PROBLEMS IN TEACHING AND LEARNING HOW TO SOLVE MATHEMATICAL PROBLEMS OF THE TYPE "DISCRETE OPTIMIZATION" (FOR MATHEMATICS COMPETITIONS)

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Резюме

Разгледани са две задачи от тип дискретна оптимизация. Втората е авторска. Засега не съществуват научни изследвания, разработени дидактически технологии или предписания как да обучаваме (или да се учим) как се решават задачи от този тип. Ако се прилагат някакви подходи от страна на обучаващите, то това става несистемно и не докрай осъзнато и целенасочено. Решенията обикновено се дават без мотивация [2]. Специалистите Сава Гроздев, Йордан Табов, Петър Бойваленков, Ивайло Кортезов, Емил Колев, Невена Събева и др. от Института по Математика и Информатика на БАН подкрепят/изказват мнения от този вид. Тук предложените решения са нови, със систематичен подход и обучаващи въпроси, последователни подобрения на хипотезите за достигане до оптималното решение [1], както и подробни инструкции как и защо така се разсъждава. Поставени са нерешени проблеми и са очертани бъдещи цели на научната работа.

Ключови думи: дискретна оптимизация, математически олимпиади, мотивация, експериментиране, евристика, обучаващи решения.

What is a problem of type Discrete Optimization (PTDO)? The mention of minimization or maximization is not always bound with Calculus. Some problems concern objects of discrete nature (see Problem 1,2) .
For example: "20 teams take part in a championship. Find ***the minimal*** number of matches which have to be played by a certain time, at which time among any three of these teams will be two which have played each other."
More on PTDO see in [4], [5] and [7].

How to solve a PTDO?
The accepted scheme for the methods for solving PTDO is principally this (see [4]):
1) ("***boundedness***") Suggest and prove an upper or lower bound;
2) ("***existence***") Find a case, for which this optimum is reached.

There is a trial in [7] to be shown the necessity of each one of these two steps. Here is given a trial for improvement of this attempt. There is required a variety of techniques to find the optimal solution as graph theory, number theory, appropriate visualization, etc [4]. To solve a competitive PTDO one may have to perform a long and purposeful sequence of experiments, analysing both the problem and his (her) unsuccessful attempts to solve it [3]. One has to reason how the latter could and how they should be improved in order to get the final solution. Every success at this could be a breakthrough for a beginner.

The absence of didactic materials on the topic is partially due to the reason that PTDO are solved until now only by small number Olympiad competitors over the world, whose mathematical abilities are high genetically. I.e. there is no very urgent need for didactization of teaching how to solve PTDO. Nevertheless the idea for creation of some didactic technologies on PTDO impressed the above mentioned specialists very much. Economizing time and efforts during students' preparation for mathematics competitions and creating systematical approach in problem solving is very important for their best performance, width and depth of their knowledge. The discrete mathematics is widely used in theoretical base of computer science and practical informatics. Therefore solving of PTDO at secondary school by upper-level students is advisedly, which would additionally increase the need of PTDO-didactization.

Problem 1 (Canadian mathematical Olympiad 1981, see [4]):
Six musicians gathered at a chamber music festival. At each scheduled concert some of them played, while the rest listened as members of the audience. What is the least number of such concerts which would be scheduled in order to enable each musician to listen to each of the rest five as a member of the audience?

It is good to leave the students first to try to solve it by themselves. Reasoning over the task, they will become more and more introduced with its scheme, the objects in it, with the relations between the statements in its text of the task, etc. To know the shortest way through a deep unknown forest, they should first ***know the forest*** *to some extent. If someone directly shows them the shortest way, they will know* ***only*** *it and they will know it badly, without motivation [2]. Attempts for motivation without space for orientation in the task may fall. If the students go through the forest without help and step away the only way they know, then they will be lost in a labyrinth. That's why the given solution of Problem 1 is guiding them through trial-errors through the forest. This solution teaches* ***how to search for and recognize*** *true ways through the forest. The best solution or a better one students can find when the task is already solved somehow.*

Solution 1 by Andy Liu - [4] (for comparison):
Boundedness: "Let the musicians be *A, B, C, D, E* and *F*. Suppose there are only three concerts. Since each of the six must perform at least once, at least one concert must feature two or more musicians. Say both A and B perform in the first concert. They must still perform for each other.
Say A performs in the second concert for B and B in the third for A. Now *C, D, E* and *F* must all perform in the second concert, since it is the only time B is in the audience.
Similarly they must all perform in the third. The first concert alone is not enough to allow *C, D, E* and *F* to perform for one another. Hence we need at least four concerts.
Existence: This is sufficient, as we may have A, B and C in the first, A, D and E in the second, B, D and F in the third and C, E and F in the fourth."

