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Pólya-Aeppli Risk Models

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Abstract. In this paper we consider three risk models. As counting processes for these risk models we use the Poisson process, the Pólya-Aeppli process and the Noncentral Pólya-Aeppli process. For the related risk models we define an exponential martingales and the corresponding martingale approximation of the ruin probabilities. In the case of exponentially distributed claims we compare these three models and for specific values of the parameters we make some conclusions for their applications in risk theory.

INTRODUCTION

The standard model of an insurance company, called *risk process* $\{X(t),\ t \geq 0\}$, defined on the complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ is given by

$$X(t) = ct - \sum_{k=1}^{N(t)} Z_k, \qquad \left(\sum_{1}^{0} = 0\right). \tag{1}$$

The constant c is a positive real constant which represents *the gross premium rate*. The sequence $\{Z_k\}_{k=1}^{\infty}$ of mutually independent and identically distributed random variables with common distribution function $F,\ F(0) = 0$ and mean value $EZ = \mu < \infty$ is independent of the counting process $\{N(t),\ t \geq 0\}$. The process $N(t)$ is interpreted as the number of claims to an insurance company during the time interval $[0, t]$. In the classical risk model, the counting process is a Poisson process with parameter λ, see Grandell [1]. In this case we use the notation $N_1(t) \sim P_oP(\lambda)$. Then the aggregated claim amount up to time t is given by the compound Poisson process $S_{N_1(t)} = \sum_{k=1}^{N_1(t)} Z_k$. The probability mass function of the process $N_1(t)$ is given by

$$P(N_1(t) = m) = \frac{(\lambda t)^m}{m!} e^{-\lambda t},\ m = 0, 1, \ldots \tag{2}$$

One of the most important properties of the Poisson process is its equidispersion i.e the Poisson mean and variance are equal. Then the corresponding Fisher index which is a ratio of the variance to mean of the process is equal to one. In many practical applications the equidispersion property of the Poisson process is not observed in the count data at hand. This fact motivated many authors to search more flexible models for counting such type of data. The most used generalization of the Poisson process is the compound Poisson process. In many risk models it is a basic counting process. Such model is called compound Poisson model and it is useful for cluster data. In Minkova [2] the compound Poisson process with geometric compounding distribution was introduced. The resulting counting process is called Pólya-Aeppli process. As application, the corresponding risk model is defined and discussion on ruin probability is given. In 2013 the Pólya-Aeppli process was characterized by Chukova and Minkova [3]. In 2015 Lazarova and Minkova [4] have defined the Noncentral Pólya-Aeppli process which is a generalization of the Pólya-Aeppli process. It is a sum of homogeneous Poisson process and Pólya-Aeppli process.

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The paper is organized as follows. In the next section we consider the counting processes in risk models. Then we analyze the corresponding risk models. The martingale approximation of the ruin probability is given. Then we consider the case of exponentially distributed claims and compare the models in terms of the approximated ruin probability.

COUNTING PROCESSES IN RISK MODELS

In real life every stochastic counting process counts the number of the events that occur up to time t. Usually the counting process $\{N(t),\ t \geq 0\}$ is a non-negative process with integer values. Every counting process is a non-decreasing process which can denote the number of the accidents on the road, the number of births and deaths and also the number of the claims to an insurance company in time interval $[0, t]$.

Definition 1 (Ross [5]) *A stochastic process* $\{N(t),\ t \geq 0\}$ *is said to be a counting process if*
(1) it starts at zero, $N(0) = 0$
(2) for all $t \geq 0$ *the process* $N(t)$ *has integer values*
(3) for $0 \leq s < t$, *the number of the events which occur in the interval* $(s, t]$ *is equal to* $N(t) - N(s)$.

