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On the Generalized Solutions of a Boundary Value Problem for Multidimensional Hyperbolic and Weakly Hyperbolic Equations

RESEARCH MONOGRAPH

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Preface

Denoting $(x,t) := (x_1, x_2, x_3, t) \in \mathbb{R}^4$, for $m \in \mathbb{R}$, $0 \le m < 2$ we consider the equation

$$L_m[u] \equiv u_{x_1x_1} + u_{x_2x_2} + u_{x_3x_3} - (t^m u_t)_t = f(x,t) \tag{0.1}$$

in the domain

$$\Omega_m := \left\{ (x,t) : \quad 0 < t < t_0, \quad \frac{2}{2-m} t^{\frac{2-m}{2}} < |x| < 1 - \frac{2}{2-m} t^{\frac{2-m}{2}} \right\},$$

where $t_0 = \left(\frac{2-m}{4}\right)^{\frac{2}{2-m}}$. Here, as usual, $|x| = \sqrt{x_1^2 + x_2^2 + x_3^2}$.

The region Ω_m is bounded by the ball

$$\Sigma_0 := \{(x,t) : \; t = 0, \quad |x| < 1\},$$

centered at the origin $O(0,0,0,0)$ and by the surfaces

$$\Sigma_1^m := \left\{ (x,t) : 0 < t < t_0, \; |x| = 1 - \frac{2}{2-m} t^{\frac{2-m}{2}} \right\},$$

$$\Sigma_2^m := \left\{ (x,t) : 0 < t < t_0, \; |x| = \frac{2}{2-m} t^{\frac{2-m}{2}} \right\}.$$

In the present monograph we are interested basically in **the case $\mathbf{0 < m < 2}$**, when equation (0.1) is hyperbolic with power-type degeneration at Σ_0, or more precisely, it is a *weakly hyperbolic equation of Keldysh type*. We study the so called Protter-Morawetz boundary value problem (or, shortly, Protter problem) for this equation, i.e. we have the following boundary conditions:

$$u|_{\Sigma_1^m} = 0, \qquad t^m u_t \to 0 \text{ as } t \to +0.$$

In this work we denote this problem by P_m.

However, we begin our investigations with the limiting **case $\mathbf{m = 0}$** (Problem P_0), when equation (0.1) becomes simply the four-dimensional *wave equation* and the boundary condition on Σ_0 turns to

$$u_t|_{\Sigma_0} = 0.$$

In this case the problem is much more simpler and, correspondingly, it is very well studied, but here we derive some more precise results on the exact behavior of the solutions. These results are helpful, since they suggest the structure of the solutions of the boundary value problem in the more general case $0 < m < 2$.

It is well known that different boundary value problems for mixed-type equations have important applications in transonic gas dynamics, such as modeling of certain flows around airfoils (see Bers [6], Morawetz [28], Kuz'min [23]). After a space symmetry assumption, the transonic potential flows in fluid dynamics are described in the hodograph plane by two-dimensional boundary value problems (such as the famous Guderley-Morawetz problem) for the Chaplygin equation

$$K(t)u_{xx} - u_{tt} = 0$$

with $tK(t) > 0$ for $t \neq 0$. Clearly, this equation is elliptic in the subsonic half-plane $t < 0$ and hyperbolic in the supersonic half-plane $t > 0$.

M. Protter [43, 44] proposed some multidimensional variants of the Guderley-Morawetz problem in a domain consisting of a subsonic ($t < 0$) and a supersonic part ($t > 0$). Restricting his investigation only in the supersonic part of the domain, he also formulated some new boundary value problems for the equation

$$t^m \sum_{j=1}^{N} u_{x_j x_j} - u_{tt} = f(x, t) \tag{0.2}$$

with $N \geq 2$, $m \geq 0$ and $x := (x_1, \ldots, x_N)$. This equation obviously is a multidimensional analogue of the Chaplygin equation with $K(t) = \operatorname{sgn}(t)\, |t|^m$. More precisely, Protter formulated his new problems in a domain bounded by two characteristics surfaces of equation (0.2) as well as by the surface $\{t = 0\}$ and he prescribed the boundary data on one of these characteristic surfaces and on the hyperplane $\{t = 0\}$. In this way, the Protter problems are multidimensional analogues of the Darboux plane problems for the Gellerstedt equation ($m > 0$) or for the wave equation

($m = 0$). (Actually, for $m = 0$, when we have Neumann boundary data on Σ_0, the corresponding four-dimensional Protter problem coincides with Problem P_0 which we study in the first chapter.)

However, while the two-dimensional Darboux problems are well posed, this is not true for the Protter problems. Actually, these problems have infinite-dimensional cokernels, which firstly were found out for the wave equation ([50]) and, after, for the Gellerstedt equation ([41, 22]). This means that for the existence of classical solutions it is necessary infinitely many orthogonality conditions on the right-hand side of the equation to be fulfilled. For this reason, Popivanov and Schneider [41] suggested the Protter problems to be studied in the frame of generalized solutions with possible big singularities. Today it is well-known that such singularities really exist (see for example [16, 33, 38, 41]). It is interesting that they are isolated at one boundary point and do not propagate along the bicharacteristics, which is not traditionally assumed for the hyperbolic equations.

Different aspects of the Protter problems and several their generalizations are studied by many authors, see for example [1, 2, 4, 8, 12, 32] and references therein. For different statements of other related problems for mixed-type equations, including nonlinear equations, see [9, 10, 21, 25, 27, 47].

The Chaplygin equation and its multidimensional variants are known as *Tricomi-type equations*, while the mixed-type equation

$$u_{xx} - K(t)u_{tt} = 0$$

and its different generalizations (including equation (0.1) for $m \in (0,2)$) are known to be *Keldysh-type equations*.

It is known that the Keldysh-type equations also play an important role in fluid mechanics, for example the equation

$$u_{xx} + t^m u_{tt} + a u_x + b u_t + cu = 0 \tag{0.3}$$

near the line $t = 0$. Keldysh [19], while studied the regularity of the solutions of 2-D elliptic equations near the boundary, showed that for the degenerating elliptic equation (0.3) the formulation of the Dirichlet problem may depend on the lower order terms (the dependence is different for different values of m).

Fichera [11] generalized Keldysh's results for multidimensional linear equations with nonnegative characteristic form and now the boundary value problems for them are well understood in the sense that boundary conditions should not be imposed on the whole boundary. A summary of Fichera's theory can be found in Radkevich [45, 46]. Keyfitz [20] examined whether the Fichera's classification could be extended to quasilinear equations and mentioned that the contrasting behavior of the characteristics of the Tricomi and the Keldysh-type equations may have implications, unexplored yet, for the solutions of some free boundary problems arising in the fluid dynamics. Otway [34, 35] and Lupo, Monticelli and Payne [24] gave a statement of some 2-D boundary value problems for elliptic-hyperbolic Keldysh-type equations with specific applications in plasma physics, including a model for analyzing the possible heating in axisymmetric cold plasmas.

In view of all these results it is interesting for us to study the multidimensional Protter problems for Keldysh-type equations.

In [17] Hristov, Popivanov and Schneider considered a three-dimensional analogue of Problem P_m $(0 < m < 2)$ involving lower order terms and proved the uniqueness of quasiregular solution.

We mention here that a specific feature of the Keldysh-type equations is that their solutions are not differentiable at the degenerate boundary $\{t = 0\}$ (see [7]). Then, in contrary to the Protter problems for Tricomi-type equations, we cannot prescribe Neumann boundary data on $\{t = 0\}$. Indeed, in our Problem P_m $(0 < m < 2)$ we have no data on the ball Σ_0. Instead of this, we have only a limitation on the growth of the possible singularity of the derivative u_t, imposed by the second condition in (2.2).

Nevertheless, in this works, based on series of our publications ([14, 15, 29, 31, 36, 37]), we find some essential similarities between Problem P_m and the Protter problems for Tricomi-type equations: they have infinite-dimensional co-kernel and they have generalized solutions with strong singularities isolated at one boundary point.

The present monograph consists of Preface, two chapters and Appendix. In the first chapter we treat a Protter problem for the wave equation and we improve a well known asymptotic formula describing the exact behavior of the singular solutions. In the second chapter we treat a Protter problem for equation (0.1) with $m \in (0, 2)$. For the case $0 < m < 4/3$ we prove the existence and uniqueness of a generalized solution of this problem at certain conditions for the right-hand side of the equation, as well as we find an asymptotic expansion of the singular solutions. The Appendix contains various formulas for the hypergeometric function and some of its generalizations as far as they are one of the basic tools which we apply in our calculations.

1. The Protter problem for the wave equation

In this chapter we study the case $m = 0$, when equation (0.1) is hyperbolic, with no degeneration on $t = 0$. An important result concerning this research we have announced in [29], this is Theorem 1.5.1 which we consider in Sections 1.5-1.6.

1.1. Statement of the problem

More precisely, here we consider the four-dimensional wave equation

$$L_0[u] \equiv u_{x_1x_1} + u_{x_2x_2} + u_{x_3x_3} - u_{tt} = f(x,t) \tag{1.1}$$

in

$$\Omega_0 := \{(x,t) : \ 0 < t < 1/2, \ t < |x| < 1 - t\} .$$

The region Ω_0 (see Fig. 1.1) is bounded by a non-characteristic surface, this is the ball

$$\Sigma_0 := \{(x,t) : \ t = 0, \ |x| < 1\},$$

and by two characteristic surfaces of equation (1.1)

$$\Sigma_1^0 := \{(x,t) : \ 0 < t < 1/2, \ |x| = 1 - t\},$$

$$\Sigma_2^0 := \{(x,t) : \ 0 < t < 1/2, \ |x| = t\}.$$

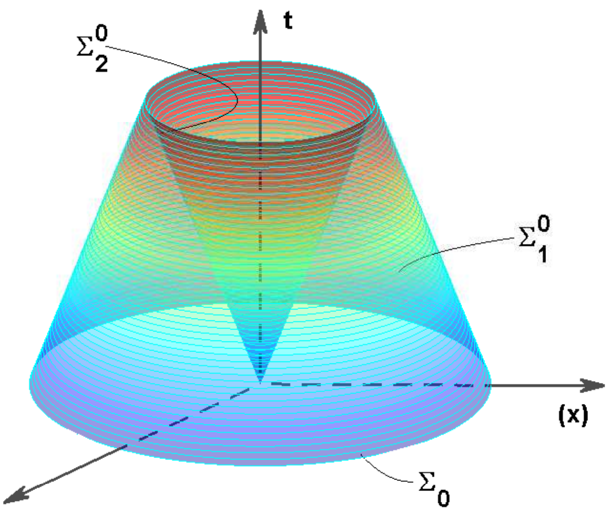

Figure 1.1.: The region Ω_0.

We consider the following boundary value problem:

Problem $\mathbf{P_0}$. *Find a solution to equation* (1.1) *in* Ω_0 *which satisfies the boundary conditions*

$$u|_{\Sigma_1^0} = 0, \qquad u_t|_{\Sigma_0} = 0.$$

The adjoint problem to P_0 is as follows:

Problem $\mathbf{P_0^*}$. *Find a solution to the self-adjoint equation* (1.1) *in* Ω_0 *which* *satisfies the boundary conditions*

$$u|_{\Sigma_2^0} = 0, \qquad u_t|_{\Sigma_0} = 0.$$

1.2. Some known results concerning Problem $\mathbf{P_0}$

Firstly, we give some well known results, developed in [38], [40].

As we mentioned in the Preface, the Protter problems are not well posed. In particular, Problem P_0 is ill-posed as well: the adjoint homogeneous Problem P_0^* has infinitely many linearly independent classical solutions.

In order to give their exact representation, for $k, n \in \mathbb{N} \cup \{0\}$ introduce the following functions:

$$\mathcal{E}_k^n(|x|, t) := \sum_{i=0}^{k} A_i^k |x|^{-n+2i-1} \left(|x|^2 - t^2\right)^{n-k-i},$$

where

$$A_i^k := (-1)^i \frac{(k-i+1)_i (n-k-i+1)_i}{i!(n+1/2-i)_i}.$$

Here $(a)_i := \Gamma(a+i)/\Gamma(a)$, which gives $(a)_i = a(a+1)\dots(a+i-1)$ for $i \in \mathbb{N}$, and $(a)_0 = 1$.

Further, let us denote by $Y_n^s(x)$, $n \in \mathbb{N} \cup \{0\},\ s = 1, 2, \dots, 2n+1$ the three-dimensional spherical functions. They are usually defined on the unit sphere $S^2 := \{x \in \mathbb{R}^3 : |x| = 1\}$, but for convenience of our discussions we extend them out of S^2 radially, keeping the same notation for the extended functions:

$$Y_n^s(x) := Y_n^s(x/|x|), \quad x \in \mathbb{R}^3 \setminus \{0\}.$$

Then the following lemma holds:

Lemma 1.2.1 ([38]). *For $k = 0, \dots, [n/2] - 2$ and $s = 1, 2, \dots, 2n + 1$ the functions*

$$v_{k,s}^n(x,t) := \begin{cases} \mathcal{E}_k^n(|x|,t) Y_n^s(x), & (x,t) \neq O, \\ 0, & (x,t) = O \end{cases} \tag{1.2}$$

are classical solutions from $C^2(\bar{\Omega}_0)$ of the homogeneous Problem P_0^*.

It is easy to see that a necessary condition for the existence of a classical solution of Problem P_0 is the orthogonality of the right-hand side function $f(x,t)$ to all these functions $v_{k,s}^n(x,t)$. Indeed

$$\begin{aligned} \int_{\Omega_m} v_{k,s}^n(x,t) f(x,t)\, dxdt &= \int_{\Omega_m} v_{k,s}^n(x,t) L_0[u](x,t)\, dxdt \\ &= \int_{\Omega_m} L_0[v_{k,s}^n](x,t) u(x,t)\, dxdt = 0. \end{aligned}$$

This means that an infinite number of orthogonality conditions $\mu_{k,s}^n = 0$ with

$$\mu_{k,s}^n := \int_{\Omega_0} v_{k,s}^n(x,t) f(x,t)\, dxdt \tag{1.3}$$

must be fulfilled.

In this case it is suitable to seek for solutions to this problem in a generalized sense. Similarly to Popivanov and Schneider [42], the generalized solutions of Problem P_0 are defined in the following way:

Definition 1.2.1 ([38]). *A function $u = u(x,t)$ is called a generalized solution of Problem P_0 in Ω_0 if:*

(1) $u \in C^1(\bar{\Omega}_0 \backslash O),\ u|_{\Sigma_1^0} = 0,\ u_t|_{\Sigma_0 \backslash O} = 0;$

(2) *the identity*

$$\int_{\Omega_0} (u_t v_t - u_{x_1} v_{x_1} - u_{x_2} v_{x_2} - u_{x_3} v_{x_3} - fv)\, dxdt = 0$$

holds for all v from

$$V_0 := \{v \in C^1(\bar{\Omega}_0) :\ v_t|_{\Sigma_0} = 0, v \equiv 0 \textit{ in a neighborhood of } \Sigma_2^0\}.$$

This definition allows the generalized solutions to have strong singularities at the point O. In the general case such singularities really exist.

Next, the following result on the existence and uniqueness of the generalized solution of Problem P_0 is valid:

Theorem 1.2.1 ([38]). *Problem P_0 has at most one generalized solution in Ω_0. Additionally, suppose that the right-hand side of* (1.1) *belongs to $C^1(\bar{\Omega}_0)$ and has the form*

$$f(x,t) = \sum_{n=0}^{l} \sum_{s=1}^{2n+1} f_n^s(|x|,t) Y_n^s(x), \tag{1.4}$$

where $l \in \mathbb{N} \cup \{0\}$. Then the unique generalized solution of the Problem P_0 in Ω_0 exists and it has the form

$$u(x,t) = \sum_{n=0}^{l} \sum_{s=1}^{2n+1} u_n^s(|x|,t) Y_n^s(x). \tag{1.5}$$

Further, we will focus on the important particular case when the right-hand side function $f(x,t)$ is of the form (1.4). Actually, it is well known that the spherical functions form a complete orthonormal system in $L_2(S^2)$ (for detailed information on the spherical functions see for example [18]).

The next result describes the asymptotic behavior of the singularities of the generalized solution:

Theorem 1.2.2 ([40]). *Suppose that the right-hand side function $f \in C^1(\bar{\Omega}_0)$ has the form* (1.4). *Then the unique generalized solution $u(x,t)$ of Problem P_0 belongs to $C^2(\bar{\Omega}_0 \backslash O)$ and has the following asymptotic expansion at the singular point O:*

$$u(x,t) = \sum_{p=1}^{l+1} (|x|^2 + t^2)^{-p/2} F_p(x,t) + F(x,t), \tag{1.6}$$

where

(i) *the function $F \in C^2(\bar{\Omega}_0 \backslash O)$ and satisfies the a priori estimate*

$$|F(x,t)| \leq C \|f\|_{C^1(\Omega_0)}, \quad (x,t) \in \Omega_0$$

with a constant C independent of f;

(ii) *the functions F_p, $p = 1, \dots, l+1$ satisfy the equalities*

$$F_p(x,t) = \sum_{k=0}^{[(l-p+1)/2]} \sum_{s=1}^{2p+4k-1} \mu_{k,s}^{p+2k-1} F_{k,s}^{p+2k-1}(x,t) \tag{1.7}$$

with functions $F_{k,s}^n \in C^2(\bar{\Omega}_0 \backslash O)$ bounded and independent of f;

(iii) *if at least one of the constants $\mu_{k,s}^{p+2k-1}$ in* (1.7) *is different from* *zero, then for the corresponding function* $F_p(x,t)$ *there exists a direction* $(\alpha, 1) := (\alpha_1, \alpha_2, \alpha_3, 1)$ *with* $(\alpha, 1)\, t \in \Sigma_2^0$ *for* $0 < t < 1/2$*, such that*

$$\lim_{t \to +0} F_p(\alpha t, t) = \text{const} \neq 0.$$

This means that in this case the order of singularity of $u(x,t)$ *will be no smaller than* p*.*

According to this theorem, the order of singularity of $u(x,t)$ can be strictly fixed by the coefficients (1.3), i.e. by choosing the right-hand side $f(x,t)$ to be orthogonal to the appropriate functions $v_{k,s}^n(x,t)$. Note also that the derived asymptotic expansion clearly illustrates the fact that, as we mentioned above, a necessary (but not sufficient) condition for the classical solvability of Problem P_0 is the orthogonality of $f(x,t)$ to all the classical solutions of the adjoint homogenous Problem P_0^*.

1.3. Two-dimensional problem corresponding to Problem $\mathbf{P_0}$

Problem P_0 in the case when the right-side function $f(x,t)$ is of the form (1.4) reduces to a two-dimensional problem ([38]).

More precisely, let us look for a solution to Problem P_0 of the form (1.5). Using the spherical coordinates $(r, \theta, \varphi, t) \in \mathbb{R}^4$, $r > 0$, $0 \leq \theta < \pi$, $0 \leq \varphi < 2\pi$ with

$$x_1 = r \sin\theta \cos\varphi, \quad x_2 = r \sin\theta \sin\varphi, \quad x_3 = r\cos\theta, \tag{1.8}$$

and later in the characteristic coordinates

$$\xi = 1 - r - t, \quad \eta = 1 - r + t,$$

for the functions

$$U(\xi,\eta) := r(\xi,\eta)u_n^s\big(r(\xi,\eta), t(\xi,\eta)\big)$$

the following Darboux-Goursat problem is obtained:

Problem $\mathbf{P_{02}}$. *Find a solution of the equation*

$$E_0[U] \equiv U_{\xi\eta} - \frac{n(n+1)}{(2-\xi-\eta)^2}U = F(\xi,\eta) \quad \textit{in} \quad D, \tag{1.9}$$

satisfying the following boundary conditions

$$U(0,\eta) = 0,\ (U_\xi - U_\eta)(\xi,\xi) = 0, \tag{1.10}$$

where

$$D := \{(\xi,\eta):\ 0 < \xi < \eta < 1\}$$

and

$$F(\xi,\eta) := \frac{1}{8}(2-\xi-\eta)f_n^s(r(\xi,\eta), t(\xi,\eta)).$$

Note that equation (1.9) involves a coefficient with singularity at the point $(\xi,\eta) = (1,1)$.

From the results in [38], [40] it is known that if $F \in C^1(\bar{D})$, then there exists an unique function $U(\xi,\eta)$, belonging to $C^2(\bar{D}\backslash\{(1,1)\})$, which is a classical solution of the considered problem in each domain $D \cap \{\xi < \delta,\ 0 < \delta < 1\}$, but it may become unbounded as $(\xi, \eta) \to (1, 1)$. In the present paper this function will be called a *generalized solution* of Problem P_{02}.

The asymptotic behavior of $U(\xi, \eta)$ at the point $(1, 1)$ is closely connected with the non-uniqueness results on the corresponding adjoint homogeneous problem

$$E_0[U] = 0 \quad \text{in } D, \tag{1.11}$$

$$U(\xi, 1) = 0, \quad (U_\eta - U_\xi)(\xi, \xi) = 0. \tag{1.12}$$

Indeed, for $k = 0, 1, \ldots, [(n-3)/2]$ the functions

$$E_k^n(\xi, \eta) := \begin{cases} \dfrac{(1-\xi)^{n-2k}(1-\eta)^{n-2k}}{(2-\xi-\eta)^{n-2k}}\, {}_2F_1\left(n-k+\dfrac{1}{2},\ -k, \dfrac{1}{2};\quad \dfrac{(\eta-\xi)^2}{(2-\xi-\eta)^2}\right), & (\xi,\eta) \neq (1,1), \\ 0, & (\xi,\eta) = (1,1), \end{cases} \tag{1.13}$$

where ${}_2F_1(a, b, c; \zeta)$ is the Gauss hypergeometric series, are classical solutions to problem (1.11)-(1.12). They can be obtained from Lemma 1.2.1. Actually, taking into account the formulas (A.4) and (A.16), one can see that the functions $E_k^n(\xi, \eta)$ are connected with the functions $\mathcal{E}_k^n(|x|, t)$ by the relation

$$E^n_k(\xi,\eta) = \gamma^n_k(2-\xi-\eta)\mathcal{E}^n_k(r(\xi,\eta),t(\xi,\eta)), \quad \gamma^n_k := \frac{(-1)^k\,(1/2-n)_k}{2^{n-2k+1}\,(1/2)_k}. \tag{1.14}$$

Further, it was derived a decomposition of the generalized solution of Problem P_{02} in terms of the functions $E^n_k(\xi,\eta)$, or more precisely, with use of the scalar products

$$\mu^n_k := \int_D E^n_k(\xi,\eta)F(\xi,\eta)\,d\xi d\eta. \tag{1.15}$$

Namely, the function $U(\xi,\eta)$ according to [38], [40] may be expanded in the following way:

$$U(\xi,\eta) = \sum_{k=0}^{[n/2]} \mu^n_k G^n_k(\xi,\eta)(2-\xi-\eta)^{2k-n} + G(\xi,\eta), \quad (\xi,\eta)\in D, \tag{1.16}$$

where $G^n_k(\xi,\eta)$ and $G(\xi,\eta)$ are bounded in $\bar{D}$ functions, such that $G^n_k(\xi,1) =$ const $\neq 0$ and

$$|G(\xi,\eta)| \le K\|F\|_{C^1(D)}(2-\xi-\eta)\,(1+|\ln(2-\xi-\eta)|), \quad (\xi,\eta)\in D \tag{1.17}$$

with a positive constant K. The functions $G^n_k(\xi,\eta)$ and the constant K are independent of $F(\xi,\eta)$.

According to this decomposition, the order of singularity of the generalized solution $U(\xi,\eta)$ at the point $(1,1)$ can be strictly fixed by the coefficients μ^n_k, i.e. by choosing the right-hand side $F(\xi,\eta)$ to be orthogonal to the appropriate functions $E^n_k(\xi,\eta)$. Obviously, the asymptotic expansion of the generalized solution $u(x,t)$ of Problem P_0 in Theorem 1.2.2 is closely related to the expansion (1.16).

1.4. Further study of the function $\mathbf{U}(\xi,\eta)$ and its restriction $\mathbf{U}(\xi,\mathbf{1})$

Firstly, although an explicit form of the generalized solution $U(\xi,\eta)$ is given in [38], [40], here we give a more simple one, with use of a Riemann-Hadamard function. Furthermore, we impose a weaker condition on the right-hand side $F(\xi,\eta)$: for all our next calculations it is sufficient $F(\xi,\eta)$ to be continuous in $\bar{D}$.

Theorem 1.4.1. *Let $F \in C(\bar{D})$. Then there exists an unique generalized solution of Problem P_{02} and it has the following integral representation at a point $(\xi_0,\eta_0) \in D$:*

$$U(\xi_0,\eta_0) = \int_0^{\xi_0} \int_{\xi}^{\eta_0} \Phi(\xi,\eta;\xi_0,\eta_0) F(\xi,\eta)\, d\eta d\xi \tag{1.18}$$

where the Riemann-Hadamard function $\Phi(\xi,\eta;\xi_0,\eta_0)$ is defined as

$$\Phi(\xi,\eta;\xi_0,\eta_0) := \begin{cases} \Phi^+(\xi,\eta;\xi_0,\eta_0), & \eta > \xi_0, \\ \Phi^-(\xi,\eta;\xi_0,\eta_0), & \eta < \xi_0 \end{cases}$$

with

$$\Phi^+(\xi,\eta;\xi_0,\eta_0) := {}_2F_1(n+1,-n,1;Y),$$

$$\Phi^-(\xi,\eta;\xi_0,\eta_0) := {}_2F_1(n+1,-n,1;Y) + {}_2F_1(n+1,-n,1;Y^*)$$

and

$$Y = Y(\xi,\eta;\xi_0,\eta_0) := \frac{-(\xi_0-\xi)(\eta_0-\eta)}{(2-\xi-\eta)(2-\xi_0-\eta_0)},$$

$$Y^* = Y(\xi,\eta;\eta_0,\xi_0) := \frac{-(\eta_0-\xi)(\xi_0-\eta)}{(2-\xi-\eta)(2-\xi_0-\eta_0)}.$$

Proof. It is well known that $\Phi^+(\xi,\eta;\xi_0,\eta_0)$ is the Riemann function for equation (1.9). Then, given a point $(\xi_0,\eta_0)\in D$, the function $\Phi^+(\xi,\eta;\xi_0,\eta_0)$ has the following properties, which we need for our considerations:

(i) $E_0[\Phi^+](\xi,\eta;\xi_0,\eta_0)=0$ in D;

(ii) $\Phi^+(\xi_0,\eta;\xi_0,\eta_0)=1, \quad \xi_0\le\eta\le 1$;

(iii) $\Phi^+(\xi,\eta_0;\xi_0,\eta_0)=1, \quad 0\le\xi\le\xi_0$.