The ***existence*** *step – finding a case, when four concerts are enough, is missing in the solution by A. Liu. Only* ***the result*** *of it is present. It is OK from mathematical, but not from didactical point of view. And how we guess to start the solution with a proof that three concerts are not enough? Why exactly three and not four?*

Solution 1' (new):
Boundedness: Denote the musicians *A, B, C, D, E* and *F*. Denote *M* the event: "A musician has listened from the audience to a performing colleague during a concert". The desired event is ***G***: "Every musician has listened to each of his colleagues from the audience". Therefore ***G*** will come true when *M* has already come true at least 6.5=30 times as each of the 6 musicians should listen to the rest 5. If we find the ***maximum*** possible number of realisations of *M* during a single concert, we might easier predict the sought number of concerts. Mention that some realisations of *M* may due to repeated

listening of some musician(s) to other(s). ***Why?*** Imagine for example that during the first concert *A* has listened to *B* and *C*, who were playing. But *B* should listen to *C* (and *C* to *B*). Let *B* join the audience for the second concert to listen to *C*. If *E* goes to the stage, *A* may stay in the audience for the second concert to listen to *E*. Thus *A* will be listening ***repeatedly*** to *C*. Now let us find the maximal possible number realisations of *M* in a single concert. Denote *k* the number of musicians, playing on the stage. The rest $6-k$ are listening to them during this concert. Thus the event *M* comes true $k(6-k)$ times. The function $f(k)=k(6-k)$ reaches maximum when $k=3$. This could be drawn by using derivatives, or by comparing the values $f(0), f(1)\dots f(6)$ {or $f(1), f(2)\dots f(5)$}. Why derivatives, isn't $f(k)=k(6-k)$ a discrete function? Yes, it is discrete. But its range is a subset of the range of the continuous function $f_{cont}(x)=x(6-x)$ and $f_{cont}(3)=9$ is the supremum of the range of $f_{cont}(x)$ and therefore an upper limit of the range of $f(x)$. As $f(3)=f_{cont}(3)$ and $f(3)$ belongs to the range of $f(x)$, 9 is ***the supremum*** of the range of $f(x)$ too. In *n* concerts *M* comes true $9n$ times. As there ***might*** be repeated listenings, then $9n \geq 30$ in order to have ***G***. From $9n \geq 30$ we have $n \geq \frac{10}{3} > 3$, i.e. $n > 3$, which is equivalent to $n \geq 4$ in the set of the integers. Are four concerts ***sufficient***? Is the solution complete? Do we need further reasoning? We have9.4=36 realizations of *M* in 4 concerts, but the number of the ***repeated*** realizations of *M* ***is unclear***. Hence we need further reasoning. In what direction should be the reasoning? The goal is to find the minimum number of concerts in order … As $n \geq 4$ our task is to find is it possible the musicians to be scheduled in four concerts so as to have ***G***? If not then is it possible for five concerts and so on and so on. Is this method economic and promising to be quick enough? Yes, because for 6 concerts there is a schedule: *ABCDE* to play for *F* in the first concert, then *ABCDF* – for *E*, then *ABCEF* for *D* … Hence we need of ***at most*** two major steps for finding the optimal number – consideration of if $n_{opt}=4$ and if not, then if $n_{opt}=5$. Even if not $n_{opt}=5$, then $n_{opt}=6$ as we have already found a successful 6-concerts schedule.

Existence: Denote with the ordered pair (*A,B*) the event "*A* has listened to *B* from the audience". Why ordered pair? Mention that $(A,B) \neq (B,A)$, because the second pair means the opposite. We will use these pairs to visualize the results. Thus we can assign an ordered pair to every realisation of *M*. Let *A*, *B* and *C* be the performers at the first concert. The result will be:

$$\begin{array}{l}(D,A),(D,B),(D,C)\\(E,A),(E,B),(E,C)\\(F,A),(F,B),(F,C).\end{array}$$

To ***completely*** avoid repeated pairs, let us interchange the musicians for the next concert. Thus *A*, *B* and *C* will be listeners. The resulting pairs will be:

$$\begin{array}{l}(A,D),(B,D),(C,D)\\(A,E),(B,E),(C,E)\\(A,F),(B,F),(C,F).\end{array}$$

Thus we have now 18 various ordered pairs. At the third concert there should play two musicians from the one triple, and one - from the other triple. Let *A*, *B* and *E* be the performers at this concert, which do not affect the generality as choosing *A*, *C* and *E*, or

A, *B* and *D*, or *B*, *D* and *E* will lead to isomorphic case. Now 5 from 9 pairs are repeated. Thus we have totally 22 different pairs in three concerts. Definitely we cannot construct a successful scheme this way. It is important that we exhausted all cases of the scheme 3:3 – then reverse 3:3 – then [(2+1):(1+2) or (1+2): (2+1)] – then reverse.