The Poisson process

The most popular counting stochastic process is the Poisson process, named after the French scientist Siméon Poisson (1781-1840). It is a Markov process in continuous time for which the only possible jumps are to the next higher state. The only changes in the process are unit jumps upwards.
Definition 2 (Ross [5]) *A counting process* $\{N_1(t),\ t \geq 0\}$ *is said to be a Poisson process with intensity* $\lambda > 0$ *if*
(1) it starts at zero, $N_1(0) = 0$
(2) $N_1(t)$ *is a process with stationary and independent increments*
(3) for each $t > 0$, *the number of arrivals* $N_1(t)$ *in any interval of length t has a Poisson distribution.*

For a Poisson process with parameter λt we use the notation $N_1(t) \sim P_oP(\lambda t)$. The probability generating function of the Poisson process is given by

$$\psi_{N_1(t)}(s) = e^{-\lambda t(1-s)}. \tag{3}$$

The mean and the variance of the Poisson process are given by

$$E(N_1(t)) = Var(N_1(t)) = \lambda t,$$

while for the Fisher index of dispersion we obtain

$$FI(N_1(t)) = \frac{Var(N_1(t))}{E(N_1(t))} = 1.$$

The Pólya-Aeppli process

Definition 4 (Chukova and Minkova [3]) *The counting process* $\{N_2(t),\ t \geq 0\}$ *is said to be a Pólya-Aeppli process if*
(1) it starts at zero, $N_2(0) = 0$
(2) $N_2(t)$ *is a process with stationary and independent increments*
(3) for each $t > 0$, $N_2(t)$ *is Pólya-Aeppli distributed.*

For the Pólya-Aeppli process with parameters λ and $\rho \in [0, 1)$ we use the notation $N_2(t) \sim PAP(\lambda, \rho)$. The probability mass function and probability generating function of the Pólya-Aeppli process are given by

$$P(N_2(t) = m \mid \lambda) = \begin{cases} e^{-\lambda t}, & m = 0, \\ e^{-\lambda t} \displaystyle\sum_{i=1}^{m} \binom{m-1}{i-1} \frac{[\lambda(1-\rho)t]^i}{i!} \rho^{m-i}, & m = 1, 2, \ldots \end{cases} \tag{4}$$

and

$$\psi_{N_2(t)}(s) = e^{-\lambda t(1-\psi_1(s))}, \tag{5}$$

where

$$\psi_1(s) = \frac{(1-\rho)s}{1-\rho s} \tag{6}$$

is the probability generating function of the shifted geometric distribution with success probability $1-\rho,\ Ge_1(1-\rho)$. The mean and the variance of the Pólya-Aeppli process are given by

$$E(N_2(t)) = \frac{\lambda t}{1-\rho} \quad \text{and} \quad Var(N_2(t)) = \frac{\lambda t(1+\rho)}{(1-\rho)^2}.$$

The Fisher index of dispersion is equal to

$$FI(N_2(t)) = \frac{1+\rho}{1-\rho} = 1 + \frac{2\rho}{1-\rho} > 1,$$

i.e for $\rho \neq 0$, the Pólya-Aeppli process is over-dispersed related to the Poisson process. This fact provides a greater flexibility in modeling count data than the standard Poisson process.

Let $T_1, T_2, \ldots$ is a sequence of non-negative, mutually independent random variables and $S_n = \sum_{i=1}^{n} T_i,\ n = 1, 2, \ldots,\ S_0 = 0$ is the corresponding renewal process. The process S_n can be interpreted as a sequence of *renewal* epochs. T_1 is the time until the first renewal epoch and $\{T_i\}_{i\geq 2}$ are the inter-arrival times. Let $N_2(t) = \sup\{n \geq 0,\ S_n \leq t\},\ t \geq 0$ be the number of the renewals occurring up to time t. The distribution of $N_2(t)$ is related to that of S_n and for any $t \geq 0$ and $n \geq 0$ the following relation holds

$$P(N_2(t) = m) = P(S_m \leq t) - P(S_{m+1} \leq t), \quad m = 0, 1, 2, \ldots \tag{7}$$