Further, a direct calculation shows that

(iv) $(\Phi^-_\xi-\Phi^-_\eta)(\xi,\xi;\xi_0,\eta_0)=0, \quad 0\le\xi\le\xi_0$

and

(v) $\Phi^-(\xi,\xi_0;\xi_0,\eta_0)-\Phi^+(\xi,\xi_0;\xi_0,\eta_0)=1, \quad 0\le\xi\le\xi_0$.

First, suppose that $U(\xi,\eta)$ is a generalized solution of Problem P_{02}.

Then applying an integration by parts into the identity

$$\int_0^{\xi_0}\int_{\xi_0}^{\eta_0} E_0[U](\xi,\eta)\Phi^+(\xi,\eta;\xi_0,\eta_0)\,d\eta d\xi \\ +\int_0^{\xi_0}\int_0^{\eta} E_0[U](\xi,\eta)\Phi^-(\xi,\eta;\xi_0,\eta_0)\,d\xi d\eta$$

$$=\int_0^{\xi_0}\int_{\xi_0}^{\eta_0} F(\xi,\eta)\Phi^+(\xi,\eta;\xi_0,\eta_0)\,d\eta d\xi \\ +\int_0^{\xi_0}\int_0^{\eta} F(\xi,\eta)\Phi^-(\xi,\eta;\xi_0,\eta_0)\,d\xi d\eta,$$

with use of the properties (i)-(v) of the function $\Phi(\xi,\eta;\xi_0,\eta_0)$ and the boundary conditions (1.10), we obtain that the function $U(\xi,\eta)$ should have the representation (1.18) at the point (ξ_0,η_0), which confirms the uniqueness.

A direct calculation shows that in $\bar{D}\backslash(1,1)$ the function $U(\xi,\eta)$, defined by (1.18), really satisfies the differential equation and the boundary conditions. To check this, take into account that the function $\Phi(\xi,\eta;\xi_0,\eta_0)$ solves the corresponding homogeneous equation not only in respect to the variables ξ and η, but also in respect to ξ_0 and η_0:

$$\Phi_{\xi_0\eta_0}-\frac{n(n+1)}{(2-\xi_0-\eta_0)^2}\,\Phi=0.$$

This confirms the existence. $\square$

Next, from here we derive a more accurate formula for the restriction $U(\xi,1)$:

Theorem 1.4.2. *Suppose that $F\in C(\bar{D})$. Then the restriction $U(\xi,1)$ of* *the generalized solution of Problem P_{02} has the following expansion on the segment* $\{0 \le \xi < 1\}$:

$$U(\xi,1) = \sum_{k=0}^{[n/2]} 2a_k^n \mu_k^n (1-\xi)^{2k-n} - \sum_{k=0}^{[n/2]} 2a_k^n J_k^n(\xi)(1-\xi)^{2k-n} + J^+(\xi), \quad (1.19)$$

where μ_k^n *are the coefficients* (1.15) *and*

$$a_k^n := \frac{(n+1)_{n-2k}(-n)_{n-2k}(1/2)_k}{(n-2k)!\,(n-k)!\,(-1)^{n+k}}, \quad (1.20)$$

$$J_k^n(\xi) := \int_\xi^1 \int_0^{\eta_1} E_k^n(\xi_1,\eta_1)F(\xi_1,\eta_1)\,d\xi_1 d\eta_1, \quad (1.21)$$

$$J^+(\xi) := \int_\xi^1 \int_0^{\xi} \Phi^+(\xi_1,\eta_1;\xi,1)F(\xi_1,\eta_1)\,d\xi_1 d\eta_1.$$

Proof. Actually, we will prove that the function $\Phi^-(\xi,\eta;\xi_0,1)$ has the following expansion in negative powers of $(1-\xi_0)$:

$$\Phi^-(\xi,\eta;\xi_0,1) = \sum_{k=0}^{[n/2]} 2a_k^n E_k^n(\xi,\eta)(1-\xi_0)^{2k-n}. \quad (1.22)$$

Then (1.19) would follow directly from Theorem 1.4.1, where $\Phi^-(\xi,\eta;\xi_0,1)$ is represented by (1.22).

With use of (A.10) the function ${}_2F_1(n+1,-n,1;a\zeta+b)$ can be expanded in Taylor series in powers of ζ:

$$\begin{aligned} &{}_2F_1(n+1,-n,1;a\zeta+b) \\ &\quad = \sum_{s=0}^{n} \frac{(n+1)_s(-n)_s}{s!\,s!}\,{}_2F_1(n+1+s,-n+s,1+s;b)\,a^s\zeta^s. \end{aligned}$$

Then, denoting for shortness

$$Y_1 := Y(\xi,\eta;\xi_0,1) = \frac{-(\xi_0-\xi)(1-\eta)}{(2-\xi-\eta)(1-\xi_0)},$$

$$Y_1^* := Y(\xi,\eta;1,\xi_0) = \frac{-(1-\xi)(\xi_0-\eta)}{(2-\xi-\eta)(1-\xi_0)},$$

we obtain:

$${}_2F_1(n+1,-n,1;Y_1) = \sum_{s=0}^{n} c_s^n Q_s^n(\xi,\eta)(1-\xi_0)^{-s}, \tag{1.23}$$

$${}_2F_1(n+1,-n,1;Y_1^*) = \sum_{s=0}^{n} c_s^n Q_s^n(\eta,\xi)(1-\xi_0)^{-s}, \tag{1.24}$$

where

$$c_s^n := \frac{(n+1)_s(-n)_s}{s!\,s!}(-1)^s, \tag{1.25}$$

$$Q_s^n(\xi,\eta) := {}_2F_1\left(n+1+s,-n+s,1+s;\frac{1-\eta}{2-\xi-\eta}\right)\frac{(1-\xi)^s(1-\eta)^s}{(2-\xi-\eta)^s}. \tag{1.26}$$

Applying the quadratic transformation (A.17)-(A.19) to the function ${}_2F_1\,(n+1+s,-n+s,1+s;\zeta)$ gives:

$$\begin{aligned}
&{}_2F_1\,(n+1+s,-n+s,1+s;\zeta)\\
&=\begin{cases}
\alpha_s^n\,{}_2F_1\left(\dfrac{n+s+1}{2},\dfrac{s-n}{2},\dfrac{1}{2};(1-2\zeta)^2\right), & n-s \text{ even},\\[2ex]
\beta_s^n(1-2\zeta)\,{}_2F_1\left(\dfrac{n+2+s}{2},\dfrac{s-n+1}{2},\dfrac{3}{2};(1-2\zeta)^2\right), & n-s \text{ odd}
\end{cases}
\end{aligned}$$

with

$$\alpha_s^n := \frac{\Gamma(\frac{1}{2})\, s!}{\Gamma(\frac{n+2+s}{2})\Gamma(\frac{-n+s+1}{2})}, \qquad \beta_s^n := \frac{\Gamma(-\frac{1}{2})\, s!}{\Gamma(\frac{n+s+1}{2})\Gamma(\frac{-n+s}{2})}. \tag{1.27}$$

From here we have:

$$c^n_{n-2k} Q^n_{n-2k}(\xi,\eta) = c^n_{n-2k} Q^n_{n-2k}(\eta,\xi) = a^n_k E^n_k(\xi,\eta), \qquad k = 0,\dots,[n/2], \tag{1.28}$$

$$c^n_{n-2k-1} Q^n_{n-2k-1}(\xi,\eta) = -c^n_{n-2k-1} Q^n_{n-2k-1}(\eta,\xi) = b^n_k H^n_k(\xi,\eta), \qquad k = 0,\dots,[(n-1)/2], \tag{1.29}$$

where

$$a^n_k := c^n_{n-2k}\alpha^n_{n-2k}, \qquad b^n_k := c^n_{n-2k-1}\beta^n_{n-2k-1} \tag{1.30}$$

and

$$H^n_k(\xi,\eta) := \big(\eta-\xi\big)\,\frac{(1-\xi)^{n-2k-1}(1-\eta)^{n-2k-1}}{(2-\xi-\eta)^{n-2k}} \times {}_2F_1\left(n-k+\frac{1}{2}, -k, \frac{3}{2}; \frac{(\eta-\xi)^2}{(2-\xi-\eta)^2}\right). \tag{1.31}$$

Then the expansions (1.23)-(1.24) can be replaced by:

$$F(n+1,-n,1;Y_1) = \sum_{k=0}^{[n/2]} a_k^n E_k^n(\xi,\eta)(1-\xi_0)^{2k-n} + \sum_{k=0}^{[(n-1)/2]} b_k^n H_k^n(\xi,\eta)(1-\xi_0)^{2k+1-n} \quad (1.32)$$

$$F(n+1,-n,1;Y_1^*) = \sum_{k=0}^{[n/2]} a_k^n E_k^n(\xi,\eta)(1-\xi_0)^{2k-n} - \sum_{k=0}^{[(n-1)/2]} b_k^n H_k^n(\xi,\eta)(1-\xi_0)^{2k+1-n}. \quad (1.33)$$

It is easy to check that the coefficients a_k^n defined by (1.30) coincide with the corresponding coefficients defined by (1.20).

Finally, the expansion (1.22) follows directly from (1.32)-(1.33), which completes the proof. □

Remark 1.4.1. *It is known* ([38], [22]) *that for* $k = 0, 1, \ldots, [n/2]-1$ *the functions* $H_k^n(\xi,\eta)$ *(see* (1.31)*), continued as* $H_k^n(1,1) := 0$*, are classical solutions to a problem analogous to* (1.11)-(1.12)*, but with a Dirichlet boundary condition* $U(\xi,\xi) = 0$ *instead of* $(U_\eta - U_\xi)(\xi,\xi) = 0$.

Remark 1.4.2. *The constants* b_k^n *in* (1.30) *can be evaluated as*

$$b_k^n = \frac{(n+1)_{n-2k-1}(-n)_{n-2k-1}(3/2)_k}{(n-2k-1)!\,(n-k-1)!\,(-1)^{n+k+1}}. \quad (1.34)$$

In our further investigation we will need to know the derivative of the function $U(\xi,1)$, given in Theorem 1.4.2.

Lemma 1.4.1. *If $F \in C(\bar{D})$, then the derivative $U_\xi(\xi,1)$ has the following representation on the segment $\{0 \le \xi < 1\}$:*

$$U_\xi(\xi,1) = \sum_{k=0}^{[(n-1)/2]} 2a_k^n(n-2k)(1-\xi)^{2k-n-1}(\mu_k^n - J_k^n(\xi))$$
$$+\int_0^\xi F(\xi_1,\xi)\,d\xi_1 + \int_\xi^1 F(\xi,\eta_1)\,d\eta_1 + \int_\xi^1\int_0^\xi \Phi_\xi^+(\xi_1,\eta_1;\xi,1)F(\xi_1,\eta_1)\,d\xi_1 d\eta_1. \tag{1.35}$$

Proof. A direct calculation gives:

$$\begin{aligned} U_\xi(\xi,1) = &\sum_{k=0}^{[(n-1)/2]} 2a_k^n(n-2k)(1-\xi)^{2k-n-1}(\mu_k^n - J_k^n(\xi)) \\ &+\sum_{k=0}^{[n/2]} 2a_k^n(1-\xi)^{2k-n}\int_0^\xi E_k^n(\xi_1,\xi)F(\xi_1,\xi)\,d\xi_1 \\ &-\int_0^\xi \Phi^+(\xi_1,\xi;\xi,1)F(\xi_1,\xi)\,d\xi_1 + \int_\xi^1 F(\xi,\eta_1)\,d\eta_1 \\ &+\int_\xi^1\int_0^\xi \Phi_\xi^+(\xi_1,\eta_1;\xi,1)F(\xi_1,\eta_1)\,d\xi_1 d\eta_1. \end{aligned} \tag{1.36}$$

According to (1.22) we have

$$\sum_{k=0}^{[n/2]} 2a_k^n(1-\xi)^{2k-n}E_k^n(\xi_1,\xi) = \Phi^-(\xi_1,\xi;\xi,1) =$$
$$\Phi^+(\xi_1,\xi;\xi,1) + \Phi^+(\xi_1,\xi;1,\xi) = \Phi^+(\xi_1,\xi;\xi,1) + 1,$$

which we substitute in (1.36) to obtain (1.35). $\square$

1.5. An improved asymptotic representation of the function $\mathbf{U}(\xi, \eta)$

In order to represent our next results in a more compact form, we introduce the functions

$$\tilde{E}_k^n(\xi,\eta) := \frac{\alpha_{n-2k}^n}{(2-\xi-\eta)^{n-2k}}\,{}_2F_1\left(n-k+\frac{1}{2}, -k, \frac{1}{2}; \frac{(\eta-\xi)^2}{(2-\xi-\eta)^2}\right),$$

$$\tilde{H}_k^n(\xi,\eta) := \frac{\beta_{n-2k-1}^n\,(\eta-\xi)}{(2-\xi-\eta)^{n-2k}}\,{}_2F_1\left(n-k+\frac{1}{2}, -k, \frac{3}{2}; \frac{(\eta-\xi)^2}{(2-\xi-\eta)^2}\right),$$

where α_s^n and β_s^n are the coefficients (1.27). These functions are obviously connected with the functions (1.13) and (1.31) by the relations

$$\alpha_{n-2k}^n\,E_k^n(\xi,\eta) = (1-\xi)^{n-2k}(1-\eta)^{n-2k}\,\tilde{E}_k^n(\xi,\eta), \tag{1.37}$$

$$\beta_{n-2k-1}^n\,H_k^n(\xi,\eta) = (1-\xi)^{n-2k-1}(1-\eta)^{n-2k-1}\,\tilde{H}_k^n(\xi,\eta). \tag{1.38}$$

In the next theorem we show another exact formula for the generalized solution $U(\xi,\eta)$, which actually gives an explicit form of the functions $G_k^n(\xi,\eta)$ and $G(\xi,\eta)$ in the asymptotic representation (1.16).

Theorem 1.5.1. *Let $F \in C(\bar{D})$. Then the generalized solution of Problem*

P_{02} has the following representation in D:

$$
\begin{aligned}
U(\xi,\eta) = & \sum_{k=0}^{[n/2]} 2a_k^n \tilde{E}_k^n(\xi,\eta) \int_0^{\xi}\int_0^{\eta_1} E_k^n(\xi_1,\eta_1)F(\xi_1,\eta_1)\,d\xi_1 d\eta_1 \\
& + \sum_{k=0}^{[(n-1)/2]} 2a_k^n E_k^n(\xi,\eta) \int_0^{\xi}\int_0^{\eta_1} \tilde{E}_k^n(\xi_1,\eta_1)F(\xi_1,\eta_1)\,d\xi_1 d\eta_1 \\
& + \sum_{k=0}^{[n/2]} a_k^n \tilde{E}_k^n(\xi,\eta) \int_{\xi}^{\eta}\int_0^{\xi} E_k^n(\xi_1,\eta_1)F(\xi_1,\eta_1)\,d\xi_1 d\eta_1 \\
& + \sum_{k=0}^{[(n-1)/2]} a_k^n E_k^n(\xi,\eta) \int_{\xi}^{\eta}\int_0^{\xi} \tilde{E}_k^n(\xi_1,\eta_1)F(\xi_1,\eta_1)\,d\xi_1 d\eta_1 \\
& + \sum_{k=0}^{[(n-1)/2]} b_k^n \tilde{H}_k^n(\xi,\eta) \int_{\xi}^{\eta}\int_0^{\xi} H_k^n(\xi_1,\eta_1)F(\xi_1,\eta_1)\,d\xi_1 d\eta_1 \\
& + \sum_{k=0}^{[n/2]-1} b_k^n H_k^n(\xi,\eta) \int_{\xi}^{\eta}\int_0^{\xi} \tilde{H}_k^n(\xi_1,\eta_1)F(\xi_1,\eta_1)\,d\xi_1 d\eta_1,
\end{aligned}
\tag{1.39}
$$

where the coefficients a_k^n and b_k^n are given by (1.20) *and* (1.34) *respectively.*

The proof of this theorem is too long and we leave it for the next section.

Corollary 1.5.1. *Let $F \in C(\bar{D})$. Then the asymptotic expansion of $U(\xi,\eta)$ given by* (1.16) *is still valid, i.e.*

$$
U(\xi,\eta) = \sum_{k=0}^{[n/2]} \mu_k^n G_k^n(\xi,\eta)(2-\xi-\eta)^{2k-n} + G(\xi,\eta), \quad (\xi,\eta) \in D,
$$

(even if $F \notin C^1(\bar{D})$). Furthermore, we may specify

$$G_k^n(\xi,\eta) = 2a_k^n \alpha_{n-2k}^n \, {}_2F_1\left(n-k+\frac{1}{2}, -k, \frac{1}{2}; \frac{(\eta-\xi)^2}{(2-\xi-\eta)^2}\right).$$

For the function $G(\xi,\eta)$ there exists a positive constant K independent of F, such that a following estimate holds

$$|G(\xi,\eta)| \le K\|F\|_{C(D)}(2-\xi-\eta), \quad (\xi,\eta)\in D, \tag{1.40}$$

which improves the corresponding estimate (1.17).

Proof. The first term in the representation (1.39) may become unbounded as $(\xi,\eta)\to(1,1)$ and the other terms are bounded. Actually, defining

$$G_k^n(\xi,\eta) := 2a_k^n(2-\xi-\eta)^{n-2k}\tilde{E}_k^n(\xi,\eta),$$

the first term may be written as

$$\begin{aligned}
&\sum_{k=0}^{[n/2]} 2a_k^n \tilde{E}_k^n(\xi,\eta)\int_0^{\xi}\int_0^{\eta_1} E_k^n(\xi_1,\eta_1)F(\xi_1,\eta_1)\,d\xi_1 d\eta_1\\
&= \sum_{k=0}^{[n/2]} 2a_k^n\mu_k^n \tilde{E}_k^n(\xi,\eta) - \sum_{k=0}^{[n/2]} 2a_k^n \tilde{E}_k^n(\xi,\eta)\int_{\xi}^{1}\int_0^{\eta_1} E_k^n(\xi_1,\eta_1)F(\xi_1,\eta_1)\,d\xi_1 d\eta_1\\
&\qquad = \sum_{k=0}^{[n/2]} \mu_k^n G_k^n(\xi,\eta)(2-\xi-\eta)^{2k-n} - \sum_{k=0}^{[n/2]} 2a_k^n\tilde{E}_k^n(\xi,\eta)J_k^n(\xi).
\end{aligned}$$

Next, defining

$$G(\xi,\eta) := U(\xi,\eta) - \sum_{k=0}^{[n/2]} \mu_k^n G_k^n(\xi,\eta)(2-\xi-\eta)^{2k-n},$$

we obtain (1.40) with use of the estimates

$$
\begin{aligned}
&|E(\xi,\eta)| \le C(1-\eta)^{n-2k}, \qquad |H(\xi,\eta)| \le C(1-\eta)^{n-2k-1},\\
&|\tilde{E}(\xi,\eta)| \le C(1-\xi)^{2k-n}, \qquad |\tilde{H}(\xi,\eta)| \le C(1-\xi)^{2k+1-n},
\end{aligned}
\tag{1.41}
$$

where $C = \text{const} > 0$. The estimates (1.41) easily follow, taking into account that the hypergeometric series of the form ${}_2F_1(a,-k,c;\zeta)$, $|\zeta| \le 1$, $k \in \mathbb{N} \cup \{0\}$ are bounded, because they are polynomials of ζ. $\square$

Remark 1.5.1. *The expansion* (1.19) *is in accordance with the derived expansion of* $U(\xi,\eta)$. *Comparing* (1.19) *with* (1.16) *we have*

$$G(\xi,1) = -\sum_{k=0}^{[n/2]} 2a_k^n J_k^n(\xi)(1-\xi)^{2k-n} + J^+(\xi)$$

and

$$G_k^n(\xi,1) = 2a_k^n.$$

1.6. Proof of Theorem 1.5.1

At first we will prove some auxiliary lemmas.

Lemma 1.6.1. *For* $p = 1,\dots,n$ *the following relation holds*:

$$\sum_{s=0}^{n} \frac{p}{s+p} \frac{c_s^n Q_s^n(\xi,\eta)}{(1-\xi)^s} = (1-\eta)^{-p} Q_p^n(\xi,\eta), \tag{1.42}$$

where c_s^n *are the constants* (1.25) *and* $Q_s^n(\xi,\eta)$ *are the functions* (1.26).

Proof. For $p = 1, \dots, n$ we have:

$$\sum_{s=0}^{n} \frac{p}{s+p} \frac{c_s^n Q_s^n(\xi,\eta)}{(1-\xi)^s} = \sum_{s=0}^{n} \sum_{j=0}^{n-s} \frac{(p)_s}{(p+1)_s} \frac{(n+1)_{j+s}(-n)_{j+s}}{(-1)^s (1)_{j+s}\, j!\, s!} \frac{(1-\eta)^{j+s}}{(2-\xi-\eta)^{j+s}}$$
$$= \sum_{m=0}^{n} \frac{(n+1)_m(-n)_m}{m!\, m!} \frac{(1-\eta)^m}{(2-\xi-\eta)^m} \sum_{s=0}^{m} \frac{(p)_s(-m)_s}{(p+1)_s\, s!},$$

where we used (A.1)-(A.2). Further, using (A.9) we see that

$${}_2F_1(p, -m, p+1; 1) = \frac{\Gamma(p+1)\Gamma(m+1)}{\Gamma(1)\Gamma(p+m+1)} = \frac{m!}{(p+1)_m}.$$

Then we obtain:

$$\sum_{s=0}^{n} \frac{p}{s+p} \frac{c_s^n Q_s^n(\xi,\eta)}{(1-\xi)^s} = {}_2F_1\left(n+1, -n, p+1; \frac{1-\eta}{2-\xi-\eta}\right).$$

Finally, by the auto transformation formula (A.13) we have:

$${}_2F_1\left(n+1, -n, p+1; \frac{1-\eta}{2-\xi-\eta}\right)$$
$$= \frac{(1-\xi)^p}{(2-\xi-\eta)^p} \, {}_2F_1\left(p+n+1, p-n, p+1; \frac{1-\eta}{2-\xi-\eta}\right),$$

which completes the proof. $\square$

Lemma 1.6.2. *Define*

$$S^{(1)}(\xi_0,\eta_0) := \\ -\sum_{s=1}^{n}\sum_{p=0}^{n} c_s^n c_p^n Q_p^n(\xi_0,\eta_0)\left(\int_0^{\xi_0}\int_0^{\eta}(1-\eta)^{-s-p}Q_s^n(\eta,\xi)F(\xi,\eta)\,d\xi d\eta \right.\\ \left. +\int_0^{\xi_0}\int_\xi^1(1-\xi)^{-s-p}Q_s^n(\xi,\eta)F(\xi,\eta)\,d\eta d\xi\right). \quad (1.43)$$

Then

$$S^{(1)}(\xi_0,\eta_0) = \mathcal{J}(\xi_0,\eta_0) - \mathcal{J}_0(\xi_0,\eta_0), \quad (1.44)$$

where

$$\mathcal{J}(\xi_0,\eta_0) := \int_0^{\xi_0}\int_0^{\eta}\Phi^+(\eta,1;\xi_0,\eta_0)Q_0^n(\eta,\xi)F(\xi,\eta)\,d\xi d\eta \\ +\int_0^{\xi_0}\int_\xi^1\Phi^+(\xi,1;\xi_0,\eta_0)Q_0^n(\xi,\eta)F(\xi,\eta)\,d\eta d\xi, \quad (1.45)$$

$$\mathcal{J}_0(\xi_0,\eta_0) := \int_0^{\xi_0}\int_0^{\eta}\Phi^+(\eta,1;\xi_0,\eta_0)F(\xi,\eta)\,d\xi d\eta \\ +\int_0^{\xi_0}\int_\xi^1\Phi^+(\xi,1;\xi_0,\eta_0)F(\xi,\eta)\,d\eta d\xi. \quad (1.46)$$

Proof. According to (1.23)

$$\sum_{s=0}^{n} c_s^n Q_s^n(\eta,\xi)(1-\eta)^{-s} = \Phi^+(\eta,\xi;\eta,1) = 1,$$
$$\sum_{s=0}^{n} c_s^n Q_s^n(\xi,\eta)(1-\xi)^{-s} = \Phi^+(\xi,\eta;\xi,1) = 1.$$

Therefore

$$\begin{aligned}\sum_{s=1}^{n} c_s^n Q_s^n(\eta,\xi)(1-\eta)^{-s} &= 1 - Q_0^n(\eta,\xi),\\ \sum_{s=1}^{n} c_s^n Q_s^n(\xi,\eta)(1-\xi)^{-s} &= 1 - Q_0^n(\xi,\eta),\end{aligned} \tag{1.47}$$

as far as $c_0^n = 1$.