Is there any successful one? Let we reason how to improve if possible the first scheme. Can we find some shortcomings in it and overcome them? Our scheme was optimal concerning the first two concerts, but its effectiveness rapidly decreased for the following ones. Let's try to gain more ***uniformity*** instead of inpatient attempts for excessive optimization. Let's try cyclic musicians' interchanging. Let the performers be enlisted for the concerts as follows: *ABC, BCD, CDE, DEF.* The results will be:

$$\begin{array}{ll} (D,A),(D,B),(D,C) & (A,D),(A,B),(A,C) \\ 1{:}(E,A),(E,B),(E,C) & 2{:}(E,D),(E,B),(E,C) \\ (F,A),(F,B),(F,C), & (F,D),(F,B),(F,C), \end{array}$$

$$\begin{array}{ll} (A,E),(A,D),(A,C) & (A,D),(B,D),(C,D) \\ 3{:}(B,E),(B,D),(B,C) & 4{:}(A,E),(B,E),(C,E) \\ (F,E),(F,D),(F,C), & (A,F),(B,F),(C,F). \end{array}$$

The total number of the different pairs is $9+5+5+5=24$ in this case.

What weakness has this scheme and how to overcome it? The last distribution didn't treat all the musicians in equal manner. *A* and *E* were playing once, while each of the rest participated in three concerts. ***Is this external characteristic essential?*** Likely yes, because the musicians who play many times are probably listened repeatedly by some of their colleagues. More likely yes, as *A* listens to *C* two times and three times to *D*. The distribution *ABC, FAB, EFA, DEF* is isomorphic (the same but in reverse order) and *AEC, BAE, DBA, FDB* is isomorphic too (it is more general as one of the first triple drops away, then another one drops away, …). So we exhausted cyclic schemes of this type. Note that all permutations of the concerts give an isomorphic solution. Why we need to think exhaustively? Because a vain trial leaves the question of existence open. If all possible trials are vain, only then we may conclude that the problem has no solution. Is there a scheme in which in every two (subsequent) concerts there is only one repeated musician? It should be something like this: *ABC, CDE, EFA* and … *ABC*.

Cyclic schemes do not work in our case. The musicians are again not equally treated. ***Here arises again the question is there any scheme, giving G in four concerts. Is there any scheme, which treats all the musicians in equal manner? And is it a solution to the given problem?*** Probably yes. Let us try. The participants, necessary for four concerts, are 3.4=12. All the musicians are 6. Thus every musician should participate in exactly two concerts in order all of them to be equally treated. For example: *ABC*, *ADE*, *BDF*, *CEF*. The total number of ordered pairs is $9+8+7+6=30$ now.

Why we need two steps? *In the* **boundedness** *step we have proved that* $n \geq 4$. *Hence the optimal value*

(*) $\quad n_{opt} \geq 4$.

In the **existence** *step we have constructed a solution for* **some** $n=4$. *Hence*

(**) $\quad n_{opt} \leq 4$.

It follows from (*) and (**) that $n_{opt}=4$. *Here the necessity of both steps is visible. These reasonings are principally the same in all PTDO.*

Solution 1'' (new):
This solution is based on simpler visualization. It is given a 7×7 matrix in the figures 1,2 and 3 (*A, B, C, D, E* and *F* are the musicians). The presentation of the event *A* has listened to *D* is by filling the cell in the row of *A* and the column of *D*:

	A	B	C	D	E	F
A				**X**		
B						
C						
D						
E						
F						

Fig. 1: *A* has listened to *D.*

Why this visualization is better? Because we have with one glance the full and the empty cells. This enables us to ***seek*** for optimal schemes for concerts with the purpose to fill the empty cells and to predict ***how many*** concerts are there necessary in order to have this. Let the playing musicians in the first two concerts be *ABC*, *DEF*. This scheme as we saw in the previous solution is not optimal. The resulting matrix is:

	A	B	C	D	E	F
A		**?**		**X**	**X**	**X**
B	**?**			**X**	**X**	**X**
C				**X**	**X**	**X**
D	**X**	**X**	**X**			
E	**X**	**X**	**X**			
F	**X**	**X**	**X**			

Fig. 2: *ABC* and *DEF* have performed (two concerts).