The Pólya-Aeppli process is characterized by the fact that T_1 is exponentially distributed with parameter λ and $\{T_2, T_3, \ldots\}$ are independent, identically distributed. The inter-arrival times $\{T_2, T_3, \ldots\}$ are zero with probability ρ and with probability $1-\rho$ are exponentially distributed with parameter λ. The probability density function of T_2 is given by

$$f_{T_2}(t) = \rho\delta_0(t) + (1-\rho)\lambda e^{-\lambda t} \quad t \geq 0,$$

where

$$\delta_0(t) = \begin{cases} 1, & t = 0 \\ 0, & t > 0. \end{cases}$$

For the cumulative distribution function we have

$$F_{T_2}(t) = 1 - (1-\rho)e^{-\lambda t},\ t \geq 0.$$

The Pólya-Aeppli process is a time-homogeneous process. In the case of $\rho = 0$ it becomes a homogeneous Poisson process. So, we have a homogeneous process with an additional parameter. The additional parameter ρ has an interpretation as correlation coefficient, see [6].

The Non-central Pólya-Aeppli process

The Non-central Pólya-Aeppli process is a sum of two stochastic counting processes, one of which is the homogeneous Poisson process and the other one is the Pólya-Aeppli process.

Definition 5 *A counting process* $\{N_3(t),\ t \geq 0\}$ *is said to be a Non-central Pólya-Aeppli process if*
(1) it starts at zero, $N_3(0) = 0$
(2) for each $t > 0$, $N_3(t)$ *is Non-central Pólya-Aeppli distributed*

For a Non-central Pólya-Aeppli process with parameters λ_1, λ_2 and $\rho \in [0, 1)$ we use the notation $N_3(t) \sim NPAP(\lambda_1, \lambda_2, \rho)$. The probability mass function and the probability generating function of the Non-central Pólya-Aeppli process are given by

$$P(N_3(t) = i) = \begin{cases} e^{-(\lambda_1+\lambda_2)t}, & i = 0, \\ e^{-(\lambda_1+\lambda_2)t}\left[\frac{(\lambda_1 t)^i}{i!} + \sum_{j=1}^{i} \frac{(\lambda_1 t)^{i-j}}{(i-j)!} \sum_{k=1}^{j} \binom{j-1}{k-1} \frac{[\lambda_2(1-\rho)t]^k}{k!} \rho^{j-k}\right], & i = 1, 2, \ldots, \end{cases} \tag{8}$$

and

$$\psi_{N_3(t)}(s) = e^{-\lambda_1 t(1-s)} e^{-\lambda_2 t(1-\psi_1(s))}, \tag{9}$$

where $\psi_1(s)$ is the probability generating function of shifted geometric distribution, given in (6). The mean and the variance of the Non-central Pólya-Aeppli process are given by

$$E(N_3(t)) = \left(\lambda_1 + \frac{\lambda_2}{1-\rho}\right) t \quad \text{and} \quad Var(N_3(t)) = \left[\lambda_1 + \lambda_2 \frac{1+\rho}{(1-\rho)^2}\right] t.$$

The related Fisher index of dispersion is equal to

$$FI(N_3(t)) = \frac{1+\rho}{1-\rho} - \frac{2\lambda_1 \rho}{\lambda_1(1-\rho) + \lambda_2} < \frac{1+\rho}{1-\rho},$$

i.e for $\rho \neq 0$ the Non-central Pólya-Aeppli process is under-dispersed related to the Pólya-Aeppli process.

PÓLYA-AEPPLI RISK MODELS

We consider the risk process $X(t)$, defined by (1), where the counting process is independent of the claim sizes $\{Z_k\}_{k=1}^{\infty}$. The relative *safety loading* θ is defined by

$$\theta = \frac{EX(t)}{E(S_{N(t)})},$$

and we consider the case of positive safety loading $\theta > 0$. Define

$$\tau(u) = \inf\{t > 0,\ u + X(t) \leq 0\}$$

the time to ruin of an insurance company having initial capital u. If for all $t > 0,\ u + X(t) > 0$, we let $\tau = \infty$, then the probability of ruin is $\Psi(u) = P(\tau(u) < \infty)$.