On the other hand, again by (1.23), we have

$$\begin{aligned}\sum_{p=0}^{n} c_p^n Q_p^n(\xi_0,\eta_0)(1-\eta)^{-p} &= \Phi^+(\xi_0,\eta_0;\eta,1) = \Phi^+(\eta,1;\xi_0,\eta_0),\\ \sum_{p=0}^{n} c_p^n Q_p^n(\xi_0,\eta_0)(1-\xi)^{-p} &= \Phi^+(\xi_0,\eta_0;\xi,1) = \Phi^+(\xi,1;\xi_0,\eta_0),\end{aligned} \tag{1.48}$$

where we take into account that

$$\Phi^+(\xi_0,\eta_0;\xi,\eta) = \Phi^+(\xi,\eta;\xi_0,\eta_0).$$

Then (1.44) follows directly from (1.47) and (1.48). □

Lemma 1.6.3. *Define*

$$\begin{aligned}&S^{(2)}(\xi_0,\eta_0) :=\\ &\sum_{p=1}^{n}\sum_{s=1}^{n} c_s^n c_p^n \frac{p}{s+p} Q_p^n(\xi_0,\eta_0)\left(\int_0^{\xi_0}\int_0^{\eta}(1-\eta)^{-s-p}Q_s^n(\eta,\xi)F(\xi,\eta)\,d\xi d\eta\right.\\ &\qquad\left. + \int_0^{\xi_0}\int_{\xi}^{1}(1-\xi)^{-s-p}Q_s^n(\xi,\eta)F(\xi,\eta)\,d\eta d\xi\right).\end{aligned} \tag{1.49}$$

Then

$$S^{(2)}(\xi_0,\eta_0) = \sum_{k=0}^{[(n-1)/2]} a_k^n E_k^n(\xi_0,\eta_0) \int_{\xi_0}^{1}\int_{0}^{\xi_0} \tilde{E}_k^n(\xi,\eta)F(\xi,\eta)\,d\xi d\eta$$
$$+ \sum_{k=0}^{[n/2]-1} b_k^n H_k^n(\xi_0,\eta_0) \int_{\xi_0}^{1}\int_{0}^{\xi_0} \tilde{H}_k^n(\xi,\eta)F(\xi,\eta)\,d\xi d\eta$$
$$+ \sum_{k=0}^{[(n-1)/2]} 2a_k^n E_k^n(\xi_0,\eta_0) \int_{0}^{\xi_0}\int_{0}^{\eta} \tilde{E}_k^n(\xi,\eta)F(\xi,\eta)\,d\xi d\eta$$
$$- \mathcal{J}(\xi_0,\eta_0) + \mathcal{I}_0(\xi_0,\eta_0),$$

where $\mathcal{J}(\xi_0,\eta_0)$ *is the function* (1.45) *from Lemma 1.6.2 and*

$$\mathcal{I}_0(\xi_0,\eta_0) := Q_0^n(\xi_0,\eta_0)\left(\int_0^{\xi_0}\int_0^{\eta} Q_0^n(\eta,\xi)F(\xi,\eta)\,d\xi d\eta + \int_0^{\xi_0}\int_{\xi}^{1} Q_0^n(\xi,\eta)F(\xi,\eta)\,d\eta d\xi\right). \quad (1.50)$$

Proof. First, recalling (1.48), we note that

$$\mathcal{J}(\xi_0,\eta_0) - \mathcal{I}_0(\xi_0,\eta_0) = \sum_{p=1}^{n} c_p^n Q_p^n(\xi_0,\eta_0)\left(\int_0^{\xi_0}\int_0^{\eta}(1-\eta)^{-p} Q_0^n(\eta,\xi)F(\xi,\eta)\,d\xi d\eta + \int_0^{\xi_0}\int_{\xi}^{1}(1-\xi)^{-p} Q_0^n(\xi,\eta)F(\xi,\eta)\,d\eta d\xi\right).$$

Then

$$S^{(2)}(\xi_0,\eta_0)+\mathcal{J}(\xi_0,\eta_0)-\mathcal{I}_0(\xi_0,\eta_0)=$$
$$\sum_{p=1}^{n}\sum_{s=0}^{n} c_s^n c_p^n \frac{p}{s+p} Q_p^n(\xi_0,\eta_0)\left(\int_0^{\xi_0}\int_0^{\eta}(1-\eta)^{-s-p}Q_s^n(\eta,\xi)F(\xi,\eta)\,d\xi d\eta\right.$$
$$\left.+\int_0^{\xi_0}\int_\xi^1 (1-\xi)^{-s-p}Q_s^n(\xi,\eta)F(\xi,\eta)\,d\eta d\xi\right).$$

Now, with Lemma 1.6.1 we come to

$$S^{(2)}(\xi_0,\eta_0)+\mathcal{J}(\xi_0,\eta_0)-\mathcal{I}_0(\xi_0,\eta_0)=$$
$$\sum_{p=1}^{n} c_p^n Q_p^n(\xi_0,\eta_0)\left(\int_0^{\xi_0}\int_0^{\eta}(1-\xi)^{-p}(1-\eta)^{-p}Q_p^n(\eta,\xi)F(\xi,\eta)\,d\xi d\eta\right.$$
$$\left.+\int_0^{\xi_0}\int_\xi^1 (1-\xi)^{-p}(1-\eta)^{-p}Q_p^n(\xi,\eta)F(\xi,\eta)\,d\eta d\xi\right). \quad (1.51)$$

Comparing the relations (1.28)-(1.30) and (1.37)-(1.38) we see that

$$Q^n_{n-2k}(\xi,\eta)=Q^n_{n-2k}(\eta,\xi)=(1-\xi)^{n-2k}(1-\eta)^{n-2k}\tilde{E}^n_k(\xi,\eta),$$
$$k=0,\dots,[n/2], \quad (1.52)$$

$$Q^n_{n-2k-1}(\xi,\eta)=-Q^n_{n-2k-1}(\eta,\xi)=(1-\xi)^{n-2k-1}(1-\eta)^{n-2k-1}\tilde{H}^n_k(\xi,\eta),$$
$$k=0,\dots,[(n-1)/2]. \quad (1.53)$$

Applying (1.28)-(1.29) and (1.52)-(1.53) into (1.51) completes the proof of the lemma. □

Lemma 1.6.4. *For $k = 0, \dots, [(n-1)/2]$ and $(\xi_0, \eta_0) \in D$ define*

$$I_k^n(\xi_0, \eta_0) := (n-2k) \int_0^{\xi_0} \big(\mu_k^n - J_k^n(\xi)\big)(1-\xi)^{2k-n-1} \Phi^+(\xi, 1; \xi_0, \eta_0)\, d\xi. \tag{1.54}$$

Then

$$I_k^n(\xi_0, \eta_0) = \tilde{E}_k^n(\xi_0, \eta_0) \int_0^{\xi_0} \int_0^{\eta} E_k^n(\xi, \eta) F(\xi, \eta)\, d\xi d\eta + P_k^n(\xi_0, \eta_0), \tag{1.55}$$

where

$$P_k^n(\xi_0, \eta_0) := \sum_{s=0}^{n} \frac{n-2k}{s+n-2k} c_s^n Q_s^n(\xi_0, \eta_0) \int_0^{\xi_0} \frac{dJ_k^n(\xi)}{d\xi} (1-\xi)^{2k-n-s}\, d\xi. \tag{1.56}$$

Proof. Since $\Phi^+(\xi, \eta; \xi_0, \eta_0) = \Phi^+(\xi_0, \eta_0; \xi, \eta)$, by (1.23) we have:

$$\Phi^+(\xi, 1; \xi_0, \eta_0) = \sum_{s=0}^{n} c_s^n Q_s^n(\xi_0, \eta_0)(1-\xi)^{-s}. \tag{1.57}$$

Then

$$I_k^n(\xi_0, \eta_0) = \sum_{s=0}^{n} \frac{n-2k}{s+n-2k} c_s^n Q_s^n(\xi_0, \eta_0) \int_0^{\xi_0} \big(\mu_k^n - J_k^n(\xi)\big) \frac{d}{d\xi}(1-\xi)^{2k-n-s}\, d\xi.$$

Integrating by parts and taking into account that $J_k^n(0) = \mu_k^n$ (see (1.21) and (1.15)), we obtain:

$$I_k^n(\xi_0, \eta_0) = P_k^n(\xi_0, \eta_0) + \sum_{s=0}^{n} \frac{n-2k}{s+n-2k} \frac{c_s^n Q_s^n(\xi_0, \eta_0)}{(1-\xi_0)^{n-2k+s}} \big(\mu_k^n - J_k^n(\xi_0)\big).$$

Then, taking into account (1.52), from the relation

$$\mu_k^n - J_k^n(\xi_0) = \int_0^{\xi_0} \int_0^{\eta} E_k^n(\xi,\eta) F(\xi,\eta)\, d\xi d\eta$$

and from Lemma 1.6.1 with $p = n - 2k$ it follows (1.55) . $\square$

Lemma 1.6.5. *Define*

$$I(\xi_0,\eta_0) := \int_0^{\xi_0} \left(\int_\xi^1 \int_0^\xi \Phi_\xi^+(\xi_1,\eta_1;\xi,1) F(\xi_1,\eta_1)\, d\xi_1 d\eta_1 \right) \Phi^+(\xi,1;\xi_0,\eta_0)\, d\xi. \tag{1.58}$$

Then

$$\begin{aligned} I(\xi_0,\eta_0) = & \sum_{k=0}^{[(n-1)/2]} a_k^n \tilde{E}_k^n(\xi_0,\eta_0) \int_{\xi_0}^1 \int_0^{\xi_0} E_k^n(\xi,\eta) F(\xi,\eta)\, d\xi d\eta \\ & + \sum_{k=0}^{[n/2]-1} b_k^n \tilde{H}_k^n(\xi_0,\eta_0) \int_{\xi_0}^1 \int_0^{\xi_0} H_k^n(\xi,\eta) F(\xi,\eta)\, d\xi d\eta \\ & + \sum_{k=0}^{[(n-1)/2]} a_k^n E_k^n(\xi_0,\eta_0) \int_{\xi_0}^1 \int_0^{\xi_0} \tilde{E}_k^n(\xi,\eta) F(\xi,\eta)\, d\xi d\eta \\ & + \sum_{k=0}^{[n/2]-1} b_k^n H_k^n(\xi_0,\eta_0) \int_{\xi_0}^1 \int_0^{\xi_0} \tilde{H}_k^n(\xi,\eta) F(\xi,\eta)\, d\xi d\eta \\ & + \sum_{k=0}^{[(n-1)/2]} 2a_k^n E_k^n(\xi_0,\eta_0) \int_0^{\xi_0} \int_0^{\eta} \tilde{E}_k^n(\xi,\eta) F(\xi,\eta)\, d\xi d\eta \\ & - \mathcal{J}_0(\xi_0,\eta_0) + \mathcal{I}_0(\xi_0,\eta_0) - \sum_{k=0}^{[(n-1)/2]} 2a_k^n P_k^n(\xi_0,\eta_0), \end{aligned}$$

where $\mathcal{J}_0(\xi_0,\eta_0)$ *is the function* (1.46) *from Lemma 1.6.2,* $\mathcal{I}_0(\xi_0,\eta_0)$ *is the function* (1.50) *from Lemma 1.6.3 and* $P_k^n(\xi_0,\eta_0)$ *are the functions* (1.56) *from Lemma 1.6.4.*

Proof. Using the expansions (1.23) and (1.57) we may write:

$$I(\xi_0,\eta_0)=\sum_{p=1}^{n}\sum_{s=0}^{n}c_s^n c_p^n\frac{p}{s+p}Q_s^n(\xi_0,\eta_0)\int_0^{\xi_0}\Upsilon_p^n(\xi)\frac{d}{d\xi}(1-\xi)^{-s-p}\,d\xi$$

with

$$\Upsilon_p^n(\xi):=\int_\xi^1\int_0^\xi Q_p^n(\xi_1,\eta_1)F(\xi_1,\eta_1)\,d\xi_1 d\eta_1.$$

Integrating by parts, we obtain:

$$I(\xi_0,\eta_0)=I^{(1)}(\xi_0,\eta_0)+I^{(2)}(\xi_0,\eta_0),$$

where

$$I^{(1)}(\xi_0,\eta_0):=\sum_{p=1}^{n}\sum_{s=0}^{n}c_s^n c_p^n\frac{p}{s+p}Q_s^n(\xi_0,\eta_0)\Upsilon_p^n(\xi_0)(1-\xi_0)^{-s-p},$$

$$I^{(2)}(\xi_0,\eta_0):=-\sum_{s=1}^{n}\sum_{p=0}^{n}c_s^n c_p^n\frac{s}{s+p}Q_p^n(\xi_0,\eta_0)\int_0^{\xi_0}\frac{d\Upsilon_s^n(\xi)}{d\xi}(1-\xi)^{-s-p}\,d\xi.$$

A. Calculation of $I^{(1)}(\xi_0,\eta_0)$. First, we apply Lemma 1.6.1 to obtain:

$$I^{(1)}(\xi_0,\eta_0)=\sum_{p=1}^{n}c_p^n\Upsilon_p^n(\xi_0)(1-\xi_0)^{-p}(1-\eta_0)^{-p}Q_p^n(\xi_0,\eta_0).$$

Then with (1.28)-(1.29) and (1.52)-(1.53) we come to

$$I^{(1)} = \sum_{k=0}^{[(n-1)/2]} a_k^n \tilde{E}_k^n(\xi_0,\eta_0) \int_{\xi_0}^1 \int_0^{\xi_0} E_k^n(\xi,\eta) F(\xi,\eta)\, d\xi d\eta \\ + \sum_{k=0}^{[n/2]-1} b_k^n \tilde{H}_k^n(\xi_0,\eta_0) \int_{\xi_0}^1 \int_0^{\xi_0} H_k^n(\xi,\eta) F(\xi,\eta)\, d\xi d\eta.$$

B. Calculation of $I^{(2)}(\xi_0,\eta_0)$. First, we calculate:

$$\frac{d\Upsilon_s^n(\xi)}{d\xi} = -\int_0^{\xi} Q_s^n(\xi_1,\xi) F(\xi_1,\xi)\, d\xi_1 + \int_{\xi}^1 Q_s^n(\xi,\eta_1) F(\xi,\eta_1)\, d\eta_1.$$

Then we have:

$$I^{(2)}(\xi_0,\eta_0) = \\ \sum_{s=1}^{n}\sum_{p=0}^{n} c_s^n c_p^n \frac{s}{s+p} Q_p^n(\xi_0,\eta_0) \left(\int_0^{\xi_0}\int_0^{\eta} (1-\eta)^{-s-p} Q_s^n(\xi,\eta) F(\xi,\eta)\, d\xi d\eta \right. \\ \left. - \int_0^{\xi_0}\int_{\xi}^{1} (1-\xi)^{-s-p} Q_s^n(\xi,\eta) F(\xi,\eta)\, d\eta d\xi \right).$$

Now, using the relations (1.28)-(1.29), we see that

$$\sum_{s=1}^{n}\sum_{p=0}^{n} c_s^n c_p^n \frac{s}{s+p} Q_p^n(\xi_0,\eta_0) \int_0^{\xi_0}\int_0^{\eta} (1-\eta)^{-s-p} Q_s^n(\xi,\eta) F(\xi,\eta)\, d\xi d\eta \\ = -\sum_{s=1}^{n}\sum_{p=0}^{n} c_s^n c_p^n \frac{s}{s+p} Q_p^n(\xi_0,\eta_0) \int_0^{\xi_0}\int_0^{\eta} (1-\eta)^{-s-p} Q_s^n(\eta,\xi) F(\xi,\eta)\, d\xi d\eta \\ - \sum_{k=0}^{[(n-1)/2]} 2a_k^n P_k^n(\xi_0,\eta_0), \quad (1.59)$$

where we take into account that

$$-\sum_{k=0}^{[(n-1)/2]} 2a_k^n P_k^n(\xi_0,\eta_0) = \sum_{k=0}^{[(n-1)/2]} \sum_{p=0}^{n} 2a_k^n \frac{n-2k}{p+n-2k} c_p^n Q_p^n(\xi_0,\eta_0)$$
$$\times \int_0^{\xi_0} \int_0^{\eta} (1-\eta)^{2k-n-p} E_k^n(\xi,\eta) F(\xi,\eta)\, d\xi d\eta,$$

since

$$\frac{dJ_k^n(\xi)}{d\xi} = -\int_0^{\xi} E_k^n(\xi_1,\xi) F(\xi_1,\xi)\, d\xi_1.$$

Consequently, $I^{(2)}(\xi_0,\eta_0)$ becomes

$$I^{(2)}(\xi_0,\eta_0) =$$
$$-\sum_{s=1}^{n}\sum_{p=0}^{n} c_s^n c_p^n \frac{s}{s+p} Q_p^n(\xi_0,\eta_0) \left(\int_0^{\xi_0} \int_0^{\eta} (1-\eta)^{-s-p} Q_s^n(\eta,\xi) F(\xi,\eta)\, d\xi d\eta \right.$$
$$\left. + \int_0^{\xi_0} \int_{\xi}^{1} (1-\xi)^{-s-p} Q_s^n(\xi,\eta) F(\xi,\eta)\, d\eta d\xi \right) - \sum_{k=0}^{[(n-1)/2]} 2a_k^n P_k^n(\xi_0,\eta_0).$$

Applying here the simple equality

$$\frac{s}{s+p} = 1 - \frac{p}{s+p},$$

we decompose $I^{(2)}(\xi_0,\eta_0)$ as

$$I^{(2)}(\xi_0,\eta_0) = S^{(1)}(\xi_0,\eta_0) + S^{(2)}(\xi_0,\eta_0) - \sum_{k=0}^{[(n-1)/2]} 2a_k^n P_k^n(\xi_0,\eta_0), \qquad (1.60)$$

where $S^{(1)}(\xi_0,\eta_0)$ and $S^{(2)}(\xi_0,\eta_0)$ are the functions (1.43) and (1.49) respectively. Finally, applying Lemma 1.6.2 and Lemma 1.6.3 into (1.60) we derive

$$\begin{aligned} I^{(2)}(\xi_0,\eta_0) = & \sum_{k=0}^{[(n-1)/2]} a_k^n E_k^n(\xi_0,\eta_0) \int_{\xi_0}^{1}\int_0^{\xi_0} \tilde{E}_k^n(\xi,\eta) F(\xi,\eta)\, d\xi d\eta \\ & + \sum_{k=0}^{[n/2]-1} b_k^n H_k^n(\xi_0,\eta_0) \int_{\xi_0}^{1}\int_0^{\xi_0} \tilde{H}_k^n(\xi,\eta) F(\xi,\eta)\, d\xi d\eta \\ & + \sum_{k=0}^{[(n-1)/2]} 2a_k^n E_k^n(\xi_0,\eta_0) \int_0^{\xi_0}\int_0^{\eta} \tilde{E}_k^n(\xi,\eta) F(\xi,\eta)\, d\xi d\eta \\ & - \mathcal{J}_0(\xi_0,\eta_0) + \mathcal{I}_0(\xi_0,\eta_0) - \sum_{k=0}^{[(n-1)/2]} 2a_k^n P_k^n(\xi_0,\eta_0) \end{aligned}$$

The proof is complete. □

Lemma 1.6.6. *Let* $\mathcal{I}_0(\xi_0,\eta_0)$ *be the function defined by* (1.50).

(i) *If* n *is an even number, then*

$$\mathcal{I}_0(\xi_0,\eta_0) = a_{n/2}^n \tilde{E}_{n/2}^n(\xi_0,\eta_0) \left(\int_0^{\xi_0}\int_0^{\eta} 2\, E_{n/2}^n(\xi,\eta) F(\xi,\eta)\, d\xi d\eta + \int_{\xi_0}^{1}\int_0^{\xi_0} E_{n/2}^n(\xi,\eta) F(\xi,\eta)\, d\xi d\eta \right).$$

(ii) *If* n *is an odd number, then*

$$\mathcal{I}_0(\xi_0,\eta_0) = b_{(n-1)/2}^n \tilde{H}_{(n-1)/2}^n(\xi_0,\eta_0) \int_{\xi_0}^{1}\int_0^{\xi_0} H_{(n-1)/2}^n(\xi,\eta) F(\xi,\eta)\, d\xi d\eta.$$

The proof of this lemma follows directly from the relations (1.28)-(1.29) and (1.52)-(1.53).

Proof of Theorem 1.5.1. Obviously, the function $U(\xi,\eta)$ is a solution of the following Goursat problem:

$$U_{\xi\eta} - \frac{n(n+1)}{(2-\xi-\eta)^2} U = F(\xi,\eta),$$

$$U(0,\eta) = 0,$$

$$U(\xi,1) = \sum_{k=0}^{[n/2]} 2a_k^n \mu_k^n (1-\xi)^{2k-n} - \sum_{k=0}^{[n/2]} 2a_k^n J_k^n(\xi)(1-\xi)^{2k-n} + J^+(\xi),$$

(see (1.9)-(1.10) and Theorem 1.4.2).

Then, solving this problem by the Riemann method, we obtain

$$U(\xi_0,\eta_0) = W(\xi_0,\eta_0) - \int_{\eta_0}^{1} \int_0^{\xi_0} \Phi^+(\xi,\eta;\xi_0,\eta_0) F(\xi,\eta)\, d\xi d\eta, \tag{1.61}$$

where

$$W(\xi_0,\eta_0) := \int_0^{\xi_0} U_\xi(\xi,1)\Phi^+(\xi,1;\xi_0,\eta_0)\, d\xi. \tag{1.62}$$

On the one hand, according to (1.61) and the representation (1.18) in Theorem 1.4.1 we have

$$W(\xi_0,\eta_0) = \int_0^{\xi_0} \int_\xi^1 \Phi(\xi,\eta;\xi_0,\eta_0) F(\xi,\eta)\, d\eta d\xi. \tag{1.63}$$

On the other hand, using Lemma 1.4.1, we can evaluate the integral in (1.62) directly:

$$W(\xi_0,\eta_0) = \sum_{k=0}^{[(n-1)/2]} 2a_k^n I_k^n(\xi_0,\eta_0) + \mathcal{J}_0(\xi_0,\eta_0) + I(\xi_0,\eta_0), \tag{1.64}$$

where $I_k^n(\xi_0,\eta_0)$, $\mathcal{J}_0(\xi_0,\eta_0)$ and $I(\xi_0,\eta_0)$ are the functions defined by (1.54), (1.46) and (1.58) respectively. Applying Lemma 1.6.4, Lemma 1.6.5 and Lemma 1.6.6 to (1.64) gives that

$$W(\xi_0,\eta_0) = \int_0^{\xi_0}\int_{\xi}^{1} \tilde{\Phi}(\xi,\eta;\xi_0,\eta_0)F(\xi,\eta)\,d\eta d\xi, \tag{1.65}$$

where

$$\tilde{\Phi}(\xi,\eta;\xi_0,\eta_0) := \begin{cases} \tilde{\Phi}^+(\xi,\eta;\xi_0,\eta_0), & \eta > \xi_0, \\ \tilde{\Phi}^-(\xi,\eta;\xi_0,\eta_0), & \eta < \xi_0 \end{cases} \tag{1.66}$$

with

$$\tilde{\Phi}^-(\xi,\eta;\xi_0,\eta_0) := \sum_{k=0}^{[n/2]} 2a_k^n \tilde{E}_k^n(\xi_0,\eta_0)E_k^n(\xi,\eta) + \sum_{k=0}^{[(n-1)/2]} 2a_k^n E_k^n(\xi_0,\eta_0)\tilde{E}_k^n(\xi,\eta), \tag{1.67}$$

$$\begin{aligned} \tilde{\Phi}^+(\xi,\eta;\xi_0,\eta_0) := & \sum_{k=0}^{[n/2]} a_k^n \tilde{E}_k^n(\xi_0,\eta_0)E_k^n(\xi,\eta) + \sum_{k=0}^{[(n-1)/2]} a_k^n E_k^n(\xi_0,\eta_0)\tilde{E}_k^n(\xi,\eta) \\ & + \sum_{k=0}^{[(n-1)/2]} b_k^n \tilde{H}_k^n(\xi_0,\eta_0)H_k^n(\xi,\eta) + \sum_{k=0}^{[n/2]-1} b_k^n H_k^n(\xi_0,\eta_0)\tilde{H}_k^n(\xi,\eta). \end{aligned} \tag{1.68}$$

Since $F(\xi,\eta)$ is an arbitrary continuous function, comparing the identities (1.63) and (1.65), we conclude that

$$\tilde{\Phi}(\xi,\eta;\xi_0,\eta_0) \equiv \Phi(\xi,\eta;\xi_0,\eta_0).$$

Finally, applying the expansion (1.66)-(1.68) into (1.18) completes the proof of the theorem. □

1.7. Asymptotic expansion of the generalized solution of Problem $\mathbf{P}_0$

Lemma 1.7.1. *For $k = 0, \dots, [n/2]$ the following relations hold:*

$$\mu_k^n = \gamma_k^n \mu_{k,s}^n,$$

where the coefficients $\mu_{k,s}^n$, γ_k^n, μ_k^n are defined by (1.3), (1.14), (1.15) *respectively.*

Proof. Denote

$$G_0 := \{(r,t) : \, 0 < t < 1/2, \, t < r < 1 - t\}.$$

Denote also by $\mathcal{Y}_n^s$ the spherical functions expressed in the spherical coordinates, i.e. $Y_n^s(x) = \mathcal{Y}_n^s(\theta(x), \varphi(x))$. Then, using the orthonormality of the spherical functions on the unit sphere S^2 and the relation (1.14), a direct calculation gives:

$$\begin{aligned}
\mu_{k,s}^n &= \int_{\Omega_0} v_{k,s}^n(x,t) f(x,t)\, dxdt \\
&= \int_0^\pi \int_0^{2\pi} \int_{G_0} \mathcal{E}_k^n(r,t) \mathcal{Y}_n^s(\theta,\varphi) \left(\sum_{p=0}^{l} \sum_{q=1}^{2p+1} f_p^q(r,t) \mathcal{Y}_p^q(\theta,\varphi) \right) \sin\theta\, r^2\, drdtd\varphi d\theta \\
&= \int_{S^2} (\mathcal{Y}_n^s)^2(\theta,\varphi)\, dS \int_{G_0} (\mathcal{E}_k^n f_n^s)(r,t)\, r^2\, drdt = \frac{\mu_k^n}{\gamma_k^n}.
\end{aligned}$$

The proof is complete. $\square$

Now, the inverse transformation from Problem P_{02} to Problem P_0 gives the following improvement of Theorem 1.2.2:

Theorem 1.7.1. *Suppose that the right-hand side function $f \in C(\bar{\Omega}_0)$ has the form* (1.4). *Then the unique generalized solution $u(x,t)$ of Problem P_0 has the following asymptotic expansion at the singular point O:*

$$u(x,t) = \sum_{p=0}^{l} |x|^{-p-1} F_p(x,t) + F(x,t), \tag{1.69}$$

where

(i) *the function $F(x,t)$ satisfies the a priori estimate*

$$|F(x,t)| \leq C\|f\|_{C(\Omega_0)}, \quad (x,t) \in \Omega_0$$

with a constant C independent of f;

(ii) *the functions F_p, $p = 0, \dots, l$ satisfy the equalities*

$$F_p(x,t) = \sum_{k=0}^{[(l-p)/2]} \sum_{s=1}^{2p+4k+1} \mu_{k,s}^{p+2k} F_{k,s}^{p+2k}(x,t),$$

where

$$F_{k,s}^{n}(x,t) = 2^{2k-n+1} \gamma_k^n a_k^n \alpha_{n-2k}^n \, {}_2F_1\left(n-k+\frac{1}{2}, -k, \frac{1}{2}; \frac{t^2}{|x|^2}\right) Y_n^s(x),$$

and the constants γ_k^n, a_k^n, α_k^n are given by (1.14), (1.20), (1.27) *respectively;*

(iii) *if at least one of the constants $\mu_{k,s}^{p+2k}$ in* (1.7) *is different from zero, then for the corresponding function $F_p(x,t)$ there exists a direction*

$(\alpha, 1) := (\alpha_1, \alpha_2, \alpha_3, 1)$ *with* $(\alpha, 1)\, t \in \Sigma_2^0$ *for* $0 < t < 1/2$*, such that*

$$\lim_{t \to +0} F_p(\alpha t, t) = \text{const} \neq 0.$$

This means that the order of singularity of $u(x,t)$ *will be no smaller than* $p+1$.

