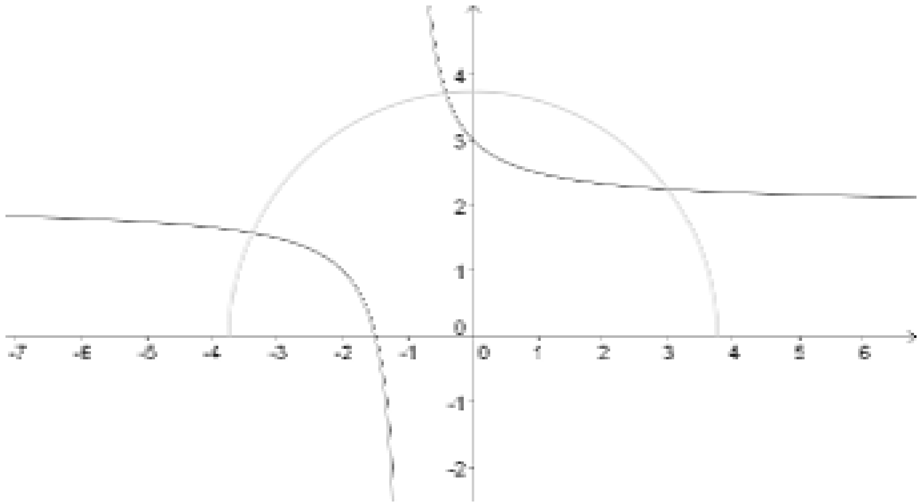

Fig. 2: Graphical presentation of the equation $\sqrt{14-x^2} = \dfrac{1}{x+1} + 2$

Problem 3. Find approximations of the roots of the equation $\sqrt{1-x^2} - \dfrac{1}{x+1} - 2 = 0$ with accuracy to tenths.

Solution 3: We transform the given equation to $\sqrt{1-x^2} = \dfrac{1}{x+1} + 2$. The graphs of $y = \sqrt{1-x^2}$ and $y = \dfrac{1}{x+1} + 2$ have no intersection points; the problem has no solution.

The problems 1, 2 and 3 can serve as basis for **graphical exploration of the number of solutions of the equation** $\sqrt{a-x^2} - \dfrac{1}{x+1} - 2 = 0$ **depending on the real parameter** ***a*.**

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