The Poisson risk model

Suppose that the counting process to the risk model in (1) is a $P_oP(\lambda)$. In this case, the relative safety loading θ is given by

$$\theta = \frac{c - \lambda\mu}{\lambda\mu} = \frac{c}{\lambda\mu} - 1.$$

In the case of positive safety loading $\theta > 0$, the premium income c should satisfy the inequality $c > \lambda\mu$.

The Pólya-Aeppli risk model

We consider the risk process $X(t)$, given in (1) with a Pólya-Aeppli counting process. It is called a Pólya-Aeppli risk model, see Minkova [2]. The interpretation of the counting process in this risk model is that the insurance company have its policies divided into homogeneous, independent and identically distributed groups. The number of the groups has a Poisson distribution. The number of the policies in each group has a shifted geometric distribution. The relative safety loading in this case θ is given by

$$\theta = \frac{c(1-\rho) - \lambda\mu}{\lambda\mu} = \frac{c(1-\rho)}{\lambda\mu} - 1,$$

and in the case of positive safety loading $\theta > 0$, the premium income c should satisfy the inequality $c > \frac{\lambda\mu}{1-\rho}$.

The Non-central Pólya-Aeppli risk model

Let us consider now the risk process $X(t)$, where the counting process is the Non-central Pólya-Aeppli process. We will call this process a Non-central Pólya-Aeppli risk model. In this case we suppose that two types of claims arrive to the insurance company. The first type of the claims are counted by the Poisson process and the second type by the Pólya-Aeppli process. The relative safety loading θ is given by

$$\theta = \frac{c(1-\rho) - \mu[\lambda_1(1-\rho) + \lambda_2]}{\mu[\lambda_1(1-\rho) + \lambda_2]} = \frac{c(1-\rho)}{\mu[\lambda_1(1-\rho) + \lambda_2]} - 1,$$

and in the case of positive safety loading $\theta > 0$, the premium income c should satisfy the inequality $c > \frac{\mu(\lambda_1(1-\rho)+\lambda_2)}{1-\rho}$.

MARTINGALE APPROXIMATION

Martingales for risk processes

The modern theory of risk [7] is related with the collective risk model introduced by Filip Lundberg. The surplus process of an insurance risk can be described as

$$U(t) = u + ct - S(t),\ t \geq 0, \tag{10}$$

where u is the initial capital of the company, c is the company's income per unit time and $S(t) = \sum_{k=1}^{N(t)} Z_k$ is the total amount of the claims, paid by the insurance company up to time t.

Let us denote by $(\mathcal{F}_t^X)$ the natural filtration generated by the stochastic process $X(t)$, given in (1). $(\mathcal{F}_t^X)$ is the smallest complete filtration to which the process $X(t)$ is adapted. As the ruin times are first entrance time to some interval we need a complete filtration in order to assure that the ruin times are stopping times. Denote by $LT_Z(r) = \int\limits_0^\infty e^{-rx} f_Z(x)dx$ the Laplace transform of the random variable Z. Then we have that the Laplace transform of $X(t)$ is given by

$$Ee^{-rX(t)} = e^{-rct}\Psi_{N(t)}(LT_Z(-r)) = e^{g(r)t},$$

where $\Psi_{N(t)}$ is the probability generating function of the counting process.

From the martingale theory [7] we take the equation

$$Ee^{-rX(t)} = e^{-rct}e^{-\lambda t[1-\psi(LT_Z(-r))]}$$

and get the following

$$g(r) = -rc - \lambda[1 - \psi(LT_Z(-r))].$$

Equating the function $g(r)$ to zero we obtain

$$rc + \lambda\left[1 - \psi(LT_Z(-r))\right] = 0. \tag{11}$$

The positive solution R of the equation (11) is called a Lundberg's exponent and the inequality $\Psi(u) \leq e^{-Ru}$ - a Lundberg's inequality.

Again from [7] it follows that the process

$$M_t = \frac{e^{-rX(t)}}{e^{-rct}\Psi_{N(t)}(LT_Z(-r))}$$

is a $(\mathcal{F}_t^X)-$ martingale.

We consider the cases of Poisson, Pólya-Aeppli and Non-central Pólya-Aeppli counting processes.