To obtain this result, we use Corollary 1.5.1 and Lemma 1.7.1.

Remark 1.7.1. *Using* (1.14) *we see that the hypergeometric functions* ${}_2F_1\left(n-k+1/2, -k, 1/2; t^2/|x|^2\right)$ *are connected with* $\mathcal{E}_k^n(|x|, t)$ *in* Ω_0 *in the following way*:

$${}_2F_1\left(n-k+\frac{1}{2}, -k, \frac{1}{2}; \frac{t^2}{|x|^2}\right) = \gamma_k^n \frac{(2|x|)^{n-2k+1}}{(|x|^2-t^2)^{n-2k}}\, \mathcal{E}_k^n(|x|, t).$$

Consequently,

$$v_{k,s}^n(x,t) = K\, F_{k,s}^n(x,t)\, \frac{(|x|^2-t^2)^{n-2k}}{|x|^{n-2k+1}}, \qquad (x,t) \in \Omega_0$$

with $K = \text{const} \neq 0$.

2. The Protter problem for Keldysh-type equations

In this chapter we study the case $0 < m < 2$, when equation (0.1) is weakly hyperbolic. For $0 < m < 4/3$ we derive some results on the generalized solvability of the considered boundary value problem, as well as we clarify the asymptotic behavior of the singularities of the generalized solutions. An essential part of this investigation we have published in [14], [15], [36], [37] and [31].

2.1. Statement of the problem

For $m \in \mathbb{R}$, $0 < m < 2$ consider the equation

$$L_m[u] \equiv u_{x_1x_1} + u_{x_2x_2} + u_{x_3x_3} - (t^m u_t)_t = f(x,t) \tag{2.1}$$

in the domain

$$\Omega_m := \left\{ (x,t) : \quad 0 < t < t_0, \ \ \frac{2}{2-m} t^{\frac{2-m}{2}} < |x| < 1 - \frac{2}{2-m} t^{\frac{2-m}{2}} \right\},$$

with $t_0 = \left((2-m)/4\right)^{2/(2-m)}$. The region Ω_m is bounded by the ball Σ_0 and by two characteristic surfaces of equation (2.1)

$$\Sigma_1^m := \left\{ (x,t) : 0 < t < t_0,\ |x| = 1 - \frac{2}{2-m} t^{\frac{2-m}{2}} \right\},$$

$$\Sigma_2^m := \left\{ (x,t) : 0 < t < t_0,\ |x| = \frac{2}{2-m} t^{\frac{2-m}{2}} \right\},$$

(see Fig. 2.1).

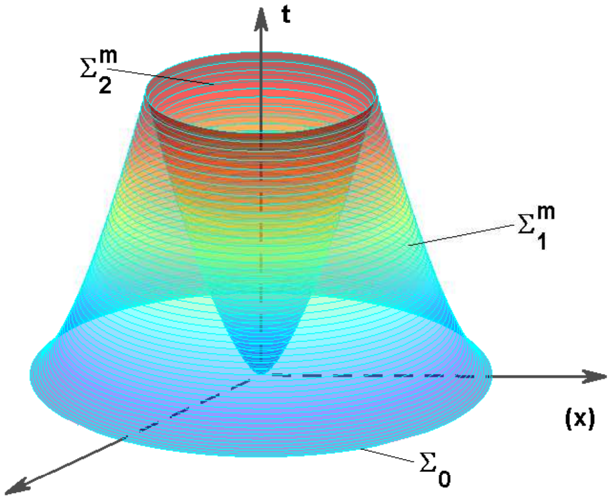

Figure 2.1.: The region Ω_m.

Note that the hyperplane $\{t = 0\}$ is tangential to the characteristics Σ_1^m and Σ_2^m and the ball Σ_0 is also a characteristic surface of equation (2.1).

We study the following boundary value problem:

Problem $\mathbf{P_m}$. *Find a solution to equation* (2.1) *in* Ω_m *which satisfies the* *boundary conditions*

$$u|_{\Sigma_1^m} = 0, \qquad t^m u_t \to 0 \text{ as } t \to +0. \tag{2.2}$$

The adjoint problem to P_m is as follows:

Problem $\mathbf{P_m^*}$. *Find a solution to the self-adjoint equation* (2.1) *in* Ω_m *which satisfies the boundary conditions*

$$u|_{\Sigma_2^m} = 0, \qquad t^m u_t \to 0 \text{ as } t \to +0. \tag{2.3}$$

Note that there is no data on the degenerate boundary Σ_0. Instead, the derivative u_t is allowed to have singularity on it up to the prescribed level.

2.2. Generalized solvability of Problem $\mathbf{P_m}$ and asymptotic behavior of the singularities of the generalized solutions

Similarly to Problem P_0, Problem P_m is not well posed, because its adjoint homogeneous Problem P_m^* has infinitely many nontrivial classical solutions.

Indeed, for $k, n \in \mathbb{N} \cup \{0\}$ let us introduce the functions

$$\mathcal{E}_k^{n,m}(|x|, t) := \sum_{i=0}^{k} A_i^{k,m} |x|^{-n+2i-1} \left(|x|^2 - \frac{4}{(2-m)^2} t^{2-m} \right)^{n-k-i-\frac{m}{2(2-m)}} \tag{2.4}$$

with

$$A_i^{k,m} := (-1)^i \frac{(k-i+1)_i(n-k-i+(4-3m)/(4-2m))_i}{i!(n+1/2-i)_i}.$$

Then the following lemma holds:

Lemma 2.2.1. *For all* $m \in \mathbb{R},\ 0 < m < 2,\ k, n \in \mathbb{N} \cup \{0\}$,
$n > N(m,k) := 2k+1+m/(2-m)$ *and* $s = 1, 2, \dots, 2n+1$, *the functions*

$$v_{k,s}^{n,m}(x,t) := \begin{cases} \mathcal{E}_k^{n,m}(|x|,t)Y_n^s(x), & (x,t) \neq O, \\ 0, & (x,t) = O \end{cases} \tag{2.5}$$

are classical solutions from $C^2(\Omega_m) \cap C(\bar{\Omega}_m)$ *of the homogeneous Problem* P_m^*.

Proof. First, we have obviously that $v_{k,s}^{n,m}(x,t) \in C^\infty(\Omega_m)$.

For $n > N(m,k)$ we see that $\mathcal{E}_k^{n,m}(|x|,t) \to 0$ as $(x,t) \to O$. Therefore $v_{k,s}^{n,m}(x,t) \in C^\infty(\Omega_m) \cap C(\bar{\Omega}_m)$.

It is easy to check that for $n > N(m,k)$ the boundary conditions (2.3) are also satisfied.

Now, let us look for solutions of the homogeneous Problem P_m^* of the form (2.5). Passing to the spherical coordinates (1.8) in the homogeneous equation (2.1) and using that the spherical functions satisfy the differential equation

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}Y_n^s\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}Y_n^s + n(n+1)Y_n^s = 0, \tag{2.6}$$

we see that the functions $\mathcal{E}_k^{n,m}$ should be solutions of the equation

$$v_{rr} + \frac{2}{r} v_r - (t^m v_t)_t - \frac{n(n+1)}{r^2} v = 0 \tag{2.7}$$

in

$$G_m := \left\{ (r,t) : 0 < t < t_0, \; \frac{2}{2-m} t^{\frac{2-m}{2}} < r < 1 - \frac{2}{2-m} t^{\frac{2-m}{2}} \right\}.$$

A direct calculation of the derivatives of $\mathcal{E}_k^{n,m}(r,t)$ shows that these functions indeed satisfy equation (2.7).

The proof is complete. □

Consequently, a necessary condition for the existence of a classical solution of Problem P_m is the orthogonality of the right-hand side function $f(x,t)$ to all these functions $v_{k,s}^{n,m}(x,t)$. Respectively, an infinite number of orthogonality conditions $\mu_{k,s}^{n,m} = 0$ with

$$\mu_{k,s}^{n,m} := \int_{\Omega_m} v_{k,s}^{n,m}(x,t) f(x,t)\, dxdt \tag{2.8}$$

must be fulfilled.

According to this situation, we consider solutions to this problem in a generalized sense. We focus on the case $0 < m < 4/3$ and we use the following definition of a generalized solution of Problem P_m:

Definition 2.2.1. *We call a function $u(x,t)$ a generalized solution of Problem P_m in Ω_m, $0 < m < 4/3$, for equation* (2.1) *if:*

(1) $u, u_{x_j} \in C(\bar{\Omega}_m \setminus O), j = 1,2,3,\ u_t \in C(\bar{\Omega}_m \setminus \bar{\Sigma}_0)$;

(2) $u|_{\Sigma_1^m} = 0$;

(3) *for each* $\varepsilon \in (0,1)$ *there exists a constant* $C(\varepsilon) > 0$*, such that*

$$|u_t(x,t)| \leq C(\varepsilon) t^{-\frac{3m}{4}} \quad \text{in} \quad \Omega_m \cap \{|x| > \varepsilon\}\,; \tag{2.9}$$

(4) *the identity*

$$\int_{\Omega_m} \{t^m u_t v_t - u_{x_1} v_{x_1} - u_{x_2} v_{x_2} - u_{x_3} v_{x_3} - fv\}\, dxdt = 0 \tag{2.10}$$

holds for all v *from*

$$V_m := \Big\{ v(x,t) : v \in C^2(\bar{\Omega}_m),\ v|_{\Sigma_2^m} = 0,\ v \equiv 0 \ \text{in a neighborhood of} \ O \Big\}.$$

We mention that the inequality (2.9) restricts the generalized solution's function space to a class which is smaller than it is allowed by the second boundary condition in (2.2).

In this paper we will prove the following results on the existence and uniqueness of a generalized solution of Problem P_m:

Theorem 2.2.1. *If* $m \in (0, 4/3)$, *then there exists at most one generalized solution of Problem* P_m *in* Ω_m.

Theorem 2.2.2. *Let* $m \in (0, 4/3)$. *Suppose that the right-hand side function* $f(x,t)$ *is of the form* (1.4) *and* $f \in C^1(\bar{\Omega}_m)$. *Then there exists an unique generalized solution* $u(x,t)$ *of Problem* P_m *in* Ω_m *and it has the form* (1.5).

The proof of Theorems 2.2.1-2.2.2 will be given Section 2.6.

We mention also that Definition 2.2.1 allows the generalized solutions to have some singularity at the point O. Indeed there exist such singular solutions to this problem and we will prove the following theorem describing their asymptotic behavior:

Theorem 2.2.3. *Let $m \in (0, \frac{4}{3})$ and the right-hand side function $f \in C^1(\bar{\Omega}_m)$ has the form (1.4). Then the unique generalized solution $u(x,t)$ of Problem P_m has the following expansion at the point O:*

$$u(x,t) = \sum_{p=0}^{l} F_p^m(x,t)|x|^{-p-1} + F^{(m)}(x,t)|x|^{-1}, \tag{2.11}$$

where

(i) *the function $F^{(m)}(x,t) \in C(\bar{\Omega}_m)$, $F^{(m)}(O) = 0$ and in the case $0 < m < 1$ it satisfies in Ω_m the a priori estimate*

$$|F^{(m)}(x,t)| \leq C\|f\|_{C(\Omega_m)}|x|^{1-\beta}\left(1 + \big|\ln|x|\big|\right), \qquad \beta = \frac{m}{2(2-m)}, \tag{2.12}$$

with a constant $C > 0$ independent of f;

(ii) *the functions $F_p^m(x,t)$, $p = 0, \dots, l$ have the following structure*

$$F_p^m(x,t) = \sum_{k=0}^{[(l-p)/2]} \sum_{s=1}^{2p+4k+1} c_k^{p+2k,m} \mu_{k,s}^{p+2k,m} H_{k,s}^{p+2k,m}(x,t), \tag{2.13}$$

where $c_k^{p+2k,m} \neq 0$ are constants independent of $f(x,t)$ and

$$H_{k,s}^{n,m}(x,t) = {}_2F_1\left(n - k + \frac{1}{2}, -k, \frac{1}{2-m}; \frac{4t^{2-m}}{(2-m)^2\,|x|^2}\right) Y_n^s(x);$$

(iii) *if at least one of the constants* $\mu_{k,s}^{p+2k,m}$ *in* (2.13) *is different from zero, then for the corresponding function* $F_p^m(x,t)$ *there exists a vector* $\alpha \in \mathbb{R}^3$, $|\alpha| = 1$, *such that*

$$\lim_{t\to+0} F_p^m(\sigma(t),t) = \text{const} \neq 0,$$

where

$$(\sigma(t),t) := \left(\frac{2}{2-m}\alpha t^{\frac{2-m}{2}}, t\right) \in \Sigma_2^m, \quad 0 < t < t_0.$$

This means that the order of singularity of $u(x,t)$ *will be no smaller than* $p+1$.

Remark 2.2.1. *In the case* $1 \leq m < 4/3$ *we prove that the estimate* (2.12) *holds at least in the subset*

$$\Omega_m \cap \left\{ |x| < \frac{6}{2-m} t^{\frac{2-m}{2}} \right\}.$$

Remark 2.2.2. *The functions* $H_{k,s}^{n,m}(x,t)$ *are connected with* $v_{k,s}^{n,m}(x,t)$ *for* $(x,t) \neq O$ *by the relation*

$$v_{k,s}^{n,m}(x,t) = K_m\, H_{k,s}^{n,m}(x,t)\, |x|^{2k-n-1} \left(|x|^2 - \frac{4}{(2-m)^2} t^{2-m} \right)^{n-2k-\frac{m}{2(2-m)}},$$

where

$$K_m = \text{const} = \frac{(-1)^k \left(\frac{1}{2-m}\right)_k}{(1/2-n)_k} \neq 0.$$

To confirm the assertions in Theorem 2.2.3 from here on we will study a two-dimensional problem related to Problem P_m.

2.3. Two-dimensional problem corresponding to Problem $\mathbf{P_m}$

In the case when the right-side function $f(x,t)$ is of the form (1.4) Problem P_m can be reduced to a two-dimensional problem.

To do this, let us look for a solution of the form (1.5). Passing to the spherical coordinates (1.8) and using that the spherical functions satisfy the differential equation (2.6), for the coefficients $u_n^s(r,t)$ corresponding to the right-hand sides $f_n^s(r,t)$ we obtain the 2-D equation

$$u_{rr} + \frac{2}{r}u_r - (t^m u_t)_t - \frac{n(n+1)}{r^2}u = f(r,t).$$

Then using the characteristic coordinates

$$\xi = 1 - r - \frac{2}{2-m}t^{\frac{2-m}{2}}, \quad \eta = 1 - r + \frac{2}{2-m}t^{\frac{2-m}{2}}, \tag{2.14}$$

for the functions

$$U(\xi,\eta) := r(\xi,\eta)u_n^s\big(r(\xi,\eta), t(\xi,\eta)\big)$$

we obtain the following Darboux-Goursat problem:

Problem $\mathbf{P_{m2}}$. *Find a solution of the equation*

$$E_\beta[U] \equiv U_{\xi\eta} + \frac{\beta}{\eta-\xi}(U_\xi - U_\eta) - \frac{n(n+1)}{(2-\xi-\eta)^2}U = F(\xi,\eta) \quad in \quad D, \tag{2.15}$$

satisfying the following boundary conditions

$$U(0,\eta)=0,\ \lim_{\eta-\xi\to+0}(\eta-\xi)^{2\beta}\Big(U_\xi-U_\eta\Big)=0, \tag{2.16}$$

where

$$D:=\{(\xi,\eta):\ 0<\xi<\eta<1\},$$

$$F(\xi,\eta):=\frac{1}{8}(2-\xi-\eta)f_n^s(r(\xi,\eta),t(\xi,\eta)), \tag{2.17}$$

and

$$\beta:=\frac{m}{2(2-m)}.$$

Remark 2.3.1. *As far as we consider Problem P_m in the case $m\in(0,4/3)$, for the parameter β we have*

$$0<\beta<1.$$

Directly from Lemma 2.2.1, with use of (A.4) and (A.16), we find that for $k=0,1,\dots,[n/2]-1$ the functions

$$E_k^{n,\beta}(\xi,\eta):=\begin{cases}\dfrac{(1-\xi)^{n-2k-\beta}(1-\eta)^{n-2k-\beta}}{(2-\xi-\eta)^{n-2k}}\,{}_2F_1\left(n-k+\dfrac{1}{2},\ -k,\dfrac{1}{2}+\beta;\quad \dfrac{(\eta-\xi)^2}{(2-\xi-\eta)^2}\right), & (\xi,\eta)\neq(1,1),\\ 0, & (\xi,\eta)=(1,1)\end{cases} \tag{2.18}$$

solve the corresponding adjoint homogeneous problem

$$E_\beta[U] = 0 \quad \text{in } D,$$

$$U(\xi, 1) = 0, \qquad \lim_{\eta-\xi\to+0} (\eta - \xi)^{2\beta} \Big(U_\xi - U_\eta \Big) = 0.$$

The functions $E_k^{n,\beta}(\xi, \eta)$ are connected with the functions $\mathcal{E}_k^{n,m}(|x|, t)$ by the relation

$$E_k^{n,\beta}(\xi, \eta) = \gamma_k^{n,\beta} (2 - \xi - \eta) \mathcal{E}_k^{n,m}(r(\xi, \eta), t(\xi, \eta)),$$
$$\gamma_k^{n,\beta} := \frac{(-1)^k \, (1/2 - n)_k}{2^{n-2k+1} \, (1/2 + \beta)_k}.$$

In conformity with Definition 2.2.1, we define a generalized solution of Problem P_{m2} in the following way:

Definition 2.3.1. *We call a function* $U(\xi, \eta)$ *a generalized solution of Problem* P_{m2} *in* D, $(0 < \beta < 1)$, *if:*

(1) U, $U_\xi + U_\eta \in C(\bar{D} \setminus (1, 1))$, $U_\xi - U_\eta \in C(\bar{D} \setminus \{\eta = \xi\})$;

(2) $U(0, \eta) = 0$;

(3) *for each* $\varepsilon \in (0, 1)$ *there exists a constant* $C(\varepsilon) > 0$, *such that*

$$|(U_\xi - U_\eta)(\xi, \eta)| \le C(\varepsilon)(\eta - \xi)^{-\beta} \quad \textit{in} \quad D \cap \{\xi < 1 - \varepsilon\}; \tag{2.19}$$

(4) *the identity*

$$\int_D (\eta - \xi)^{2\beta} \left\{ U_\xi V_\eta + U_\eta V_\xi + \frac{2n(n+1)}{(2 - \xi - \eta)^2} UV + 2FV \right\} d\xi \, d\eta = 0 \tag{2.20}$$

holds for all

$$V \in V^{(2)} := \{V(\xi,\eta) : V \in C^2(\bar{D}), V(\xi,1) = 0,$$
$$V \equiv 0 \ \textit{in a neighborhood of} \ (1,1)\}.$$

2.4. Riemann-Hadamard function associated to Problem $\mathbf{P_{m2}}$

Using a Riemann-Hadamard function associated to Problem P_{m2}, we give an explicit integral representation of the generalized solution $U(\xi,\eta)$. The Riemann-Hadarmard function can be represented in the following way:

$$\Psi(\xi,\eta;\xi_0,\eta_0) = \begin{cases} \Psi^+(\xi,\eta;\xi_0,\eta_0), & \eta > \xi_0, \\ \Psi^-(\xi,\eta;\xi_0,\eta_0), & \eta < \xi_0, \end{cases} \tag{2.21}$$

where

$$\Psi^+(\xi,\eta;\xi_0,\eta_0) := \left(\frac{\eta-\xi}{\eta_0-\xi_0}\right)^{\beta} F_3(\beta, n+1, 1-\beta, -n, 1; X, Y),$$
$$\Psi^-(\xi,\eta;\xi_0,\eta_0) := \gamma \left(\frac{\eta-\xi}{\eta_0-\xi_0}\right)^{\beta} X^{-\beta} H_2\left(\beta,\beta,-n,n+1,2\beta;\frac{1}{X},-Y\right), \tag{2.22}$$

$$X = X(\xi,\eta;\xi_0,\eta_0) := \frac{(\xi_0-\xi)(\eta_0-\eta)}{(\eta-\xi)(\eta_0-\xi_0)}, \tag{2.23}$$

$$Y = Y(\xi,\eta;\xi_0,\eta_0) := \frac{-(\xi_0-\xi)(\eta_0-\eta)}{(2-\xi-\eta)(2-\xi_0-\eta_0)}, \tag{2.24}$$

$$\gamma = \frac{\Gamma(\beta)}{\Gamma(1-\beta)\Gamma(2\beta)}.$$

Here $F_3(a_1, a_2, b_1, b_2, c; x, y)$ is the Appell series (A.21) which, in the general case, converges absolutely for $|x| < 1$, $|y| < 1$ and $H_2(a_1, a_2, b_1, b_2, c; x, y)$ is the Horn series (A.22) which in the general case converges absolutely for $|x| < 1$, $|y|(1 + |x|) < 1$. For basic information on the Appell and the Horn series, see [5], pp. 220 - 228.

We mention however that in our particular case in the series (A.21)-(A.22) we have finite sums with respect to i, because $n \in \mathbb{N} \cup \{0\}$. More precisely, as it will be seen further (Lemma 2.4.1), these series involve a finite number of hypergeometric series ${}_2F_1(a, b, c; x)$. Consequently, in our case we have an absolute convergence for $|y| < \infty$ and $|x| < 1$.

Remark 2.4.1. *The function $\Psi(\xi, \eta; \xi_0, \eta_0)$ is closely connected to the Riemann-Hadamard function announced in* [51] *(p. 25, example 7), which is associated to a Cauchy-Goursat problem for an equation connected with* (2.15) *with some appropriate substitutions.*

According to Gellerstedt [13] and the results of Nakhushev mentioned in the book of Smirnov [49], for $(\xi_0, \eta_0) \in D$ the Riemann-Hadamard function $\Psi(\xi, \eta; \xi_0, \eta_0)$ should have the following main properties:

(i) The function Ψ as a function of (ξ_0, η_0) satisfies

$$\frac{\partial^2 \Psi}{\partial \xi_0 \partial \eta_0} + \frac{\beta}{\eta_0 - \xi_0}\left(\frac{\partial \Psi}{\partial \xi_0} - \frac{\partial \Psi}{\partial \eta_0}\right) - \frac{n(n+1)}{(2 - \xi_0 - \eta_0)^2}\Psi = 0 \tag{2.25}$$

and with respect to the first pair of variables (ξ, η)

$$\frac{\partial^2 \Psi}{\partial \xi \partial \eta} - \frac{\partial}{\partial \xi}\left(\frac{\beta \Psi}{\eta - \xi}\right) + \frac{\partial}{\partial \eta}\left(\frac{\beta \Psi}{\eta - \xi}\right) - \frac{n(n+1)}{(2 - \xi - \eta)^2}\Psi = 0 \tag{2.26}$$

for $0 < \xi < \xi_0,\ \xi < \eta < \eta_0,\ \eta \neq \xi_0$;

(ii) $\Psi^+(\xi_0, \eta_0; \xi_0, \eta_0) = 1$;

(iii) $\Psi^+(\xi, \eta_0; \xi_0, \eta_0) = \left(\dfrac{\eta_0 - \xi}{\eta_0 - \xi_0}\right)^\beta$;

(iv) $\Psi^+(\xi_0, \eta; \xi_0, \eta_0) = \left(\dfrac{\eta - \xi_0}{\eta_0 - \xi_0}\right)^\beta$;

(v) Ψ^- vanishes on the line $\{\eta = \xi\}$ of power 2β;

(vi) the jump of the function Ψ on the line $\{\eta = \xi_0\}$ is

$$
\begin{aligned}
[[\Psi]] &:= \lim_{\delta \to +0} \{\Psi^-(\xi, \xi_0 - \delta; \xi_0, \eta_0) - \Psi^+(\xi, \xi_0 + \delta; \xi_0, \eta_0)\} \\
&= \cos \pi\beta \lim_{\delta \to +0} \{\Psi^+(\xi, \xi_0 + \delta; \xi_0, \xi_0 + \delta)\Psi^+(\xi_0, \xi_0 + \delta; \xi_0, \eta_0)\} \\
&= \cos \pi\beta \left(\frac{\xi_0 - \xi}{\eta_0 - \xi_0}\right)^\beta .
\end{aligned}
$$

The series $\Psi^+(\xi, \eta; \xi_0, \eta_0)$ at the points where it converges coincides with the Riemann function for equation (2.15).