The cells with question marks may be filled as *A* performs for *B* and then *B* for *A* – hence at least two concerts to fill the matrix. ***Generalization:*** if we have a pair empty orthogonally symmetric to each other with respect to the main diagonal cells, then we need at least two concerts to fill them. Let us form a schedule of performers-listeners for the third concert. Let *A* performs for *B.* If *A* performs for *C* too, then *B* and *C* are in the audience and they should perform for each other in fourth and ***fifth*** concerts. If *C* is a performer too, the others except *B* do not need to listen to him as his column shows, nor to *A*. if they play, *B* has already listened to them, as his row shows. Let *A, D, E* perform (*D, E* for *F* and vice versa in the next concert). I.e.:

	A	B	C	D	E	F
A				X	X	X
B	X		?	X	X	X
C	X	?		X	X	X
D	X	X	X			
E	X	X	X			
F	X	X	X	X	X	

Fig. 3: *ABC, DEF, ADE* have performed (3 concerts).

Question-marks show that at least two concerts are necessary, i.e. at least five totally. It is an isomorphic visualization if we present the musicians as vertices of a hexagon and every realisation of M with a directed segment from the one (listening musician) to the other one (performing musician) or the opposite direction. A graph will be obtained. It must be gradually built to a full graph. But I prefer the matrix visualisation, because if the directed segment $\overrightarrow{AB}$ appears on the graph, the opposite $\overrightarrow{BA}$ might be forgotten as the edge *AB* is already present on the picture. Another disadvantage of the usage of graph for representation is the multitude of segments crossed over each other, as the all sides and all diagonals of the hexagon there must be included. There is no such risk with the matrix as the events *A* performs for *B* and *B* performs for *A* are associated with different cells. In order to exhaust all possible events *X* listened to *Y*, one should fill all white cells in the matrix (outside the main diagonal), which is quite easy. In order to prevent students from mistakes when filling the matrix cells, it is advisable to write first the ordered couples as in Solution 1' and then to systematize them in the matrix. Studies on different types of representations and visualizations and their pluses and minuses are made by Athanasios Gagatsis [8], Dufur-Janvier, Fennel, Rowan, Kaput and many others. It is a further task to study their works, compare with this example, implement and cite their concrete results in further research.

Problem 2:

Find the greatest possible number of straight lines on a plane, equidistant to four different non-collinear points on the same plane.

Solution 2:

Since the four points are non-collinear, it's possible to choose three of them (denote *A, B, C*), also non-collinear. All the lines, equidistant to the points *A, B, C* are the lines, joining the midpoints of the sides of the triangle *ABC*. TO BE equidistant to all the points *A, B, C* and *D*, a line should be equidistant to *A, B* and *D*. Hence:

(1) $D \in AB$ or D is a point on the line through *C*, parallel to AB

And

(2) D lies on AC or on the line through *B*, parallel to AC

Therefore $D \in \{A, B, C\}$ or D is the fourth vertex of a parallelogram $ABDC$. Since all the points are different, $D \notin \{A, B, C\}$. Hence if and only if the figure $ABDC$ is a parallelogram, there exist lines equidistant to all the points *A, B, C,* D. These lines are the two ones joining the midpoints of the opposite sides of the parallelogram.

Notes and open questions: When the four points are collinear, then every line parallel to the four-points-line is equidistant to these points. If the requirement the points to be non-collinear is omitted, then the problem has no solution (no maximum) as the parallel lines are countless. There are missing PTDO with no solution among the ones which students solve. This practice might create expectations in students that every PTDO is solvable. It is advisable to put to the test and develop the ability of the students to exhaust all possible cases systematically and Problem 2 without the collinearity-requirement might be convenient for this purpose.
As we speak about exhausting of all possibilities, we should mention that the solutions made to PTDO usually include proofs by examples, not by regularity or a law. Questions about the number of the solutions and how to obtain each of them remain open.

Plans and current activities: Now I work under the guidance of/in collaboration with Sava Grozdev, Jordan Tabov, Nevena Sybeva and others, which work now or have worked with Olympiad competitors with purpose to collect, systematize and widen their experience. What do they share with me? Sava Grozdev said that necessary condition for acquiring abilities to solve PTDO is the recognition of the finite elements. During his experience as a coach of Bulgarian Olympiad mathematics team he noticed that the competitors used a variety of approaches to solve PTDO. According to Jordan Tabov it is a good practice to group PTDO in classes with respect to the solving method (of course, there are several methods to solve some PTDO). Since this is a general postulate in didactics of mathematics I have no doubt that it holds for the subtypes of PTDO and I do not support this reasonable view with additional references. [1] and [3] are enough. I do not like enough my solution of Problem 1 even if it might look good, because there is neither simple systematization nor visualization of either the all possible approaches to solve the problem or all optimal solutions of it. I plan at a given moment to observe the work of my senior colleagues with the young mathematical competitors, when they teach them how to solve PTDO. The "didactization" of teaching how to solve PTDO might include the implementation of interactive software environment for experimentation if not something simpler [5]. We think that there are stated in [6] many common with our observations and ideas concerning optimization, didactical software and combinatorial skills, which are necessary for solving PTDO.

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