Case 1: Martingale approach to the Poisson risk model

When the counting process is a Poisson process, then the function $g(r)$ is given by

$$g(r) = rc + \lambda[1 - LT_Z(-r)]. \tag{12}$$

Using the probability generating function given in formula (3) we obtain the following martingale

$$M_t = \frac{e^{rS(t)}}{e^{-\lambda t[1-LT(-r)]}}.$$

Case 2: Martingale approach to the Pólya-Aeppli risk model
When the counting process is a Pólya-Aeppli process, then the function $g(r)$ is given by

$$g(r) = rc + \lambda \left[\frac{1 - LT_Z(-r)}{1 - \rho LT_Z(-r)} \right]. \tag{13}$$

Using the probability generating function in formula (5) and (6) we obtain the following martingale

$$M_t = \frac{e^{rS(t)}}{e^{-\lambda t\left[1 - \frac{1-LT(-r)}{1-\rho LT(-r)}\right]}}.$$

Case 3: Martingale approach to the Non-central Pólya-Aeppli risk model
When the counting process is a Non-central Pólya-Aeppli process, then the function $g(r)$ is given by

$$g(r) = rc + \lambda_1[1 - LT_Z(-r)] + \lambda_2 \left[\frac{1 - LT_Z(-r)}{1 - \rho LT_Z(-r)} \right]. \tag{14}$$

Using the probability generating function given in formula (9), where $\psi_1(s)$ is the probability generating function of shifted geometric distribution, given in (6) we obtain the following martingale

$$M_t = \frac{e^{rS(t)}}{e^{-\lambda_1 t[1-LT(-r)]} e^{-\lambda_2 t\left[\frac{1-LT(-r)}{1-\rho LT(-r)}\right]}}.$$

Let the process $M_t = \frac{e^{-rX(t)}}{e^{-rct}\Psi_{N(t)}(LT_Z(-r))}$, $t \geq 0$ be a martingale relative the σ - algebras $\mathcal{F}^X$, generated by the process $X(t)$ and $\tau = inf\{t \geq 0 : X(t) < 0\}$ be the time to ruin for an insurance company. Applying the martingale stopping time theorem on the martingale M_t we obtain that

$$\begin{aligned}
1 &= M_0 = E[M(\tau)|\mathcal{F}_0] = E[M(t_0 \wedge \tau)] \\
&= E[M(t_0 \wedge \tau), \tau \leq t_0].P(\tau \leq t_0) + E[M(t_0 \wedge \tau), \tau > t_0]P(\tau > t_0) \\
&= E[e^{-r.X(\tau)} \tfrac{e^{rc\tau}}{\Psi_{N(\tau)}(LT_Z(-r))} | \tau \leq t_0].P(\tau \leq t_0) + E[e^{-r.X(t_0)} \tfrac{e^{rct_0}}{\Psi_{N(t_0)}(LT_Z(-r))} | \tau > t_0]P(\tau > t_0) \\
&\geq E[M(t_0 \wedge \tau), \tau \leq t_0] = E[e^{-rX(\tau)} \tfrac{e^{rc\tau}}{\Psi_{N(\tau)}(LT_Z(-r))} | \tau \leq t_0]P(\tau \leq t_0) \\
&\geq e^{ru} E[\tfrac{e^{rc\tau}}{\Psi_{N(\tau)}(LT_Z(-r))} | \tau \leq t_0]P(\tau \leq t_0).
\end{aligned}$$

The process $X(\tau) \leq 0$ for $\tau < \infty$. Then the inequality $e^{-ru} \geq 1$ holds. Since t_0 in the above calculations was arbitrary selected, then for every t we have

$$P(\tau \leq t) \leq \frac{e^{-ru}}{E[\frac{e^{rc\tau}}{\Psi_{N(\tau)}(LT_Z(-r))} | \tau \leq t]} \leq e^{-ru}.$$