These properties will be justified below, but before this we will give a special decomposition of the function $\Psi(\xi, \eta; \xi_0, \eta_0)$, which will be useful for our further considerations.

Define the functions

$$
H(\xi, \eta; \xi_0, \eta_0) := \begin{cases} H^+(\xi, \eta; \xi_0, \eta_0), & \eta > \xi_0, \\ H^-(\xi, \eta; \xi_0, \eta_0), & \eta < \xi_0, \end{cases} \tag{2.27}
$$

where

$$H^{+}(\xi,\eta;\xi_0,\eta_0) := \left(\frac{\eta-\xi}{\eta_0-\xi_0}\right)^{\beta} {}_2F_1(\beta,1-\beta,1;X), \tag{2.28}$$

$$H^{-}(\xi,\eta;\xi_0,\eta_0) := \gamma\left(\frac{\eta-\xi}{\eta_0-\xi_0}\right)^{\beta} X^{-\beta}\, {}_2F_1\left(\beta,\beta,2\beta;\frac{1}{X}\right) \tag{2.29}$$

and

$$G(\xi,\eta;\xi_0,\eta_0) := \begin{cases} G^{+}(\xi,\eta;\xi_0,\eta_0), & \eta>\xi_0, \\ G^{-}(\xi,\eta;\xi_0,\eta_0), & \eta<\xi_0, \end{cases} \tag{2.30}$$

where

$$G^{+}(\xi,\eta;\xi_0,\eta_0) := \left(\frac{\eta-\xi}{\eta_0-\xi_0}\right)^{\beta} \sum_{i=1}^{n} c_i Y^i\, {}_2F_1(\beta,1-\beta,i+1;X), \tag{2.31}$$

$$G^{-}(\xi,\eta;\xi_0,\eta_0) := \gamma\left(\frac{\eta-\xi}{\eta_0-\xi_0}\right)^{\beta} X^{-\beta} \sum_{i=1}^{n} d_i Y^i\, {}_2F_1\left(\beta-i,\beta,2\beta;\frac{1}{X}\right), \tag{2.32}$$

$$c_i := \frac{(n+1)_i(-n)_i}{i!\,i!}, \quad d_i := \frac{(n+1)_i(-n)_i}{(1-\beta)_i\,i!}. \tag{2.33}$$

Actually, the function $H(\xi,\eta;\xi_0,\eta_0)$ is the Riemann-Hadamard function associated to Problem P_{m2} in the case $n=0$ (see Gellerstedt [13] and M. Smirnov [49]).

Lemma 2.4.1. *For* $(\xi_0,\eta_0)\in D$ *and* $0<\xi<\xi_0,\ \xi<\eta<\eta_0,\ \eta\neq\xi_0$ *the function* $\Psi(\xi,\eta;\xi_0,\eta_0)$ *has the following decomposition*

$$\Psi(\xi,\eta;\xi_0,\eta_0) = H(\xi,\eta;\xi_0,\eta_0) + G(\xi,\eta;\xi_0,\eta_0). \tag{2.34}$$

Proof. For $(\xi_0, \eta_0) \in D$ and $0 < \xi < \xi_0$, $\xi < \eta < \eta_0$, $\eta \neq \xi_0$ we have $|Y| < \infty$, $|X| < 1$ if $\eta > \xi_0$ and $|1/X| < 1$ if $\eta < \xi_0$, therefore the functions $H(\xi, \eta; \xi_0, \eta_0)$ and $G(\xi, \eta; \xi_0, \eta_0)$ are well defined.

In view of (A.21) we have

$$F_3(\beta, n+1, 1-\beta, -n, 1; X, Y) = \sum_{i=0}^{n} \sum_{j=0}^{\infty} \frac{(\beta)_j (1-\beta)_j (n+1)_i (-n)_i}{(1)_{i+j}\, i!\, j!} X^j Y^i.$$

Since $(1)_{i+j} = (i+j)! = i!\,(i+1)_j$ for $i, j \in \mathbf{N} \cup \{0\}$, we obtain from (2.22)

$$\Psi^+(\xi, \eta; \xi_0, \eta_0) = H^+(\xi, \eta; \xi_0, \eta_0) + G^+(\xi, \eta; \xi_0, \eta_0).$$

Next, in view of (A.22) we have

$$H_2\left(\beta, \beta, -n, n+1, 2\beta; \frac{1}{X}, Y\right) := \\ \sum_{i=0}^{n} \sum_{j=0}^{\infty} \frac{(\beta)_{j-i} (\beta)_j (-n)_i (1-n)_i}{(2\beta)_j i! j!} X^{-j} (-Y)^i.$$

For $0 < \beta < 1$ and $i, j \in \mathbf{N} \cup \{0\}$ we have

$$(\beta)_{j-i} = \frac{\Gamma(\beta + j - i)}{\Gamma(\beta)} = \frac{\Gamma(\beta - i)}{\Gamma(\beta)} (\beta - i)_j.$$

Using this and also the relation $\Gamma(z)\Gamma(1-z) = \pi / \sin \pi z$ we calculate

$$(\beta)_{j-i} (1-\beta)_i = (\beta - i)_j \frac{\Gamma(\beta - i)\Gamma(1 - \beta + i)}{\Gamma(\beta)\Gamma(1-\beta)} \\ = (\beta - i)_j \frac{\sin \beta\pi}{\sin(\beta - i)\pi} = (-1)^i (\beta - i)_j.$$

Then from (2.22) it follows

$$\Psi^-(\xi,\eta;\xi_0,\eta_0) = H^-(\xi,\eta;\xi_0,\eta_0) + G^-(\xi,\eta;\xi_0,\eta_0),$$

which completes the proof. □

The properties (i)-(vi) of the function $\Psi(\xi,\eta;\xi_0,\eta_0)$ listed above can be confirmed in the following way:

(i) Using the systems of differential equations that F_3 and H_2 satisfy (see [5], p. 227 - 228) with straightforward calculation we check that the function $\Psi(\xi,\eta;\xi_0,\eta_0)$ satisfies the equations (2.25) and (2.26).

(ii)-(iv) Since $X(\xi_0,\eta;\xi_0,\eta_0) = X(\xi,\eta_0;\xi_0,\eta_0) = 0$ and $Y(\xi_0,\eta;\xi_0,\eta_0) = Y(\xi,\eta_0;\xi_0,\eta_0) = 0$ we see that the function $\Psi(\xi,\eta;\xi_0,\eta_0)$ has the properties (ii), (iii) and (iv).

(v) The property (v) easily follows from the fact that on the line $\{\eta = \xi\}$ we have $1/X = 0$.

(vi) Using (A.9), for $i \in \mathbb{N}$ we calculate

$$c_i\, {}_2F_1(\beta, 1-\beta, i+1; 1) = \gamma\, d_i\, {}_2F_1(\beta - i, \beta, 2\beta; 1) = \frac{(n+1)_i(-n)_i}{i\,\Gamma(1-\beta+i)\Gamma(\beta+i)}.$$

Applying this into (2.31) and (2.32), we see that

$$G^+(\xi,\xi_0;\xi_0,\eta_0) = G^-(\xi,\xi_0;\xi_0,\eta_0),$$

i.e. the function $G(\xi,\eta;\xi_0,\eta_0)$ has no jump on the line $\{\eta = \xi_0\}$. Then, in view of (2.34), we have $[[\Psi]] = [[H]]$. It is well known that $[[H]] =$

$\cos \pi\beta \left(\frac{\xi_0-\xi}{\eta_0-\xi_0}\right)^{\beta}$ (see Gellerstedt [13]), which confirms the property (vi).

Remark 2.4.2. *The Riemann-Hadamard function* $\Phi(\xi, \eta; \xi_0, \eta_0)$ *associated to Problem* P_{02}*, which we introduced in Theorem 1.4.1, can be obtained as*

$$\Phi(\xi, \eta; \xi_0, \eta_0) = \lim_{\beta\to 0} \Psi(\xi, \eta; \xi_0, \eta_0).$$

Indeed, using that ${}_2F_1(0, b, c; \zeta) = 1$, we have that

$$\Phi^+(\xi, \eta; \xi_0, \eta_0) = \lim_{\beta\to 0} \Psi^+(\xi, \eta; \xi_0, \eta_0),$$

because

$$\lim_{\beta\to 0} H^+(\xi, \eta; \xi_0, \eta_0) = 1, \qquad \lim_{\beta\to 0} G^+(\xi, \eta; \xi_0, \eta_0) = \sum_{i=1}^{n} c_i Y^i.$$

Further, we have

$$\Psi^-(\xi, \eta; \xi_0, \eta_0) =$$
$$\frac{1}{\Gamma(1-\beta)} \left(\frac{\eta-\xi}{\eta_0-\xi_0}\right)^{\beta} \sum_{i=0}^{n} d_i Y^i \sum_{j=0}^{\infty} \frac{(\beta-i)_j \Gamma(\beta+j)}{\Gamma(2\beta+j)\, j!} X^{-j-\beta} =$$
$$\frac{1}{\Gamma(1-\beta)} \left(\frac{\eta-\xi}{\eta_0-\xi_0}\right)^{\beta} \sum_{i=0}^{n} d_i Y^i \left[\frac{\Gamma(\beta)}{\Gamma(2\beta)} X^{-\beta} + \sum_{j=1}^{\infty} \frac{(\beta-i)_j \Gamma(\beta+j)}{\Gamma(2\beta+j)\, j!} X^{-j-\beta}\right].$$

Using the well known relation $\Gamma(2z)\Gamma(1/2) = 2^{2z-1}\Gamma(z)\Gamma(z+1/2)$, we have

$$\lim_{\beta\to 0} \frac{\Gamma(\beta)}{\Gamma(2\beta)} = \lim_{\beta\to 0} \frac{\Gamma(1/2)\Gamma(\beta)}{2^{2\beta-1}\Gamma(\beta)\Gamma(\beta+1/2)} = 2$$

and consequently, with use of (A.11), we calculate

$$\lim_{\beta\to 0} \Psi^-(\xi,\eta;\xi_0,\eta_0) = \sum_{i=0}^{n} c_i Y^i \left[2 + \sum_{j=1}^{\infty} \frac{(-i)_j}{j!} X^{-j}\right]$$
$$= \sum_{i=0}^{n} c_i Y^i \left[1 + (1 - 1/X)^i\right]$$
$$= F(n+1,-n,1;Y) + F(n+1,-n,1;Y(1-1/X)) = \Phi^-(\xi,\eta;\xi_0,\eta_0).$$

2.5. Estimates for integrals involving the Riemann-Hadamard function

In order to prove an existence result for Problem P_{m2} we need to obtain a priori estimates for some integrals involving the function $\Psi(\xi,\eta;\xi_0,\eta_0)$. First, we estimate the functions $H(\xi,\eta;\xi_0,\eta_0)$ and $G(\xi,\eta;\xi_0,\eta_0)$ and their first derivatives.

Lemma 2.5.1. *Let* $0 < \beta < 1$ *and* $(\xi_0,\eta_0) \in D$*. Then there exists a constant* $C_H > 0$ *such that*

$$|H^+(\xi,\eta;\xi_0,\eta_0)| \le C_H(\eta-\xi_0)^{-\beta}, \tag{2.35}$$
$$|H^+_{\eta_0}(\xi,\eta;\xi_0,\eta_0)| \le C_H\frac{(\eta-\xi_0)^{-\beta}}{\eta_0-\xi_0} \tag{2.36}$$

for $0 < \xi < \xi_0,\ \xi_0 < \eta < \eta_0$ *and*

$$|H^-(\xi,\eta;\xi_0,\eta_0)| \le C_H(\xi_0-\eta)^{-\beta}, \tag{2.37}$$
$$|H^-_{\eta_0}(\xi,\eta;\xi_0,\eta_0)| \le C_H\frac{(\xi_0-\eta)^{-\beta}}{\eta_0-\eta} \tag{2.38}$$

for $0 < \xi < \eta < \xi_0$.

Proof. First, using (A.6) we find that for each $\alpha > 0$ there exists a constant $c(\alpha) > 0$ such that

$$
\begin{aligned}
|H^{+}(\xi,\eta;\xi_0,\eta_0)| &\le c(\alpha)\frac{(\eta-\xi)^{\alpha+\beta}(\eta_0-\xi_0)^{\alpha-\beta}}{(\eta-\xi_0)^{\alpha}(\eta_0-\xi)^{\alpha}},\\
|H^{-}(\xi,\eta;\xi_0,\eta_0)| &\le c(\alpha)\frac{(\eta-\xi)^{2\beta}(\xi_0-\xi)^{\alpha-\beta}(\eta_0-\eta)^{\alpha-\beta}}{(\eta_0-\xi)^{\alpha}(\xi_0-\eta)^{\alpha}}.
\end{aligned}
\tag{2.39}
$$

From here choosing $\alpha = \beta$ we obtain the estimates (2.35), (2.37).

Next, using (A.10) for the derivatives with respect to η_0 we obtain

$$
H^{+}_{\eta_0} = \frac{-\beta}{\eta_0-\xi_0}H^{+} + \beta(1-\beta)\left(\frac{\eta-\xi}{\eta_0-\xi_0}\right)^{\beta} X_{\eta_0}\, {}_2F_1(1+\beta, 2-\beta, 2; X)
$$

and

$$
\begin{aligned}
H^{-}_{\eta_0} = {} & \frac{\beta H^{-}}{\eta-\eta_0}\\
& + \frac{\gamma\beta(\eta-\xi)^{2\beta}}{2(\xi_0-\xi)^{\beta}(\eta_0-\eta)^{\beta}}\left(\frac{1}{X}\right)_{\eta_0} {}_2F_1\left(1+\beta, 1+\beta, 1+2\beta; \frac{1}{X}\right).
\end{aligned}
$$

Now with use of (A.7) we obtain the estimates (2.36), (2.38). □

Lemma 2.5.2. *Let* $0 < \beta < 1$ *and* $(\xi_0, \eta_0) \in D$. *Then there exists a constant* $C_G > 0$ *such that*

$$
|G^{+}(\xi,\eta;\xi_0,\eta_0)| \le C_G(\eta_0-\xi_0)^{-\beta}, \tag{2.40}
$$

$$
\left|G^{+}_{\xi_0}(\xi,\eta;\xi_0,\eta_0)\right| \le C_G\frac{(\eta-\xi_0)^{-\beta}}{2-\xi_0-\eta_0}, \tag{2.41}
$$

$$
\left|G^{+}_{\eta_0}(\xi,\eta,\xi_0,\eta_0)\right| \le C_G\frac{(\eta_0-\xi_0)^{-\beta}}{2-\xi_0-\eta_0} \tag{2.42}
$$

for $0 < \xi < \xi_0,\ \xi_0 < \eta < \eta_0$ *and*

$$
\begin{aligned}
|G^-(\xi,\eta;\xi_0,\eta_0)| &\le C_G(2-\xi_0-\eta_0)^{-n}, &(2.43)\\
\left|G^-_{\xi_0}(\xi,\eta;\xi_0,\eta_0)\right| &\le C_G\frac{(\xi_0-\eta)^{-\beta}}{(2-\xi_0-\eta_0)^{n+1}}, &(2.44)\\
\left|G^-_{\eta_0}(\xi,\eta,\xi_0,\eta_0)\right| &\le C_G\frac{(\eta_0-\eta)^{-\beta}}{(2-\xi_0-\eta_0)^{n+1}} &(2.45)
\end{aligned}
$$

for $0 < \xi < \eta < \xi_0$.

Proof. First, let $0 < \xi < \xi_0,\ \xi_0 < \eta < \eta_0$.

According to (A.8), for $i = 1, \ldots, n$ we have

$$|\,{}_2F_1(\beta, 1-\beta, i+1; X)| \le \text{const}.$$

Applying this in the expression (2.31) for $G^+(\xi,\eta;\xi_0,\eta_0)$, we see that the estimate (2.40) holds.

Now with use of (A.10) we calculate the first derivatives of $G^+(\xi,\eta;\xi_0,\eta_0)$:

$$
\begin{aligned}
G^+_{\xi_0} = \left(\frac{\eta-\xi}{\eta_0-\xi_0}\right)^{\beta}\Bigg\{&\sum_{i=1}^{n} c_i\left[\frac{\beta\,Y^i}{\eta_0-\xi_0} + i\,Y^{i-1}Y_{\xi_0}\right]{}_2F_1(\beta,1-\beta,i+1;X)\\
&+\beta(1-\beta)\sum_{i=1}^{n}\frac{c_i}{i+1}Y^iX_{\xi_0}\,{}_2F_1(\beta+1,2-\beta,i+2;X)\Bigg\},
\end{aligned}
$$

$$
\begin{aligned}
G^+_{\eta_0} = \left(\frac{\eta-\xi}{\eta_0-\xi_0}\right)^{\beta}\Bigg\{&\sum_{i=1}^{n} c_i\left[-\frac{\beta\,Y^i}{\eta_0-\xi_0} + i\,Y^{i-1}Y_{\eta_0}\right]{}_2F_1(\beta,1-\beta,i+1;X)\\
&+\beta(1-\beta)\sum_{i=1}^{n}\frac{c_i}{i+1}Y^iX_{\eta_0}\,{}_2F_1(\beta+1,2-\beta,i+2;X)\Bigg\}.
\end{aligned}
$$

According to (A.8) and (A.6) for the hypergeometric functions in the expressions for $G^+_{\xi_0}$ and $G^+_{\eta_0}$ we have

$$|\,{}_2F_1(\beta+1, 2-\beta, 3; X)| \le c(\alpha)\left(\frac{\eta_0-\xi_0}{\eta-\xi_0}\right)^{\alpha}, \quad \alpha > 0,$$

$$|\,{}_2F_1(1+\beta, 2-\beta, i+2; X)| \le \text{const}, \quad i = 2, 3, \dots, n.$$

Using this and taking $\alpha = \beta$ we obtain the estimates (2.41) and (2.42).

Next, let $0 < \xi < \eta < \xi_0$.

According to (A.8) for $i = 1, \dots, n$ we have

$$|\,{}_2F_1(\beta - i, \beta, 2\beta; 1/X)| \le \text{const}.$$

Applying this into (2.32) leads to the estimate (2.43).

Let us calculate the first derivatives of $G^-(\xi, \eta; \xi_0, \eta_0)$:

$$\begin{aligned}
G^-_{\xi_0} = {} & \frac{\gamma(\eta-\xi)^{2\beta}}{(\xi_0-\xi)^{\beta}(\eta_0-\eta)^{\beta}} \\
& \times \left\{ \sum_{i=1}^{n} d_i \left[-\frac{\beta Y^i}{\xi_0-\xi} + i\,Y^{i-1}Y_{\xi_0} \right] {}_2F_1\left(\beta-i, \beta, 2\beta; \frac{1}{X}\right) \right. \\
& \left. + \frac{1}{2}\sum_{i=1}^{n} (\beta-i) d_i Y^i \left(\frac{1}{X}\right)_{\xi_0} {}_2F_1\left(\beta-i+1, \beta+1, 2\beta+1; \frac{1}{X}\right) \right\},
\end{aligned}$$

$$\begin{aligned}
G^-_{\eta_0} = {} & \frac{\gamma(\eta-\xi)^{2\beta}}{(\xi_0-\xi)^{\beta}(\eta_0-\eta)^{\beta}} \\
& \times \left\{ \sum_{i=1}^{n} d_i \left[-\frac{\beta Y^i}{\eta_0-\eta} + i\,Y^{i-1}Y_{\eta_0} \right] {}_2F_1\left(\beta-i, \beta, 2\beta; \frac{1}{X}\right) \right. \\
& \left. + \frac{1}{2}\sum_{i=1}^{n} (\beta-i) d_i\, Y^i \left(\frac{1}{X}\right)_{\eta_0} {}_2F_1\left(\beta-i+1, \beta+1, 2\beta+1; \frac{1}{X}\right) \right\}.
\end{aligned}$$

Now with (A.8) and (A.6) we estimate

$$\left| {}_2F_1\left(\beta, \beta+1, 2\beta+1; \frac{1}{X}\right)\right| \le c(\alpha)\left(\frac{\eta_0-\eta}{\xi_0-\eta}\right)^{\alpha}, \quad \alpha > 0,$$
$$| {}_2F_1\left(\beta-i+1, 1+\beta, 1+2\beta; \frac{1}{X}\right)| \le \text{const}, \quad i = 2, 3, \ldots, n.$$

Using this and taking $\alpha = \beta$ we come the estimates (2.44) and (2.45). □

Now we are ready to estimate some integrals involving the Riemann-Hadamard function and its first derivatives.

Lemma 2.5.3. *Suppose that* $0 < \beta < 1$ *and* $(\xi_0, \eta_0) \in D$. *Then*

$$I_H(\xi_0, \eta_0) := \int_0^{\xi_0}\int_{\xi}^{\eta_0} |H_{\eta_0}(\xi, \eta; \xi_0, \eta_0)|\, d\eta\, d\xi \le k\, \xi_0(\eta_0-\xi_0)^{-\beta}, \quad (2.46)$$

where $k = \text{const} > 0$.

Proof. Using the estimates (2.36) and (2.38) we have

$$I_H \le C_H\left\{\int_0^{\xi_0}\int_{\xi}^{\xi_0}\frac{(\xi_0-\eta)^{-\beta}}{\eta_0-\eta}\, d\eta\, d\xi + \int_0^{\xi_0}\int_{\xi_0}^{\eta_0}\frac{(\eta-\xi_0)^{-\beta}}{\eta_0-\xi_0}\, d\eta\, d\xi\right\}. \quad (2.47)$$

Making a substitution $\eta = \xi + (\xi_0 - \xi)\sigma$ and applying (A.5) we get

$$\int_{\xi}^{\xi_0}\frac{(\xi_0-\eta)^{-\beta}}{\eta_0-\eta}\, d\eta = \frac{(\xi_0-\xi)^{1-\beta}}{\eta_0-\xi}\int_0^1\frac{(1-\sigma)^{-\beta}}{1-\zeta\sigma}\, d\sigma$$
$$= \frac{(\xi_0-\xi)^{1-\beta}}{\eta_0-\xi}\frac{\Gamma(1-\beta)}{\Gamma(2-\beta)}\, {}_2F_1(1, 1, 2-\beta; \zeta) \quad (2.48)$$

with $\zeta = (\xi_0 - \xi)/(\eta_0 - \xi)$. Now, according to (A.7), we have

$$|\,{}_2F_1(1, 1, 2-\beta; \zeta)| \leq \text{const}\frac{(\eta_0 - \xi)^\beta}{(\eta_0 - \xi_0)^\beta}. \tag{2.49}$$

Substituting (2.48) and (2.49) into (2.47) immediately leads to the estimate (2.46). □

Theorem 2.5.1. *Define the function*

$$U^H(\xi_0, \eta_0) := \int_0^{\xi_0} \int_{\xi}^{\eta_0} F(\xi, \eta) H(\xi, \eta; \xi_0, \eta_0)\, d\eta\, d\xi \tag{2.50}$$

with $0 < \beta < 1$ *and* $F \in C^1(\bar{D})$. *Then* U^H, $U^H_{\xi_0} + U^H_{\eta_0} \in C(\bar{D} \setminus (1,1))$, $U^H_{\eta_0} \in C(\bar{D} \setminus \{\eta_0 = \xi_0\})$ *and for* $(\xi_0, \eta_0) \in D$ *the following estimates hold*:

$$|U^H(\xi_0, \eta_0)| \leq K_1 M_F \xi_0, \tag{2.51}$$

$$|U^H_{\xi_0} + U^H_{\eta_0}|(\xi_0, \eta_0) \leq K_1 M_F, \tag{2.52}$$

$$\left|U^H_{\eta_0}(\xi_0, \eta_0)\right| \leq K_1 M_F \xi_0 (\eta_0 - \xi_0)^{-\beta}, \tag{2.53}$$

where $K_1 > 0$ *is a constant, independent of* F *and*

$$M_F := \max\left\{\max_{\bar{D}} |F|, \max_{\bar{D}} |F_\xi + F_\eta|\right\}. \tag{2.54}$$

Proof. Using the estimates (2.35) and (2.37) immediately we obtain

$$|U^H(\xi_0, \eta_0)| \leq M_F \int_0^{\xi_0} \int_{\xi}^{\eta_0} H(\xi, \eta; \xi_0, \eta_0)\, d\eta\, d\xi \leq K_1 M_F \xi_0,$$

which confirms (2.51).

Next, differentiating (2.50) with respect to η_0 gives

$$U^H_{\eta_0}(\xi_0,\eta_0) = \int_0^{\xi_0}\int_\xi^{\eta_0} F(\xi,\eta)H_{\eta_0}(\xi,\eta;\xi_0,\eta_0)\,d\eta\,d\xi + \int_0^{\xi_0} F(\xi,\eta_0)\frac{(\eta_0-\xi)^\beta}{(\eta_0-\xi_0)^\beta}\,d\xi.$$

With use of the estimate (2.46) from Lemma 2.5.3 we come to the estimate (2.53).