EXPONENTIALLY DISTRIBUTED CLAIMS

Let us suppose that the claim sizes $\{Z_k\}_{k=1}^{\infty}$ are exponentially distributed i.e $Z \sim exp(\mu)$ with distribution function $F_Z(x) = 1 - e^{-\frac{x}{\mu}},\ x \geq 0,\ \mu > 0$ and mean value $EZ = \mu < \infty$. Then for the Laplace transform of the exponentially distributed claims we have the following expression

$$LT_Z(-r) = E(e^{rZ}) = \int_0^{\infty} e^{rx} dF_Z(x) = \int_0^{\infty} e^{rx} d(1 - e^{-\frac{x}{\mu}}) = \frac{1}{\mu} \int_0^{\infty} e^{rx} e^{-\frac{x}{\mu}} dx = \frac{1}{1 - \mu r},$$

where $r < \frac{1}{\mu}$.

It is known that $M_Z(r) = \frac{1}{1-\mu r}$ is the moment generating function of the exponential distribution.

Model 1: The Poisson Risk model

The counting process $N(t)$ in this risk model is a Poisson process with an intensity λ. In the case of exponentially distributed claims for this risk model, the function $g(r)$ is given by

$$cr + \lambda\left[1 - \frac{1}{1-\mu r}\right] = 0. \tag{15}$$

The equation (15) have two roots. The first root is $R_1 = 0$ and the second one is $R_2 = \frac{c-\lambda\mu}{c\mu}$. As the safety loading θ for this risk model is a positive one, we have that the premium income per unit time c satisfies the inequality $c > \lambda\mu$. This condition determines that the root R_2 is a positive root of the equation (15).

Model 2: The Pólya-Aeppli Risk model

The counting process $N(t)$ in this risk model is a Pólya-Aeppli process with an intensity λ. In the case of exponentially distributed claims for this risk model, the function $g(r)$ is given by

$$cr - \lambda\left[\frac{\mu r}{1-\mu r-\rho}\right] = 0. \tag{16}$$

In this case the equation (16) have two roots. The first root is $R_1 = 0$ and the second one is $R_2 = \frac{c(1-\rho)-\lambda\mu}{c\mu}$. As the safety loading θ for this risk model is a positive one, we have that for the Pólya-Aeppli risk model the premium income per unit time c satisfies the inequality $c > \frac{\lambda\mu}{1-\rho}$. This condition determines that the root R_2 is a positive root of the equation (16).

Model 3: The Non-central Pólya-Aeppli Risk model

The counting process $N(t)$ is a sum of two independent processes. The first one is a homogeneous Poisson process with intensity λ_1 i.e $N(t) \sim P_oP(\lambda_1)$ and the second one is a Pólya-Aeppli process with parameters λ_2 and ρ i.e $N(t) \sim PAP(\lambda_2, \rho)$.

In the case of exponentially distributed claims for this risk model, the function $g(r)$ is given by

$$rc - \lambda_1\frac{\mu r}{1-\mu r} - \lambda_2\frac{\mu r}{1-\rho-\mu r} = 0 \tag{17}$$

or

$$r\left[c - \lambda_1\frac{\mu}{1-\mu r} - \lambda_2\frac{\mu}{1-\rho-\mu r}\right] = 0.$$

We see that the last equation has a zero root $R_1 = 0$. The other roots are found by solving the equation

$$c - \lambda_1\frac{\mu}{1-\mu r} - \lambda_2\frac{\mu}{1-\rho-\mu r} = 0.$$

This leads to the following quadratic equation

$$c\mu^2 r^2 + \left[(\lambda_1+\lambda_2)\mu^2 - c\mu(2-\rho)\right]r + c(1-\rho) + \lambda_1\mu\rho - (\lambda_1+\lambda_2)\mu = 0. \tag{18}$$

The quadratic equation's discriminant has the form

$$D = [(\lambda_1+\lambda_2)]^2\,\mu^4 + 2c\mu^3\rho(\lambda_2-\lambda_1) + c^2\mu^2\rho^2. \tag{19}$$

The roots of the equation (18) are as follows

$$R_2 = \frac{c\mu(2-\rho) - (\lambda_1+\lambda_2)\mu^2 + \sqrt{(\lambda_1+\lambda_2)^2\mu^4 + 2c\mu^3\rho(\lambda_2-\lambda_1) + c^2\mu^2\rho^2}}{2c\mu^2}$$

and

$$R_3 = \frac{c\mu(2-\rho) - (\lambda_1+\lambda_2)\mu^2 - \sqrt{(\lambda_1+\lambda_2)^2\mu^4 + 2c\mu^3\rho(\lambda_2-\lambda_1) + c^2\mu^2\rho^2}}{2c\mu^2}.$$

Our main task is to compare the given three risk models. For the purpose we take arbitrary values of the parameters $\mu,\ \lambda_1 = \lambda_2 = \lambda$ and ρ. The value of the premium income per unit time c is chosen so that the safety loading θ is a positive one. Since the value of the Lundberg's exponent R gives the risk measure of the company's business we are interested of the maximum positive root of the equations (15), (16) and (17).

From the Lundberg's inequality $\Psi(u) \le e^{-Ru}$ we can conclude that the risk model with greater R is better than the rest.

The tables below give the values of the Lundberg's exponent R for $\lambda_1 = \lambda_2 = 2$ and $\mu = 1$ calculated for each of the given risk models. For the Pólya-Aeppli risk model and for the Non-central Pólya-Aeppli risk model we have an additional parameter $\rho \in [0,1)$. We give five values for the parameter ρ i.e $\rho = 0.1,\ 0.3,\ 0.5,\ 0.7$ and 0.9. We take c under the conditions $c > \lambda\mu,\ c > \frac{\lambda\mu}{1-\rho}$ and $c > \frac{\mu(\lambda_1(1-\rho)+\lambda_2)}{1-\rho}$.

Poisson Risk model

$c = 5$	$R_{P_o} = 0.6$
$c = 7$	$R_{P_o} = 0.71$
$c = 9$	$R_{P_o} = 0.78$
$c = 23$	$R_{P_o} = 0.91$

Pólya-Aeppli Risk model

$c = 5$	$\rho = 0.1$	$R_{PA} = 0.5$
$c = 5$	$\rho = 0.3$	$R_{PA} = 0.3$
$c = 7$	$\rho = 0.5$	$R_{PA} = 0.21$
$c = 9$	$\rho = 0.7$	$R_{PA} = 0.08$
$c = 23$	$\rho = 0.9$	$R_{PA} = 0.01$

Non-central Pólya-Aeppli Risk model

$c = 5$	$\rho = 0.1$	$R_{NPA} = 0.95$
$c = 5$	$\rho = 0.3$	$R_{NPA} = 0.88$
$c = 7$	$\rho = 0.5$	$R_{NPA} = 0.84$
$c = 9$	$\rho = 0.7$	$R_{NPA} = 0.84$
$c = 23$	$\rho = 0.9$	$R_{NPA} = 0.92$

The Lundberg's exponent R for the Poisson risk model and the Pólya-Aeppli risk model is calculated for one and the same value of the intensity λ. Taking the values of $c = 5, \rho = 0.1, \lambda_1 = \lambda_2 = 2$ and $\mu = 1$ we obtained the following values of the Lundberg's exponent R:

$$R_{P_o} = 0.6,\ R_{PA} = 0.5,\ R_{NPA} = 0.95.$$

As the Lundberg's exponent R gives the risk measure of the insurance company's business we can conclude that for these specific values of the parameters, the Non-central Pólya-Aeppli risk model is the best than the other ones. It is a suitable risk model because its Lundberg's exponent R is greater than the Lundbergs' exponents calculated for the Poisson and the Pólya-Aeppli risk models. Similarly if we see the other values in the tables above and the obtained results for the positive root R we may conclude the same.

CONCLUDING REMARKS

In this paper we introduced three risk models: a Poisson risk model, a Pólya-Aeppli risk model and a Non-central Pólya-Aeppli risk model. For these models we obtained a martingale approach and a corresponding estimation of the ruin probability. We compared them in the case of exponentially distributed claims and give some recommendations for choosing a risk model in the insurance.

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