Next, a direct calculation shows that the derivatives $H^+_{\xi_0}$, $H^-_{\xi_0}$ have singularities on the line $\{\eta=\xi_0\}$, which are not integrable. S. Gellerstedt [13] and E. Moiseev [26] suggested to differentiate (2.50) after appropriate substitutions of variables. In this way one can find integral representations for the first derivatives of the solution, which do not involve the first derivatives of the function $H(\xi,\eta;\xi_0,\eta_0)$. In order to do this, following Moiseev [26], introduce the new variables

$$\tilde\xi := \frac{\xi_0-\xi}{\eta_0-\xi_0},\ \tilde\eta := \frac{\eta_0-\eta}{\eta_0-\xi_0}. \tag{2.55}$$

Defining

$$\tilde H^+(\tilde\xi,\tilde\eta) := H^+(\xi,\eta;\xi_0,\eta_0),\quad \tilde H^-(\tilde\xi,\tilde\eta) := H^-(\xi,\eta;\xi_0,\eta_0),$$

we have

$$U^H(\xi_0,\eta_0) = (\eta_0-\xi_0)^2 \times \int_0^{\frac{\xi_0}{\eta_0-\xi_0}} \int_0^{1+\tilde{\xi}} F\big(\xi_0-(\eta_0-\xi_0)\tilde{\xi}, \eta_0-(\eta_0-\xi_0)\tilde{\eta}\big)\,\tilde{H}(\tilde{\xi},\tilde{\eta})\,d\tilde{\eta}\,d\tilde{\xi}$$

and

$$\begin{aligned}\left(U^H_{\xi_0}+U^H_{\eta_0}\right)(\xi_0,\eta_0) &= (\eta_0-\xi_0)^2 \\ &\times \int_0^{\frac{\xi_0}{\eta_0-\xi_0}} \int_0^{1+\tilde{\xi}} (F_\xi+F_\eta)\big(\xi_0-(\eta_0-\xi_0)\tilde{\xi}, \eta_0-(\eta_0-\xi_0)\tilde{\eta}\big)\,\tilde{H}(\tilde{\xi},\tilde{\eta})\,d\tilde{\eta}\,d\tilde{\xi} \\ &+ (\eta_0-\xi_0)\int_0^{\frac{\eta_0}{\eta_0-\xi_0}} F\big(0,\eta_0-(\eta_0-\xi_0)\tilde{\eta}\big)\,\tilde{H}\left(\frac{\xi_0}{\eta_0-\xi_0},\tilde{\eta}\right)\,d\tilde{\eta}.\end{aligned}$$

Now the inverse transform of (2.55) gives

$$\begin{aligned}\left(U^H_{\xi_0}+U^H_{\eta_0}\right)(\xi_0,\eta_0) &= \int_0^{\xi_0}\int_\xi^{\eta_0} (F_\xi+F_\eta)(\xi,\eta)H(\xi,\eta;\xi_0,\eta_0)\,d\eta\,d\xi \\ &+ \int_0^{\eta_0} F(0,\eta)H(0,\eta;\xi_0,\eta_0)\,d\eta.\end{aligned}$$

Finally, taking into account the estimates (2.35) and (2.37), we see that (2.52) holds. □

Theorem 2.5.2. *Define the function*

$$U^G(\xi_0,\eta_0) := \int_0^{\xi_0}\int_\xi^{\eta_0} F(\xi,\eta)G(\xi,\eta;\xi_0,\eta_0)\,d\eta\,d\xi \tag{2.56}$$

with $0<\beta<1$ *and* $F\in C^1(\bar{D})$*. Then* U^G, $U^G_{\xi_0}$, $U^G_{\eta_0} \in C(\bar{D}\setminus(1,1))$, *and* *for* $(\xi_0, \eta_0) \in D$ *the following estimates hold*:

$$|U^G(\xi_0, \eta_0)| \leq K_2 M_F \xi_0 (2 - \xi_0 - \eta_0)^{-n}, \tag{2.57}$$

$$\left|U^G_{\xi_0}(\xi_0, \eta_0)\right| \leq K_2 M_F \xi_0 (2 - \xi_0 - \eta_0)^{-n-1}, \tag{2.58}$$

$$\left|U^G_{\eta_0}(\xi_0, \eta_0)\right| \leq K_2 M_F \xi_0 (2 - \xi_0 - \eta_0)^{-n-1}, \tag{2.59}$$

where $K_2 > 0$ *is a constant independent of* F *and* M_F *is the constant defined by* (2.54).

Proof. First, applying the estimates (2.40) and (2.43) into (2.56) we obtain :

$$\begin{aligned} |U^G(\xi_0, \eta_0)| = \Bigg| & \int_0^{\xi_0} \int_{\xi}^{\xi_0} F(\xi, \eta) G^-(\xi, \eta; \xi_0, \eta_0)\, d\eta\, d\xi \\ & + \int_0^{\xi_0} \int_{\xi_0}^{\eta_0} F(\xi, \eta) G^+(\xi, \eta; \xi_0, \eta_0)\, d\eta\, d\xi \Bigg| \leq K_2 M_F \xi_0 (2 - \xi_0 - \eta_0)^{-n}, \end{aligned}$$

which confirms the estimate (2.57).

Next, we calculate

$$U^G_{\xi_0}(\xi_0, \eta_0) = \int_0^{\xi_0} \int_{\xi}^{\eta_0} F(\xi, \eta) G_{\xi_0}(\xi, \eta, \xi_0, \eta_0)\, d\eta\, d\xi.$$

Here we do not have integrals on the boundaries because $Y = 0$ on the line $\{\xi = \xi_0\}$ and the function $G(\xi, \eta, \xi_0, \eta_0)$ has no jump on the line $\{\eta = \xi_0\}$.

Applying to this integral the estimates (2.41) and (2.44) gives:

$$\begin{aligned}\left|U^G_{\xi_0}(\xi_0,\eta_0)\right| \leq \frac{M_F C_G}{(2-\xi_0-\eta_0)^{n+1}} \int_0^{\xi_0}\int_\xi^{\xi_0} (\xi_0-\eta)^{-\beta}\, d\eta\, d\xi \\ + \frac{M_F C_G}{2-\xi_0-\eta_0} \int_0^{\xi_0}\int_{\xi_0}^{\eta_0} (\eta-\xi_0)^{-\beta}\, d\eta\, d\xi.\end{aligned}$$

From here it easily follows the estimate (2.58).

Finally, we calculate:

$$U^G_{\eta_0}(\xi_0,\eta_0) = \int_0^{\xi_0}\int_\xi^{\eta_0} F(\xi,\eta) G_{\eta_0}(\xi,\eta;\xi_0,\eta_0)\, d\eta\, d\xi,$$

where we used that $Y = 0$ on the line $\{\eta = \eta_0\}$. Analogously, applying to the last integral the estimates (2.42) and (2.45) for the derivative G_{η_0}, which are even better than (2.41) and (2.44), we obtain the estimate (2.59).

□

Corollary 2.5.1. *Define the function*

$$U(\xi_0,\eta_0) := \int_0^{\xi_0}\int_\xi^{\eta_0} F(\xi,\eta)\Psi(\xi,\eta;\xi_0,\eta_0)\, d\eta\, d\xi \tag{2.60}$$

with $0<\beta<1$ *and* $F\in C^1(\bar{D})$. *Then* U, $U_\xi + U_\eta \in C(\bar{D}\setminus(1,1))$, $U_\eta \in C(\bar{D}\setminus\{\eta=\xi\})$ *and for* $(\xi_0,\eta_0)\in D$ *the following estimates hold*

$$\begin{aligned} |U(\xi,\eta)| &\leq K M_F \xi (2-\xi-\eta)^{-n}, \\ |(U_\xi+U_\eta)(\xi,\eta)| &\leq K M_F (2-\xi-\eta)^{-n-1}, \\ |U_\eta(\xi,\eta)| &\leq K M_F \xi(\eta-\xi)^{-\beta}(2-\xi-\eta)^{-n-1}, \end{aligned} \tag{2.61}$$

where $K>0$ *is a constant independent of* F *and* M_F *is the constant defined* *by* (2.54).

This assertion is a direct consequence of Theorem 2.5.1 and Theorem 2.5.2, because $U(\xi,\eta) = U^H(\xi,\eta) + U^G(\xi,\eta)$.

2.6. Existence and uniqueness results

In this section we prove the existence and uniqueness of a generalized solution of Problem P_{m2} at certain conditions.

Theorem 2.6.1. *Let* $0 < \beta < 1$ *and* $F \in C(\bar{D})$. *Then each generalized solution of Problem* P_{m2} *has the following integral representation in* D:

$$U(\xi_0,\eta_0) = \int_0^{\xi_0}\int_{\xi}^{\eta_0} F(\xi,\eta)\Psi(\xi,\eta;\xi_0,\eta_0)\,d\eta\,d\xi. \tag{2.62}$$

Proof. Let $U(\xi,\eta)$ be a generalized solution of Problem P_{m2} in D. For any arbitrary function $\psi(\xi,\eta)$ belonging to $V^{(2)}$ from (2.20) we obtain the identity

$$\int_D (\eta-\xi)^{2\beta}\left\{U_{\xi\eta} + \frac{\beta}{\eta-\xi}(U_\xi - U_\eta) - \frac{n(n+1)}{(2-\xi-\eta)^2}U - F\right\}\psi\,d\xi\,d\eta = 0,$$

where $U_{\xi\eta}$ is the weak derivative of U. Therefore

$$U_{\xi\eta} = F + \frac{n(n+1)}{(2-\xi-\eta)^2}U - \frac{\beta}{\eta-\xi}(U_\xi - U_\eta) \in C(D),$$

since F, U, $U_\xi - U_\eta \in C(D)$. From here it follows that $U_{\xi\eta}$ is a classical derivative of U and $U(\xi,\eta)$ satisfies the differential equation (2.15) in D in a classical sense.

Now, using the properties of the Riemann-Hadamard function, we obtain the integral representation (2.62) for the generalized solution of Problem P_{m2} integrating by parts the identity

$$E_\beta[U(\xi,\eta)]\Psi(\xi,\eta;\xi_0,\eta_0) = F(\xi,\eta)\Psi(\xi,\eta;\xi_0,\eta_0)$$

over a triangle

$$T_\delta := \{(\xi,\eta) : 0 < \xi < \xi_0 - 2\delta,\ \xi + \delta < \eta < \xi_0 - \delta\}$$

and then over the rectangle

$$\Pi_\delta := \{(\xi,\eta) : 0 < \xi < \xi_0 - 2\delta,\ \xi_0 + \delta < \eta < \eta_0\}$$

with $\delta > 0$ small enough, and finally letting $\delta \to 0$. $\square$

Actually, Theorem 2.6.1 claims the uniqueness of a generalized solution of Problem P_{m2}.

Next, if additionally $F \in C^1(\bar{D})$, the function $U(\xi,\eta)$ defined by (2.62) obviously coincides with the function (2.60) estimated in Corollary (2.5.1) and we will prove that this function is a generalized solution of Problem P_{m2} in D.

Theorem 2.6.2. *Let* $0 < \beta < 1$ *and* $F \in C^1(\bar{D})$*. Then there exists one and only one generalized solution of Problem* P_{m2} *in* D*, which has integral representation* (2.60) *and it satisfies the estimates* (2.61).

Proof. Let $U(\xi,\eta)$ be the function from Corollary 2.5.1. Then U, $U_\xi + U_\eta \in C(\bar{D} \setminus (1,1))$, $U_\eta \in C(\bar{D} \setminus \{\eta = \xi\})$, i.e. $U(\xi,\eta)$ satisfies the property (1) in Definition 2.3.1. From the estimates (2.61) it follows that the condition $U(0,\eta) = 0$ in Definition 2.3.1 and the estimate (2.19) hold as well.

Finally, we have to prove that $U(\xi,\eta)$ satisfies the identity (2.20). To do this we need three steps.

Step 1. We prove that $U(\xi,\eta)$ satisfies the differential equation (2.15) in a classical sense and $(U_\xi)_\eta \in C(D)$.

Following Smirnov [49], we find another representation formula for the function $U^H(\xi,\eta)$ from Theorem 2.5.1. Introduce the function

$$R_0(\xi,\eta;\xi_0,\eta_0) := \begin{cases} R_0^+(\xi,\eta;\xi_0,\eta_0), & \eta > \xi_0, \\ R_0^-(\xi,\eta;\xi_0,\eta_0), & \eta < \xi_0, \end{cases}$$

where

$$R_0^+(\xi,\eta;\xi_0,\eta_0) := \left(\frac{\eta_0-\eta}{\eta_0-\xi_0}\right)^{\beta}\left(\frac{\eta_0-\eta}{\eta_0-\xi}\right)^{1-\beta} \times F_1\left(1-\beta,\beta,1-\beta,2;\frac{\eta_0-\eta}{\eta_0-\xi_0},\frac{\eta_0-\eta}{\eta_0-\xi}\right),$$

$$R_0^-(\xi,\eta;\xi_0,\eta_0) := \lambda\left(\frac{\eta-\xi}{\xi_0-\xi}\right)^{\beta}\left(\frac{\eta-\xi}{\eta_0-\xi}\right)^{\beta} \times F_1\left(\beta,\beta,\beta,1+2\beta;\frac{\eta-\xi}{\xi_0-\xi},\frac{\eta-\xi}{\eta_0-\xi}\right).$$

Here

$$\lambda = \frac{-\Gamma(\beta)}{\Gamma(1-\beta)\Gamma(1+2\beta)}$$

and $F_1(a, b_1, b_2, c; x, y)$ is the hypergeometric function (A.20) of two variables. This series converges absolutely for $|x| < 1$, $|y| < 1$. For more properties of F_1 see [5], pp. 219 - 223.

From [49] it is known that for $0 < \beta < 1/2$ the function $R_0(\xi, \eta; \xi_0, \eta_0)$ solves

$$\begin{gathered} \frac{\partial R_0}{\partial \eta} = -(\eta-\xi)^{-1} H(\xi, \eta; \xi_0, \eta_0) \quad \text{for} \quad 0 < \xi < \xi_0,\ \xi < \eta < \eta_0,\ \eta \neq \xi_0, \\ R_0|_{\eta=\eta_0} = 0, \qquad R_0|_{\eta=\xi} = 0, \end{gathered} \tag{2.63}$$

where $(\xi_0, \eta_0) \in D$. Here we verify that in the more general case $0 < \beta < 1$ this is still valid.

Further, it is known that the jump of $R_0(\xi, \eta; \xi_0, \eta_0)$ on the line $\{\eta = \xi_0\}$ is

$$[[R_0]] = -\frac{1}{\beta}. \tag{2.64}$$

Using (2.63) and (2.64), after integration by parts we come to the following integral representation:

$$\begin{aligned} U^H(\xi_0, \eta_0) := & \int_0^{\xi_0} \int_{\xi}^{\eta_0} \frac{\partial}{\partial \eta}[(\eta-\xi)F(\xi, \eta)] R_0(\xi, \eta; \xi_0, \eta_0)\, d\eta\, d\xi \\ & + \frac{1}{\beta} \int_0^{\xi_0} (\xi_0 - \xi) F(\xi, \xi_0)\, d\xi. \end{aligned} \tag{2.65}$$

Differentiating (2.65) we obtain that U^H satisfies the differential equation

$$\left(U^H_{\xi_0}\right)_{\eta_0} + \frac{\beta}{\eta_0 - \xi_0}\left(U^H_{\xi_0} - U^H_{\eta_0}\right) = F(\xi_0, \eta_0), \tag{2.66}$$

where all derivatives are in a classical sense and they are continuous in D.

Since $H(\xi, \eta; \xi_0, \eta_0)$ satisfies the differential equation (2.25) with $n = 0$ and $\Psi = H + G$ satisfies (2.25) with $n \geq 0$, for the difference $G = \Psi - H$ we obtain

$$G_{\xi_0\eta_0} + \frac{\beta}{\eta_0 - \xi_0}\left(G_{\xi_0} - G_{\eta_0}\right) - \frac{n(n+1)}{(2-\xi_0-\eta_0)^2}G = \frac{n(n+1)}{(2-\xi_0-\eta_0)^2}H.$$

Now, for the function $U^G(\xi_0, \eta_0)$ from Theorem 2.5.2 we calculate:

$$\begin{aligned}
&\left(U^G_{\xi_0}\right)_{\eta_0} + \frac{\beta}{\eta_0 - \xi_0}\left(U^G_{\xi_0} - U^G_{\eta_0}\right) - \frac{n(n+1)}{(2-\xi_0-\eta_0)^2}U^G \\
&\qquad = \int_0^{\xi_0}\int_{\xi}^{\eta_0} F(\xi,\eta)\Big[G_{\xi_0\eta_0} + \frac{\beta}{\eta_0-\xi_0}\left(G_{\xi_0} - G_{\eta_0}\right) \\
&\qquad\qquad - \frac{n(n+1)}{(2-\xi_0-\eta_0)^2}G\Big](\xi,\eta;\xi_0,\eta_0)\,d\eta\,d\xi \\
&\qquad = \frac{n(n+1)}{(2-\xi_0-\eta_0)^2}\int_0^{\xi_0}\int_{\xi}^{\eta_0} F(\xi,\eta)H(\xi,\eta;\xi_0,\eta_0)\,d\eta\,d\xi \\
&\qquad = \frac{n(n+1)}{(2-\xi_0-\eta_0)^2}U^H,
\end{aligned} \tag{2.67}$$

where all derivatives are in a classical sense and they are continuous in D.

Since $U = U^H + U^G$, from (2.66) and (2.67) we find that $U(\xi_0, \eta_0)$ satisfies the differential equation

$$\left(U_{\xi_0}\right)_{\eta_0} + \frac{\beta}{\eta_0 - \xi_0}\left(U_{\xi_0} - U_{\eta_0}\right) - \frac{n(n+1)}{(2-\xi_0-\eta_0)^2}U = F(\xi_0, \eta_0) \tag{2.68}$$

in a classical sense. But, since F, U, $U_{\xi_0} - U_{\eta_0} \in C(D)$, it follows that $(U_{\xi_0})_{\eta_0} \in C(D)$.

Step 2. We will prove that identity (2.20) holds for all $V(\xi,\eta) \in V^{(2)}$, which, in addition, are equivalent to zero in a neighborhood of $\{\eta = \xi\}$.

Define

$$I_V := \int_D (\eta-\xi)^{2\beta} \left\{ U_\xi V_\eta + U_\eta V_\xi + \frac{2n(n+1)}{(2-\xi-\eta)^2} UV + 2FV \right\} d\xi\, d\eta. \tag{2.69}$$

Using that the derivatives U_ξ, U_η and $(U_\xi)_\eta$ are continuous in D, we integrate by parts in I_V in the following way:

$$\int_D (\eta-\xi)^{2\beta} U_\xi V_\eta \, d\xi\, d\eta = -\int_D (\eta-\xi)^{2\beta} \left[(U_\xi)_\eta + \frac{2\beta}{\eta-\xi} U_\xi \right] V\, d\xi\, d\eta \tag{2.70}$$

and

$$\int_D (\eta-\xi)^{2\beta} U_\eta V_\xi \, d\xi\, d\eta = -\int_D (\eta-\xi)^{2\beta} \left[(V_\xi)_\eta + \frac{2\beta}{\eta-\xi} V_\xi \right] U\, d\xi\, d\eta.$$

There are not integrals on the boundary of D, because $V(\xi,\eta) \equiv 0$ in a neighborhood of $\{\eta = \xi\}$ and $V(\xi,1) = 0,\ V_\xi(\xi,1) = 0$.

Further, since $V \in C^2(\bar{D})$, we have $(V_\xi)_\eta = (V_\eta)_\xi$. Then

$$\begin{aligned} \int_D (\eta-\xi)^{2\beta} U_\eta V_\xi \, d\xi\, d\eta &= -\int_D (\eta-\xi)^{2\beta} \left[(V_\eta)_\xi + \frac{2\beta}{\eta-\xi} V_\xi \right] U\, d\xi\, d\eta \\ &= \int_D (\eta-\xi)^{2\beta} \left[U_\xi V_\eta - \frac{2\beta}{\eta-\xi} (V_\xi + V_\eta) U \right] d\xi\, d\eta \\ &= -\int_D (\eta-\xi)^{2\beta} \left[(U_\xi)_\eta - \frac{2\beta}{\eta-\xi} U_\eta \right] V\, d\xi\, d\eta. \end{aligned} \tag{2.71}$$

Again there are not integrals on the boundary of D in view of the boundary condition $U(0,\eta)=0$ and the properties of the function $V(\xi,\eta)$.

Now, putting (2.70) and (2.71) into (2.69) we obtain

$$I_V = -2\int_D (\eta-\xi)^{2\beta}\left\{(U_\xi)_\eta + \frac{\beta}{\eta-\xi}(U_\xi - U_\eta) - \frac{n(n+1)}{(2-\xi-\eta)^2}U - F\right\} V\, d\xi\, d\eta = 0.$$

Step 3. Finally, we will prove that identity (2.20) holds for all $V(\xi,\eta)\in V^{(2)}$.

Let $\chi(s)$ be a function having the properties $\chi(s)\in C^\infty(\mathbf{R}^1)$, $\chi(s)=1$ for $s\geq 2$, $\chi(s)=0$ for $s\leq 1$ and let $V(\xi,\eta)$ be an arbitrary function belonging to $V^{(2)}$.

If $k\in\mathbb{N}$, then the functions

$$V_k(\xi,\eta) := V(\xi,\eta)\,\chi\left(k\left[\eta-\xi\right]\right)$$

belong to $V^{(2)}$ and $V_k(\xi,\eta)\equiv 0$ in a neighborhood of $\{\eta=\xi\}$. Therefore the identity (2.20) holds with $V(\xi,\eta)$ replaced by $V_k(\xi,\eta)$. More precisely, we may write:

$$\int_D (\eta-\xi)^{2\beta}\left\{U_\xi V_\eta + U_\eta V_\xi + \frac{2n(n+1)}{(2-\xi-\eta)^2}UV + 2FV\right\}\chi\left(k\left[\eta-\xi\right]\right)\, d\xi\, d\eta$$
$$+\int_D k(\eta-\xi)^{2\beta}\left\{U_\xi - U_\eta\right\}\chi'\left(k\left[\eta-\xi\right]\right)V\, d\xi d\eta =: I_{1,k} + I_{2,k} = 0. \quad (2.72)$$

Obviously $I_{1,k}\to I_V$ as $k\to\infty$.

Further, $\operatorname{supp} \chi'(k[\eta-\xi])$ is contained in $\{1 \le k[\eta-\xi] \le 2\}$, so on $\operatorname{supp} \chi'(k[\eta-\xi])$ the functions

$$W_k(\xi,\eta) := k(\eta-\xi)^{2\beta} \{U_\xi - U_\eta\} \chi'\left(k\left[\eta-\xi\right]\right) V(\xi,\eta)$$

satisfy the estimate

$$|W_k(\xi,\eta)| \le \text{const}\,(\eta-\xi)^{\beta-1},$$

where we take into account that the estimates (2.61) hold and that $V \equiv 0$ in a neighborhood of $(1,1)$ by definition. Then the sequence $W_k(\xi,\eta)$ converges pointwise almost everywhere to zero and it is dominated by a Lebesgue integrable function in D for $0<\beta<1$. Consequently, according to the Lebesgue dominated convergence theorem, $I_{2,k} \to 0$ as $k \to \infty$.

Now, letting $k \to \infty$ in (2.72) we obtain that the identity (2.20) holds for all $V \in V^{(2)}$. Consequently, the function $U(\xi,\eta)$ is a generalized solution of Problem P_{m2}. □

From the existence and uniqueness of a generalized solution of Problem P_{m2} it follows the existence and uniqueness of a generalized solution of Problem P_m, stated in Theorems 2.2.1-2.2.2.

Proof of Theorem 2.2.1. Let $u_1(x,t)$ and $u_2(x,t)$ be two different generalized solutions of Problem P_m, which means that

$$u(x,t) := u_1(x,t) - u_2(x,t)$$

is a generalized solution of the homogeneous Problem P_m. Then we claim that $u(x,t) \equiv 0$ in Ω_m, i.e. all the coefficients $u_n^s(|x|,t)$ in the Fourier expansion

$$u(x,t) = \sum_{n=0}^{\infty} \sum_{s=1}^{2n+1} u_n^s(|x|,t) Y_n^s(x) \tag{2.73}$$

are equivalent to zero in Ω_m.

Indeed, the identity (2.10) with $f \equiv 0$ holds for all test functions $v \in V_m$ of the form

$$v(x,t) = w(|x|,t) Y_n^s(x). \tag{2.74}$$

Now substitute (2.73) and (2.74) into (2.10) with $f \equiv 0$. Passing to the spherical coordinates, using the orthogonality of the spherical functions and the differential equation (2.6) that they satisfy, we find that the functions

$$U_n^s(\xi,\eta) := r(\xi,\eta)\, u_n^s\big(r(\xi,\eta), t(\xi,\eta)\big), \quad n = 0,1,2\dots, \quad s = 1,\dots,2n+1$$

with

$$\xi = 1 - r - \frac{2}{2-m} t^{\frac{2-m}{2}}, \quad \eta = 1 - r + \frac{2}{2-m} t^{\frac{2-m}{2}}$$

should be generalized solutions of the homogeneous Problem P_{m2}. According to Theorem 2.6.1, the homogeneous Problem P_{m2} has only the trivial solution, which confirms our assertion. □

Proof of Theorem 2.2.2. Let

$$f(x,t) = \sum_{n=0}^{l} \sum_{s=1}^{2n+1} f_n^s(|x|,t) Y_n^s(x) \in C^1(\bar{\Omega}_m).$$

Define

$$u(x,t) := \sum_{n=0}^{l} \sum_{s=1}^{2n+1} u_n^s(|x|,t) Y_n^s(x),$$

where $u_n^s(r,t)$ are such that the functions

$$U_n^s(\xi,\eta) := r(\xi,\eta)\, u_n^s\big(r(\xi,\eta), t(\xi,\eta)\big), \qquad n = 0,\dots,l, \quad s = 1,\dots,2n+1$$

are the generalized solutions of Problem P_{m2} with right-hand side functions

$$F_n^s(\xi,\eta) := \frac{1}{4} r(\xi,\eta)\, f_n^s\big(r(\xi,\eta), t(\xi,\eta)\big).$$

Then we check that $u(x,t)$ satisfies the properties (1)-(3) of Definition 2.2.1 and satisfy the identity (2.10) for the test functions $v \in V_m$ of the form (2.74). But these functions are dense in V_m and therefore $u(x,t)$ satisfies the property (4) of Definition 2.2.1 at all. Hence $u(x,t)$ is a generalized solution of Problem P_m with a right-hand side function $f(x,t)$. $\square$

2.7. Decomposition of the function $\boldsymbol{\Psi}^-(\xi,\eta;\boldsymbol{\xi_0},\mathbf{1})$

Next, according to the estimates (2.61), the generalized solution $U(\xi,\eta)$ of Problem P_{m2} is allowed to have a singularity of order no greater than n at the point $(1,1)$. But it is still not clear if such a singularity really exists and how it depends on the right-hand side of the equation. From here we begin to study the asymptotic behavior of the function $U(\xi,\eta)$ near the singular point $(1,1)$.

Firstly, we find an asymptotic expansion of the restriction $U(\xi,1)$ on the segment $0 \le \xi < 1$. To do this, we derive a special decomposition of the Riemann-Hadamard function on the line $\{\eta_0 = 1\}$. We start with some auxiliary lemmas.

Lemma 2.7.1. *Let $a > 0$ and $k \in \mathbb{N} \cup \{0\}$. Then*

$$ {}_2F_1(a, -N, 2a; 2) = \begin{cases} 0, & N = 2k+1, \\ \dfrac{(1/2)_k}{(1/2+a)_k}, & N = 2k. \end{cases} \tag{2.75}$$

Proof. According to the integral representation (A.5) we have

$$ {}_2F_1(a, -N, 2a; 2) = \frac{\Gamma(2a)}{\Gamma(a)\Gamma(a)} \int_0^1 t^{a-1}(1-t)^{a-1}(1-2t)^N \, dt. \tag{2.76}$$

Then for $k \in \mathbb{N} \cup \{0\}$ we have

$$ {}_2F_1(a, -2k-1, 2a; 2) = 0, $$

because the function $h(t) := t^{a-1}(1-t)^{a-1}(1-2t)^{2k+1}$ is antisymmetric in respect to the point $t = 1/2$, i.e. $h(1/2 - t) = -h(1/2 + t)$.

In the case when N is an even number we proceed by the induction method. For $k = 0$ (resp. $N = 0$) (2.75) holds obviously.

For $N = 2, 4, 6, \ldots$ from (2.76) we get

$$\begin{aligned}\frac{\Gamma(a)\Gamma(a)}{\Gamma(2a)}\, {}_2F_1(a, -N, 2a; 2) &= \frac{1}{a}\int_0^1 (1-2t)^{N-1}\, d(t-t^2)^a \\ &= \frac{2(N-1)}{a}\int_0^1 t^a(1-t)^a(1-2t)^{N-2}\, dt \\ &= \frac{2(N-1)\Gamma(a+1)\Gamma(a+1)}{a\Gamma(2a+2)}\, {}_2F_1(a+1, 2-N, 2a+2; 2),\end{aligned}$$

or more simply

$$ {}_2F_1(a, -N, 2a; 2) = \frac{(N-1)}{(2a+1)}\, {}_2F_1(a+1, 2-N, 2(a+1); 2). \tag{2.77}$$

Our induction hypothesis is that for some $k \in \mathbb{N} \cup \{0\}$ the equality

$$ {}_2F_1(a, -2k, 2a; 2) = \frac{(1/2)_k}{(a+1/2)_k}$$

holds. But then for $k+1$ this equality will also hold, because according to (2.77) we have

$$ {}_2F_1(a, -2k-2, 2a; 2) = \frac{(2k+1)(1/2)_k}{(2a+1)(a+3/2)_k} = \frac{(1/2)_{k+1}}{(a+1/2)_{k+1}}.$$

The proof is complete. □

Lemma 2.7.2. *Let $n, p \in \mathbb{N} \cup \{0\}$, $p \leq n$, $0 < \beta < 1$ and*

$$Q_{n,p}(z) := \sum_{j=0}^{n-p} a_j\, b_j\, z^j\, {}_2F_1\left(n+p+j+1, p-n+j, p+j+1; \frac{1-z}{2}\right), \tag{2.78}$$

where

$$a_j := \frac{(\beta)_j}{(2\beta)_j\, j!}, \quad b_j := \frac{(n+p+1)_j (p-n)_j}{(p+1)_j}.$$

Then

$$Q_{n,p}(z) = \begin{cases} 0, & n-p \text{ odd}, \\ c_{n,p}\, {}_2F_1\left(\dfrac{n+p+1}{2}, \dfrac{p-n}{2}, \dfrac{1}{2}+\beta; z^2\right), & n-p \text{ even}, \end{cases} \tag{2.79}$$

where

$$c_{n,p} := \frac{\Gamma(1/2)\,\Gamma(p+1)}{\Gamma\left(\frac{n+p+2}{2}\right)\Gamma\left(\frac{p-n+1}{2}\right)}. \tag{2.80}$$

Proof. First, we expand the function ${}_2F_1$ from (2.78) in Taylor series in powers of z:

$$\begin{aligned} &{}_2F_1\left(n+p+j+1, p-n+j, p+j+1; \frac{1-z}{2}\right) \\ &= \sum_{s=0}^{n-p-j} \frac{(n+p+j+1)_s (p-n+j)_s}{(p+j+1)_s\, s!} \left(\frac{-z}{2}\right)^s \\ &\times {}_2F_1\left(n+p+j+s+1, p-n+s+j, p+j+s+1; \frac{1}{2}\right), \end{aligned}$$

where we use (A.10) to compute the corresponding derivatives in the series. By (A.17)-(A.19) we have that

$$\begin{aligned} &{}_2F_1\left(n+p+N+1, p-n+N, p+N+1; \frac{1}{2}\right) \\ &= A_N := \begin{cases} \dfrac{\Gamma(1/2)\,\Gamma(p+N+1)}{\Gamma(\frac{n+p+N+2}{2})\Gamma(\frac{p-n+N+1}{2})}, & p-n+N \neq -1, -3, \ldots, \\ 0, & p-n+N = -1, -3, \ldots. \end{cases} \end{aligned} \tag{2.81}$$

Then $Q_{n,p}(z)$, using also (A.1), becomes

$$Q_{n,p}(z) = \sum_{j=0}^{n-p} \sum_{s=0}^{n-p-j} a_j \, b_{j+s} \, A_{j+s} \frac{(-1)^s}{2^s \, s!} z^{j+s}.$$

Now set $N = j + s$:

$$Q_{n,p}(z) = \sum_{N=0}^{n-p} b_N \, A_N \, z^N \sum_{j=0}^{N} a_j \frac{(-1)^{N-j}}{2^{N-j} \, (N-j)!}.$$

Since $(N-j)! = (-1)^j \, N!/(-N)_j$, for $Q_{n,p}(z)$ we obtain:

$$Q_{n,p}(z) = \sum_{N=0}^{n-p} {}_2F_1(\beta, -N, 2\beta; 2) \, b_N \, A_N \frac{(-z)^N}{2^N \, N!}. \tag{2.82}$$

There are two different cases:

A. Let $n-p$ be an odd number. In this case (2.82) becomes

$$Q_{n,p}(z) \equiv 0,$$

because:

a) for even indexes N according to (2.81) we have $A_N = 0$;

b) for odd indexes N Lemma 2.7.1 with $a = \beta$ gives ${}_2F_1(\beta, -N, 2\beta; 2) = 0$.

B. Let $n-p$ be an even number. In this case, according to (2.81), we have nonzero coefficients A_N in (2.82) only for even indexes N. Then we set $N = 2k$ and by Lemma 2.7.1 we have

$$ {}_2F_1(\beta, -2k, 2\beta; 2) = \frac{(1/2)_k}{(1/2+\beta)_k}. \tag{2.83} $$

Now with (A.3) we calculate:

$$ (n+p+1)_{2k} = 2^{2k} \left(\frac{n+p+1}{2}\right)_k \left(\frac{n+p+2}{2}\right)_k, \tag{2.84} $$

$$ (p-n)_{2k} = 2^{2k} \left(\frac{p-n}{2}\right)_k \left(\frac{p-n+1}{2}\right)_k, \tag{2.85} $$

$$ (2k)! = 2^{2k} \left(\frac{1}{2}\right)_k k!. \tag{2.86} $$

Applying the equalities (2.83)-(2.86) into (2.82) with $N = 2k$ and simplifying the derived expression we obtain:

$$ \begin{aligned} Q_{n,p}(z) &= \sum_{k=0}^{(n-p)/2} \frac{(1/2)_k}{(1/2+\beta)_k} \frac{b_{2k}\, A_{2k}}{2^{2k}\,(2k)!} z^{2k} \\ &= \Gamma(1/2)\Gamma(p+1) \sum_{k=0}^{(n-p)/2} \frac{(1/2)_k}{(1/2+\beta)_k} \frac{(n+p+1)_{2k}(p-n)_{2k}}{\Gamma(\frac{n+p+2k+2}{2})\Gamma(\frac{p-n+2k+1}{2})} \frac{z^{2k}}{2^{2k}\,(2k)!} \\ &= c_{n,p}\, {}_2F_1\left(\frac{n+p+1}{2}, \frac{p-n}{2}, \frac{1}{2}+\beta; z^2\right). \end{aligned} $$

The proof is complete. □

Lemma 2.7.3. *Define the function*

$$ \psi_1(\xi,\eta;\xi_0) := \sum_{i=0}^{n}\sum_{j=0}^{i} d_i\, b_{i,j} P_{i,j}(\xi,\xi_0) \left(\frac{(1-\xi)(1-\eta)}{1-\xi_0}\right)^{i-j} \frac{(-1)^i\,(\eta-\xi)^j}{(2-\xi-\eta)^i}, \tag{2.87} $$

where

$$P_{i,j}(\xi,\xi_0) := \sum_{q=i-j+1}^{\infty} \frac{(j-i+\beta)_q}{q!}\left(\frac{1-\xi_0}{1-\xi}\right)^q, \tag{2.88}$$

$$b_{i,j} := \frac{(\beta-i)_j(\beta)_j}{(2\beta)_j\, j!} \tag{2.89}$$

and d_i are the constants from (2.33)*. Then the following estimate holds*:

$$|\psi_1(\xi,\eta;\xi_0)| \le k_1 \frac{1-\xi_0}{(\xi_0-\xi)^{\beta}(1-\xi)^{1-\beta}} \tag{2.90}$$

for $(\xi,\eta)\in D\cap\{\eta<\xi_0\}$, where $k_1 = \mathrm{const} > 0$.

Proof. In $P_{i,j}(\xi,\xi_0)$ we set the new index $N = q+j-i-1$ instead of q to obtain:

$$\begin{aligned} P_{i,j}(\xi,\xi_0) &= (-1)^{i-j+1}(-\beta)_{i-j+1}\sum_{N=0}^{\infty}\frac{(1+\beta)_N}{(1+N)_{i-j+1}\, N!}\left(\frac{1-\xi_0}{1-\xi}\right)^{N+i-j+1} \\ &= (-1)^{i-j}(1-\beta)_{i-j}\sum_{N=0}^{\infty}\frac{(\beta+N)(\beta)_N}{(1+N)_{i-j+1}\, N!}\left(\frac{1-\xi_0}{1-\xi}\right)^{N+i-j+1}, \end{aligned}$$

where we used (A.1) and (A.4). Since

$$\frac{\beta+N}{(1+N)_{i-j+1}} < 1, \quad 0<\beta<1,\ j=0,1,\ldots,i,$$

for the function $P_{i,j}(\xi,\xi_0)$ it follows the estimate

$$\begin{aligned} |P_{i,j}(\xi,\xi_0)| &\le (1-\beta)_{i-j}\left(\frac{1-\xi_0}{1-\xi}\right)^{i-j+1}\sum_{N=0}^{\infty}\frac{(\beta)_N}{N!}\left(\frac{1-\xi_0}{1-\xi}\right)^N \\ &= (1-\beta)_{i-j}\frac{(1-\xi_0)^{i-j+1}}{(1-\xi)^{i-j+1-\beta}(\xi_0-\xi)^{\beta}} \end{aligned} \tag{2.91}$$

for $(\xi, \eta) \in D \cap \{\eta < \xi_0\}$. For the last equality we used (A.11).

Applying this estimate into (2.87) gives the final result (2.90). $\square$

Lemma 2.7.4. *Define the function*

$$\psi_2(\xi, \eta; \xi_0) := \left(\frac{1-\xi}{\xi_0 - \xi} \right)^{\beta} \sum_{i=0}^{n} d_i \, Y^i(\xi, \eta; \xi_0, 1) Q_i(\xi, \eta; \xi_0), \tag{2.92}$$

with

$$Q_i(\xi, \eta; \xi_0) := \sum_{j=i+1}^{\infty} b_{i,j} \, X^{-j}(\xi, \eta; \xi_0, 1), \tag{2.93}$$

where $X(\xi, \eta; \xi_0, \eta_0)$ and $Y(\xi, \eta; \xi_0, \eta_0)$ are given by (2.23) *and* (2.24) *respectively, $b_{i,j}$ are the constants* (2.89) *and d_i are the constants from* (2.33). *Then the following estimate holds*:

$$|\psi_2(\xi, \eta; \xi_0)| \le k_2 \frac{1-\xi_0}{(1-\eta)^{1-\beta}(\xi_0 - \eta)^{\beta}} \tag{2.94}$$

for $(\xi, \eta) \in D \cap \{\eta < \xi_0\}$, where $k_2 = \mathrm{const} > 0$.

Proof. Setting $j = N + i + 1$, with use of (A.1) and (A.4) we compute

$$\begin{aligned} Q_i(\xi, \eta; \xi_0) =& (-1)^i (1-\beta)_i \\ & \times \sum_{N=0}^{\infty} \frac{(\beta+N)(\beta)_N(\beta)_N(\beta+N)_{i+1}}{(2\beta)_N(2\beta+N)_{i+1}(1+N)_{i+1}\, N!} X^{-N-i-1}(\xi, \eta; \xi_0, 1). \end{aligned} \tag{2.95}$$

Since

$$\frac{(\beta+N)(\beta+N)_{i+1}}{(2\beta+N)_{i+1}(1+N)_{i+1}} < 1, \quad i = 0, \ldots, n, \quad 0 < \beta < 1,$$

from here it follows the estimate

$$|Q_i| \le (1-\beta)_i X^{-i-1}(\xi,\eta;\xi_0,1)\, {}_2F_1\left(\beta,\beta,2\beta;\frac{1}{X(\xi,\eta;\xi_0,1)}\right) \tag{2.96}$$

for $(\xi,\eta) \in D \cap \{\eta < \xi_0\}$. By (A.6) with $\alpha = \beta$ we have

$$\left| {}_2F_1\left(\beta,\beta,2\beta;\frac{1}{X(\xi,\eta;\xi_0,1)}\right)\right| \le c(\beta)\frac{(\xi_0-\xi)^\beta(1-\eta)^\beta}{(1-\xi)^\beta(\xi_0-\eta)^\beta}. \tag{2.97}$$

Applying (2.96)-(2.97) into (2.92) and taking into account that

$$\left|\frac{Y}{X}\right| \le 1, \quad \frac{1}{X} \le \frac{1-\xi_0}{1-\eta}, \quad 0 < \xi < \eta < \xi_0, \tag{2.98}$$

we come to the estimate (2.94). $\square$

Now we are ready to prove the special decomposition of the function $\Psi^-(\xi,\eta;\xi_0,1)$.

Theorem 2.7.1. *The trace of the function $\Psi^-(\xi,\eta;\xi_0,1)$ on the line $\{\eta_0 = 1\}$ can be decomposed in the following way*:

$$\Psi^-(\xi,\eta;\xi_0,1) = \Psi_1^-(\xi,\eta;\xi_0) + \Psi_2^-(\xi,\eta;\xi_0) \tag{2.99}$$

with

$$\Psi_1^-(\xi,\eta;\xi_0) := (\eta-\xi)^{2\beta}\sum_{k=0}^{[n/2]}\lambda_k^n(1-\xi_0)^{2k-n}E_k^{n,\beta}(\xi,\eta) \tag{2.100}$$

and

$$\Psi_2^-(\xi,\eta;\xi_0) := \frac{\gamma\,(\eta-\xi)^{2\beta}}{(1-\xi)^{\beta}(1-\eta)^{\beta}}\big\{\psi_1(\xi,\eta;\xi_0)+\psi_2(\xi,\eta;\xi_0)\big\}, \qquad (2.101)$$

where $\psi_1(\xi,\eta;\xi_0)$ *and* $\psi_2(\xi,\eta;\xi_0)$ *are the functions* (2.87) *and* (2.92) *from Lemma 2.7.3 and Lemma 2.7.4 respectively,* $E_k^{n,\beta}(\xi,\eta)$ *are the functions defined by* (2.18) *and* $\lambda_k^n = \text{const} \neq 0$. *The function* $\Psi_2^-(\xi,\eta;\xi_0)$ *satisfies for* $(\xi,\eta) \in D \cap \{\eta < \xi_0\}$ *the following estimate*

$$|\Psi_2^-(\xi,\eta;\xi_0)| \le k\frac{1-\xi_0}{(\xi_0-\eta)^{\beta}(1-\eta)}, \qquad k = \text{const} > 0. \qquad (2.102)$$

Proof. For $\Psi^-(\xi,\eta;\xi_0,1)$ from (2.27)-(2.34) we obtain

$$\Psi^-|_{\eta_0=1} = \frac{\gamma\,(\eta-\xi)^{2\beta}}{(\xi_0-\xi)^{\beta}(1-\eta)^{\beta}}\sum_{i=0}^{n}\sum_{j=0}^{\infty} d_i\, b_{i,j}\left(\frac{Y^i}{X^j}\right)(\xi,\eta;\xi_0,1),$$

where $b_{i,j}$ are the constants (2.89).

For $0<\xi<\xi_0<1$, using (A.11), we have

$$\left(\frac{\xi_0-\xi}{1-\xi}\right)^{i-j-\beta} = \sum_{q=0}^{\infty}\frac{(j-i+\beta)_q}{q!}\left(\frac{1-\xi_0}{1-\xi}\right)^{q}$$

and according to this we find that

$$\begin{aligned}\Psi^-(\xi,\eta;\xi_0,1) &= \frac{\gamma\,(\eta-\xi)^{2\beta}}{(1-\xi)^{\beta}(1-\eta)^{\beta}}\big\{\psi(\xi,\eta;\xi_0)+\psi_1(\xi,\eta;\xi_0)+\psi_2(\xi,\eta;\xi_0)\big\}\\ &= \frac{\gamma\,(\eta-\xi)^{2\beta}}{(1-\xi)^{\beta}(1-\eta)^{\beta}}\psi(\xi,\eta;\xi_0)+\Psi_2^-(\xi,\eta;\xi_0), \quad (2.103)\end{aligned}$$

where

$$\psi(\xi,\eta;\xi_0) :=$$
$$\sum_{i=0}^{n}\sum_{j=0}^{i}\sum_{q=0}^{i-j} d_i\, b_{i,j} \frac{(j-i+\beta)_q}{q!} \frac{(-1)^i(1-\xi)^{i-j-q}(1-\eta)^{i-j}(\eta-\xi)^j}{(2-\xi-\eta)^i(1-\xi_0)^{i-j-q}}. \quad (2.104)$$

Next, we aim to extract in $\psi(\xi,\eta;\xi_0)$ the negative powers of $1-\xi_0$. To do this, we introduce the new index $p = i-j-q$ instead of i:

$$\psi(\xi,\eta;\xi_0) = \sum_{p=0}^{n}\left(\frac{(1-\xi)(1-\eta)}{(2-\xi-\eta)(1-\xi_0)}\right)^p$$
$$\times \sum_{j=0}^{n-p}\sum_{q=0}^{n-p-j} d_{p+j+q}\, b_{p+j+q,j} \frac{(\beta-p-q)_q}{(-1)^{p+j+q}q!}\left(\frac{\eta-\xi}{2-\xi-\eta}\right)^j\left(\frac{1-\eta}{2-\xi-\eta}\right)^q.$$

Using (A.1) and (A.4) we simplify

$$\frac{(\beta-p-j-q)_j(\beta-p-q)_q}{(1-\beta)_{p+j+q}} = \frac{(-1)^{j+q}}{(1-\beta)_p}$$

and we derive:

$$\psi(\xi,\eta;\xi_0) = \sum_{p=0}^{n}(-1)^p\, d_p\left(\frac{(1-\xi)(1-\eta)}{(2-\xi-\eta)(1-\xi_0)}\right)^p$$
$$\times \sum_{j=0}^{n-p}\frac{(p-n)_j(n+p+1)_j}{(p+1)_j}\frac{(\beta)_j}{(2\beta)_j\, j!}\left(\frac{\eta-\xi}{2-\xi-\eta}\right)^j$$
$$\times \sum_{q=0}^{n-p-j}\frac{(p-n+j)_q(n+p+j+1)_q}{(p+j+1)_q\, q!}\left(\frac{1-\eta}{2-\xi-\eta}\right)^q,$$

which actually gives

$$\psi(\xi,\eta;\xi_0) = \sum_{p=0}^{n} (-1)^p \, d_p \left(\frac{(1-\xi)(1-\eta)}{(2-\xi-\eta)(1-\xi_0)} \right)^p Q_{n,p}\left(\frac{\eta-\xi}{2-\xi-\eta} \right),$$

where $Q_{n,p}(z)$ is the function (2.78) from Lemma 2.7.2.

Now, according to (2.79) we have non-zero terms in the sum only for indexes p of the same parity as n. For this reason we introduce the new index $k = (n-p)/2$ and by Lemma 2.7.2 we obtain:

$$\begin{aligned}\psi(\xi,\eta;\xi_0) = &\sum_{k=0}^{[n/2]} (-1)^n \, d_{n-2k} \left(\frac{(1-\xi)(1-\eta)}{(2-\xi-\eta)(1-\xi_0)} \right)^{n-2k} \\ &\times c_{n,n-2k} \, {}_2F_1\left(n-k+\frac{1}{2}, -k, \frac{1}{2}+\beta; \left(\frac{\eta-\xi}{2-\xi-\eta} \right)^2 \right) \\ &= (1-\xi)^\beta (1-\eta)^\beta \sum_{k=0}^{[n/2]} (-1)^n \, d_{n-2k} c_{n,n-2k} \, E_k^{n,\beta}(\xi,\eta).\end{aligned}$$

Putting this into (2.103), we conclude that (2.99)-(2.101) hold with

$$\lambda_k^n := \gamma \, (-1)^n c_{n,n-2k} \, d_{n-2k} \neq 0. \tag{2.105}$$

Finally, the estimate (2.102) follows directly from the estimates (2.90) and (2.94). □

2.8. Asymptotic expansion of the restriction $\mathrm{U}(\xi, 1)$ at the point $\xi = 1$

Now, introduce the scalar products

$$\mu_k^{n,\beta} := \int_D (\eta - \xi)^{2\beta} E_k^{n,\beta}(\xi, \eta) F(\xi, \eta)\, d\xi d\eta. \tag{2.106}$$

Theorem 2.8.1. *Suppose that $F \in C^1(\bar{D})$. Then the restriction $U(\xi, 1)$ of the generalized solution of Problem P_{m2} has the following expansion on the segment $\{0 \le \xi < 1\}$:*

$$U(\xi, 1) = \sum_{k=0}^{[n/2]} \lambda_k^n \mu_k^{n,\beta} (1-\xi)^{2k-n} - \sum_{k=0}^{[n/2]} \lambda_k^n J_k^{n,\beta}(\xi)(1-\xi)^{2k-n} + g(\xi), \tag{2.107}$$

where

$$J_k^{n,\beta}(\xi) := \int_\xi^1 \int_0^{\eta_1} (\eta_1 - \xi_1)^{2\beta} E_k^{n,\beta}(\xi_1, \eta_1) F(\xi_1, \eta_1)\, d\xi_1 d\eta_1, \tag{2.108}$$

$\lambda_k^n = \text{const} \neq 0$ *(see* (2.105)*), $g(\xi) \in C^1([0, 1))$ and*

$$|g(\xi)| \le C \|F\|_{C(D)} \xi (1 - \xi)^{1-\beta}, \tag{2.109}$$

with a constant $C > 0$ independent of F.

Proof. According to Theorem 2.6.2 the condition $F \in C^1(\bar{D})$ assures that there exists an unique generalized solution $U(\xi, \eta)$ of Problem P_{m2} and according to Definition 2.3.1 we see that the restriction $U(\xi, 1)$ should belong to $C^1([0, 1))$.

Next, according to Theorem 2.6.2 the generalized solution at each point $(\xi_0, \eta_0) \in D$ has the representation (2.62), but we find that by continuity for $\eta_0 = 1,\ 0 < \xi_0 < 1$ this equality still holds, i.e.

$$U(\xi_0, 1) = \int_0^{\xi_0} \int_{\xi}^{1} F(\xi, \eta)\Psi(\xi, \eta; \xi_0, 1)\, d\eta d\xi.$$

Using the decomposition of $\Psi^-(\xi, \eta; \xi_0, 1)$ given in Theorem 2.7.1 we obtain

$$\begin{aligned}U(\xi_0, 1) = &\int_0^{\xi_0} \int_{\xi}^{\xi_0} F(\xi, \eta)\Psi_1^-(\xi, \eta; \xi_0)\, d\eta d\xi \\ &+ \int_0^{\xi_0} \int_{\xi}^{\xi_0} F(\xi, \eta)\Psi_2^-(\xi, \eta; \xi_0)\, d\eta d\xi + \int_0^{\xi_0} \int_{\xi_0}^{1} F(\xi, \eta)\Psi^+(\xi, \eta; \xi_0, 1)\, d\eta d\xi \\ &=: J_1(\xi_0) + J_2(\xi_0) + J_3(\xi_0).\end{aligned}$$

According to (2.100) we have

$$\begin{aligned}J_1(\xi_0) = &\sum_{k=0}^{[n/2]} \lambda_k^n (1 - \xi_0)^{2k-n} \int_0^{\xi_0} \int_{\xi}^{\xi_0} (\eta - \xi)^{2\beta} F(\xi, \eta) E_k^{n,\beta}(\xi, \eta)\, d\eta d\xi \\ &= \sum_{k=0}^{[n/2]} \lambda_k^n \big(\mu_k^{n,\beta} - J_k^{n,\beta}(\xi_0)\big)(1 - \xi_0)^{2k-n}.\end{aligned}$$

From here it is easy to check that $J_1(\xi) \in C^1([0, 1))$.

For $J_2(\xi_0)$, using the estimate (2.102) from Theorem 2.7.1, we obtain

$$|J_2(\xi_0)| \le k\|F\|_{C(D)}(1 - \xi_0) \int_0^{\xi_0} \int_{\xi}^{\xi_0} (\xi_0 - \eta)^{-\beta}(1 - \eta)^{-1}\, d\eta d\xi. \quad (2.110)$$

Using the calculations (2.48)-(2.49) with $\eta_0 = 1$, we have

$$\left| \int_{\xi}^{\xi_0} (\xi_0 - \eta)^{-\beta} (1 - \eta)^{-1} \, d\eta \right| \leq \text{const} \, (1 - \xi_0)^{-\beta}. \tag{2.111}$$

Consequently, applying this into (2.110) we obtain

$$|J_2(\xi_0)| \leq \text{const} \, \|F\|_{C(D)} \xi_0 (1 - \xi_0)^{1-\beta}. \tag{2.112}$$

According to the estimates (2.35) and (2.40), we have an estimate

$$|\Psi^+(\xi, \eta; \xi_0, 1)| \leq \text{const} \, (\eta - \xi_0)^{-\beta}.$$

From here for $J_3(\xi_0)$ we have

$$|J_3(\xi_0)| \leq \text{const} \, \|F\|_{C(D)} \xi_0 (1 - \xi_0)^{1-\beta}. \tag{2.113}$$

Finally, defining

$$\begin{aligned} & g(\xi_0) := J_2(\xi_0) + J_3(\xi_0) \\ & = \int_0^{\xi_0} \int_{\xi}^{\xi_0} F(\xi, \eta) \Psi_2^-(\xi, \eta; \xi_0) \, d\eta d\xi + \int_0^{\xi_0} \int_{\xi_0}^{1} F(\xi, \eta) \Psi^+(\xi, \eta; \xi_0, 1) \, d\eta d\xi, \end{aligned} \tag{2.114}$$

we see that the expansion (2.107) holds, where (2.109) follows from the estimates (2.112), (2.113) and, obviously, $g(\xi) \in C^1([0, 1))$, because

$$g(\xi) = U(\xi, 1) - J_1(\xi).$$

The proof is complete. $\square$

Remark 2.8.1. *For $k = 0, \dots, [n/2]$ the functions $J_k^{n,\beta}(\xi)(1-\xi)^{2k-n}$ are bounded on the segment $0 \le \xi < 1$ and satisfy the estimate*

$$|J_k^{n,\beta}(\xi)(1-\xi)^{2k-n}| \le C\|F\|_{C(D)}(1-\xi)^{1-\beta} \tag{2.115}$$

with $C = \text{const} > 0$ independent of F. This means that the coefficients $\mu_k^{n,\beta}$ in the expansion (2.107) control entirely the singular part of the function $U(\xi, 1)$.

Indeed, the functions $E_k^{n,\beta}(\xi,\eta)$, given by (2.18), can be estimated in D in the following way:

$$|E_k^{n,\beta}(\xi,\eta)| \le \text{const}\,(1-\xi)^{-\beta}(1-\eta)^{n-2k-\beta}, \tag{2.116}$$

because the hypergeometric function in (2.18) is bounded. Then we have

$$\begin{aligned} |J_k^{n,\beta}(\xi_0)| &\le C_1\|F\|_{C(D)} \int_{\xi_0}^1 \int_0^\eta (\eta-\xi)^{2\beta}(1-\xi)^{-\beta}(1-\eta)^{n-2k-\beta}\, d\xi d\eta \\ &\le C\|F\|_{C(D)}(1-\xi_0)^{n-2k+1-\beta}, \quad C_1 = \text{const} > 0. \end{aligned}$$

From here it follows the estimate (2.115).

2.9. The derivative $U_\xi(\xi, 1)$

For our further considerations in this section we study the derivative $U_\xi(\xi, 1)$.

Lemma 2.9.1. *Let $\psi_1(\xi,\eta;\xi_0)$ be the function* (2.87) *from Lemma 2.7.3. Then the following estimate holds for $(\xi,\eta)\in D\cap\{\eta<\xi_0\}$:*

$$\left|\frac{\partial\psi_1}{\partial\xi_0}(\xi,\eta;\xi_0)\right|\le C\frac{(1-\xi)^\beta}{(\xi_0-\xi)^{1+\beta}},\quad C=\mathrm{const}>0. \tag{2.117}$$

Proof. For the function $P_{i,j}(\xi,\xi_0)$ (see (2.88)), with use of (A.1) and (A.4), we calculate

$$\begin{aligned}\frac{\partial P_{i,j}}{\partial\xi_0}(\xi,\xi_0)&=\frac{-1}{1-\xi}\sum_{q=i-j+1}^{\infty}\frac{(j-i+\beta)_q}{(q-1)!}\left(\frac{1-\xi_0}{1-\xi}\right)^{q-1}\\&=\frac{(-1)^{i-j}(-\beta)_{i-j+1}}{1-\xi}\sum_{N=0}^{\infty}\frac{(1+\beta)_N}{(1+N)_{i-j}\,N!}\left(\frac{1-\xi_0}{1-\xi}\right)^{N+i-j}.\end{aligned}$$

Taking into account (A.11), we see that for $j=0,\dots,i$ it is fulfilled

$$\begin{aligned}0&<\sum_{N=0}^{\infty}\frac{(1+\beta)_N}{(1+N)_{i-j}\,N!}\left(\frac{1-\xi_0}{1-\xi}\right)^{N}\\&\le\sum_{N=0}^{\infty}\frac{(1+\beta)_N}{N!}\left(\frac{1-\xi_0}{1-\xi}\right)^{N}=\left(\frac{1-\xi}{\xi_0-\xi}\right)^{1+\beta},\end{aligned}$$

where $0<\beta<1,\ 0<\xi<\xi_0<1$. Then we have the estimate

$$\left|\frac{\partial P_{i,j}}{\partial\xi_0}(\xi,\xi_0)\right|\le C_1\left(\frac{1-\xi_0}{1-\xi}\right)^{i-j}\frac{(1-\xi)^\beta}{(\xi_0-\xi)^{1+\beta}},\quad C_1=\mathrm{const}>0 \tag{2.118}$$

for $0<\beta<1,\ 0<\xi<\xi_0<1$.

Next, we calculate:

$$\frac{\partial \psi_1}{\partial \xi_0}(\xi,\eta;\xi_0) =$$
$$\sum_{i=0}^{n}\sum_{j=0}^{i} d_i\, b_{i,j} P_{i,j}(\xi,\xi_0)\frac{i-j}{1-\xi_0}\left(\frac{(1-\xi)(1-\eta)}{1-\xi_0}\right)^{i-j}\frac{(-1)^i\,(\eta-\xi)^j}{(2-\xi-\eta)^i}$$
$$+\sum_{i=0}^{n}\sum_{j=0}^{i} d_i\, b_{i,j}\frac{\partial P_{i,j}}{\partial \xi_0}(\xi,\xi_0)\left(\frac{(1-\xi)(1-\eta)}{1-\xi_0}\right)^{i-j}\frac{(-1)^i\,(\eta-\xi)^j}{(2-\xi-\eta)^i}.$$

Applying here the estimates (2.91) and (2.118) we obtain the final result (2.117). □

Lemma 2.9.2. *Define the function*

$$\psi_3(\xi,\eta;\xi_0) := \psi_2(\xi,\eta;\xi_0) - \left(\frac{1-\xi}{\xi_0-\xi}\right)^{\beta} Q_0(\xi,\eta;\xi_0)$$
$$= \left(\frac{1-\xi}{\xi_0-\xi}\right)^{\beta}\sum_{i=1}^{n} d_i\, Y^i(\xi,\eta;\xi_0,1)Q_i(\xi,\eta;\xi_0), \quad (2.119)$$

where $\psi_2(\xi,\eta;\xi_0)$ *and* $Q_i(\xi,\eta;\xi_0)$ *are the functions* (2.92) *and* (2.93) *respectively from Lemma 2.7.4. Then the following estimates hold for* $(\xi,\eta)\in D\cap\{\eta<\xi_0\}$:

$$|\psi_3(\xi,\eta;\xi_0)| \le C\,\frac{(1-\xi_0)(1-\xi)^{\beta}}{(1-\eta)(\xi_0-\xi)^{\beta}} \qquad (2.120)$$

$$\left|\frac{\partial \psi_3}{\partial \xi_0}(\xi,\eta;\xi_0)\right| \le \frac{C\,(1-\xi)}{(1-\eta)^{1-\beta}(\xi_0-\xi)(\xi_0-\eta)^{\beta}} \qquad (2.121)$$

with $C = \text{const} > 0$.

Proof. In the proof of Lemma 2.7.4 we have obtained an estimate for the function $Q_i(\xi,\eta;\xi_0)$ with $i=0,\dots,n$ (see (2.96)), but if exclude $i=0$, we can improve this estimate. Actually, substituting in (2.95) the relation

$$(2\beta)_N(2\beta+N)_{i+1} = 2\beta(2\beta+1)_N(2\beta+1+N)_i,$$

we obtain

$$\begin{aligned} Q_i(\xi,\eta;\xi_0) = & \frac{(-1)^i(1-\beta)_i}{2\beta} \\ & \times \sum_{N=0}^{\infty} \frac{(\beta+N)(\beta)_N(\beta)_N(\beta+N)_{i+1}}{(2\beta+1)_N(2\beta+1+N)_i(1+N)_{i+1}\,N!} X^{-N-i-1}(\xi,\eta;\xi_0,1). \end{aligned} \tag{2.122}$$

Now we have

$$\frac{(\beta+N)(\beta+N)_{i+1}}{(2\beta+1+N)_i(1+N)_{i+1}} < 1, \quad i=1,\dots,n, \quad 0<\beta<1,$$

and respectively

$$\begin{aligned} |Q_i| \le \frac{(1-\beta)_i}{2\beta} X^{-i-1}(\xi,\eta;\xi_0,1)\, {}_2F_1\left(\beta,\beta,2\beta+1;\frac{1}{X(\xi,\eta;\xi_0,1)}\right) \\ \le C_1 X^{-i-1}(\xi,\eta;\xi_0,1), \quad C_1 = \text{const} > 0 \end{aligned} \tag{2.123}$$

for $(\xi,\eta) \in D \cap \{\eta < \xi_0\}$, because the hypergeometric function here is bounded according to (A.8). Applying (2.123) into (2.119) and recalling the estimates (2.98), we find that (2.120) holds.

Next, we calculate

$$\frac{\partial\psi_3}{\partial\xi_0}(\xi,\eta;\xi_0) = \psi_{3,1}(\xi,\eta;\xi_0) + \psi_{3,2}(\xi,\eta;\xi_0) + \psi_{3,3}(\xi,\eta;\xi_0)$$

with

$$\psi_{3,1}(\xi,\eta;\xi_0) := \frac{-\beta\,\psi_3(\xi,\eta;\xi_0)}{\xi_0-\xi},$$

$$\psi_{3,2}(\xi,\eta;\xi_0) := \left(\frac{1-\xi}{\xi_0-\xi}\right)^{\beta}\sum_{i=1}^{n} i\,d_i\,(Y^{i-1}Y_{\xi_0})(\xi,\eta;\xi_0,1)Q_i(\xi,\eta;\xi_0),$$

$$\psi_{3,3}(\xi,\eta;\xi_0) := \left(\frac{1-\xi}{\xi_0-\xi}\right)^{\beta}\sum_{i=1}^{n} d_i\,Y^{i}(\xi,\eta;\xi_0,1)\frac{\partial Q_i}{\partial\xi_0}(\xi,\eta;\xi_0). \qquad (2.124)$$

A. Estimation of the function $\psi_{3,1}(\xi,\eta;\xi_0)$. Using the estimate (2.120), we immediately obtain:

$$|\psi_{3,1}(\xi,\eta;\xi_0)| \le C_2\,\frac{(1-\xi_0)(1-\xi)^{\beta}}{(1-\eta)(\xi_0-\xi)^{1+\beta}} \le \frac{C_2\,(1-\xi)}{(1-\eta)^{1-\beta}(\xi_0-\xi)(\xi_0-\eta)^{\beta}}, \qquad (2.125)$$

where $C_2=\text{const}>0$, $(\xi,\eta)\in D\cap\{\eta<\xi_0\}$.

B. Estimation of the function $\psi_{3,2}(\xi,\eta;\xi_0)$. Using the estimates (2.98), (2.123) and

$$|Y_{\xi_0}(\xi,\eta;\xi_0,1)| = \frac{(1-\xi)(1-\eta)}{(2-\xi-\eta)(1-\xi_0)^2} \le \frac{1-\eta}{(1-\xi_0)^2}$$

in $D\cap\{\eta<\xi_0\}$, we obtain

$$|\psi_{3,2}(\xi,\eta;\xi_0)| \le C_3\,\frac{(1-\xi)^{\beta}}{(1-\eta)(\xi_0-\xi)^{\beta}} \le \frac{C_3\,(1-\xi)}{(1-\eta)^{1-\beta}(\xi_0-\xi)(\xi_0-\eta)^{\beta}}, \qquad (2.126)$$

where $C_3=\text{const}>0$, $(\xi,\eta)\in D\cap\{\eta<\xi_0\}$.

C. Estimation of the function $\psi_{3,3}(\xi,\eta;\xi_0)$. First, from (2.122) we calculate:

$$\begin{aligned}\frac{\partial Q_i}{\partial \xi_0}(\xi,\eta;\xi_0) &= \frac{(-1)^i(1-\beta)_i}{2\beta} \\ &\times \sum_{N=0}^{\infty} \frac{(\beta+N)(\beta)_N(\beta)_N(\beta+N)_{i+1}}{(2\beta+1)_N(2\beta+1+N)_i(1+N)_i\,N!}\,X^{-N-i}(1/X)_{\xi_0}(\xi,\eta;\xi_0,1) \\ &= \frac{1}{2}(-1)^i(1-\beta)_i \sum_{N=0}^{\infty} \frac{(\beta+1)_N(\beta)_N}{(2\beta+1)_N\,N!}\frac{(\beta+N+1)_i(\beta+N)}{(2\beta+1+N)_i(1+N)_i} \\ &\qquad \times X^{-N-i}(1/X)_{\xi_0}(\xi,\eta;\xi_0,1).\end{aligned}$$

From here it follows

$$\begin{aligned}\left|\frac{\partial Q_i}{\partial \xi_0}(\xi,\eta;\xi_0)\right| &\le \frac{1}{2}(1-\beta)_i\big|X^{-i}(1/X)_{\xi_0}(\xi,\eta;\xi_0,1)\big| \\ &\times {}_2F_1\left(\beta+1,\beta,2\beta+1;\frac{1}{X(\xi,\eta;\xi_0,1)}\right)\end{aligned} \tag{2.127}$$

for $0<\beta<1$, $i=1,\ldots,n$ and $(\xi,\eta)\in D\cap\{\eta<\xi_0\}$. By (A.6) with $\alpha=\beta$ we have

$$\left|{}_2F_1\left(\beta+1,\beta,2\beta+1;\frac{1}{X(\xi,\eta;\xi_0,1)}\right)\right| \le c(\beta)\frac{(\xi_0-\xi)^\beta(1-\eta)^\beta}{(1-\xi)^\beta(\xi_0-\eta)^\beta}. \tag{2.128}$$

Now, applying (2.127)-(2.128) into (2.124) and taking into account that

$$\big|(1/X)_{\xi_0}(\xi,\eta;\xi_0,1)\big| = \frac{(1-\xi)(\eta-\xi)}{(1-\eta)(\xi_0-\xi)^2} \le \frac{1-\xi}{(1-\eta)(\xi_0-\xi)},$$

we obtain the following estimate

$$|\psi_{3,3}(\xi,\eta;\xi_0)| \le \frac{C_4\,(1-\xi)}{(1-\eta)^{1-\beta}(\xi_0-\xi)(\xi_0-\eta)^\beta}, \tag{2.129}$$

where $C_4 = \mathrm{const} > 0$, $(\xi, \eta) \in D \cap \{\eta < \xi_0\}$.

Finally, (2.121) follows directly from the estimates (2.125), (2.126) and (2.129). □

Theorem 2.9.1. *Suppose that $F \in C^1(\bar{D})$. Then the derivative $U_\xi(\xi, 1)$ of the generalized solution of Problem P_{m2} has the following representation on the segment $\{0 \le \xi < 1\}$:*

$$U_\xi(\xi, 1) = \sum_{k=0}^{[(n-1)/2]} (n-2k)\lambda_k^n \big(\mu_k^{n,\beta} - J_k^{n,\beta}(\xi)\big)(1-\xi)^{2k-n-1}$$
$$+ \sum_{k=0}^{[n/2]} \lambda_k^n (1-\xi)^{2k-n} \int_0^\xi (\xi - \xi_1)^{2\beta} E_k^{n,\beta}(\xi_1, \xi) F(\xi_1, \xi)\, d\xi_1 + g'(\xi), \quad (2.130)$$

where $\lambda_k^n \neq 0$ are the constants and $g(\xi)$ is the function from Theorem 2.8.1. The derivative $g'(\xi)$ satisfies the estimate

$$|g'(\xi)| \le C\|F\|_{C(D)}(1-\xi)^{-\beta}, \quad (2.131)$$

with a constant $C > 0$ independent of F.

Proof. Obviously, formula (2.130) is obtained by a straightforward differentiation of (2.107) and we only have to prove the estimate (2.131).

Recall that an explicit form of the function $g(\xi)$ is given by (2.114).

Now, by (2.101) we have that

$$\begin{aligned}\Psi_2(\xi,\eta;\xi_0) &= \frac{\gamma\,(\eta-\xi)^{2\beta}}{(1-\xi)^{\beta}(1-\eta)^{\beta}}\{\psi_1(\xi,\eta;\xi_0)+\psi_2(\xi,\eta;\xi_0)\}\\ &= \frac{\gamma\,(\eta-\xi)^{2\beta}}{(1-\xi)^{\beta}(1-\eta)^{\beta}}\left\{\psi_1(\xi,\eta;\xi_0)+\psi_3(\xi,\eta;\xi_0)-\frac{(1-\xi)^{\beta}}{(\xi_0-\xi)^{\beta}}\right\}\\ &\qquad + H^{-}(\xi,\eta;\xi_0,1),\end{aligned}$$

where the functions $\psi_1(\xi,\eta;\xi_0)$, $\psi_1(\xi,\eta;\xi_0)$ and $\psi_3(\xi,\eta;\xi_0)$ are defined in Lemmas 2.7.3, 2.7.4 and 2.9.2 respectively and $H^{-}(\xi,\eta;\xi_0,\eta_0)$ is given by (2.29). Applying this equality into (2.114) we obtain:

$$g(\xi_0)=g_1(\xi_0)+g_2(\xi_0)+g_3(\xi_0)$$

with

$$g_1(\xi_0):=\int_0^{\xi_0}\int_{\xi}^{\xi_0}\frac{\gamma\,(\eta-\xi)^{2\beta}}{(1-\xi)^{\beta}(1-\eta)^{\beta}}\psi_1(\xi,\eta;\xi_0)F(\xi,\eta)\,d\eta d\xi,$$

$$g_2(\xi_0):=\int_0^{\xi_0}\int_{\xi}^{\xi_0}\frac{\gamma\,(\eta-\xi)^{2\beta}}{(1-\xi)^{\beta}(1-\eta)^{\beta}}\left(\psi_3(\xi,\eta;\xi_0)-\frac{(1-\xi)^{\beta}}{(\xi_0-\xi)^{\beta}}\right)F(\xi,\eta)\,d\eta d\xi,$$

$$\begin{aligned}g_3(\xi_0):=&\int_0^{\xi_0}\int_{\xi}^{\xi_0}F(\xi,\eta)H^{-}(\xi,\eta;\xi_0,1)\,d\eta d\xi\\ &\qquad+\int_0^{\xi_0}\int_{\xi_0}^{1}F(\xi,\eta)\Psi^{+}(\xi,\eta;\xi_0,1)\,d\eta d\xi.\end{aligned}$$

A. Estimation of the function $g_1'(\xi_0)$. First, we calculate

$$g_1'(\xi_0) = \int_0^{\xi_0}\int_\xi^{\xi_0} \frac{\gamma\,(\eta-\xi)^{2\beta}}{(1-\xi)^\beta(1-\eta)^\beta}\frac{\partial\psi_1}{\partial\xi_0}(\xi,\eta;\xi_0)F(\xi,\eta)\,d\eta d\xi \\ + \int_0^{\xi_0} \frac{\gamma\,(\xi_0-\xi)^{2\beta}}{(1-\xi)^\beta(1-\xi_0)^\beta}\psi_1(\xi,\xi_0;\xi_0)F(\xi,\xi_0)\,d\xi.$$

Now, with use of the estimates (2.90) and (2.117) we get:

$$|g_1'(\xi_0)| \le C_1\|F\|_{C(D)}\left\{\int_0^{\xi_0}\int_\xi^{\xi_0} \frac{(\eta-\xi)^{2\beta}}{(1-\eta)^\beta(\xi_0-\xi)^{1+\beta}}\,d\eta d\xi \right. \\ \left. + \int_0^{\xi_0} \frac{(\xi_0-\xi)^\beta}{(1-\xi_0)^\beta}\,d\xi\right\} \le C_2\|F\|_{C(D)}(1-\xi_0)^{-\beta},$$

where C_1, $C_2 = \text{const} > 0$ are independent of F.

B. Estimation of the function $g_2'(\xi_0)$. Next, we compute

$$g_2'(\xi_0) = \\ \int_0^{\xi_0}\int_\xi^{\xi_0} \frac{\gamma\,(\eta-\xi)^{2\beta}}{(1-\xi)^\beta(1-\eta)^\beta}\left(\frac{\partial\psi_3}{\partial\xi_0}(\xi,\eta;\xi_0) + \frac{\beta\,(1-\xi)^\beta}{(\xi_0-\xi)^{1+\beta}}\right)F(\xi,\eta)\,d\eta d\xi \\ + \int_0^{\xi_0} \frac{\gamma\,(\xi_0-\xi)^{2\beta}}{(1-\xi)^\beta(1-\xi_0)^\beta}\left(\psi_3(\xi,\xi_0;\xi_0) - \frac{(1-\xi)^\beta}{(\xi_0-\xi)^\beta}\right)F(\xi,\xi_0)\,d\xi.$$

From the estimate (2.120) in Lemma 2.9.2 we conclude that by continuity we have

$$|\psi_3(\xi,\xi_0;\xi_0)| \le C\,\frac{(1-\xi)^\beta}{(\xi_0-\xi)^\beta}, \qquad 0 < \xi < \xi_0.$$

Using this together with the estimate (2.121) from Lemma 2.9.2, we obtain

$$|g_2'(\xi_0)| \le C_3 \|F\|_{C(D)} \left\{ \int_0^{\xi_0} \int_\xi^{\xi_0} \frac{(\eta-\xi)^{2\beta}(1-\xi)^{1-\beta}}{(1-\eta)(\xi_0-\xi)(\xi_0-\eta)^\beta} \, d\eta d\xi \right. \\ \left. + \int_0^{\xi_0} \frac{(\xi_0-\xi)^\beta}{(1-\xi_0)^\beta} \, d\xi \right\}$$

with $C_3 = \mathrm{const} > 0$ independent of F.

Taking into account that

$$\frac{(\eta-\xi)^{2\beta}}{\xi_0-\xi} \le (\xi_0-\xi)^{2\beta-1}, \qquad 0 < \xi < \eta < \xi_0$$

and using the inequality (2.111), we come to the estimate

$$|g_2'(\xi_0)| \le C_4 \|F\|_{C(D)} (1-\xi_0)^{-\beta}$$

with $C_4 = \mathrm{const} > 0$ independent of F.

C. Estimation of the function $g_3'(\xi_0)$. We may rewrite the function $g_3(\xi_0)$ in the following way:

$$g_3(\xi_0) = \int_0^{\xi_0} \int_\xi^{\xi_0} F(\xi,\eta) H^-(\xi,\eta;\xi_0,1) \, d\eta d\xi \\ + \int_0^{\xi_0} \int_{\xi_0}^1 F(\xi,\eta) H^+(\xi,\eta;\xi_0,1) \, d\eta d\xi + \int_0^{\xi_0} \int_{\xi_0}^1 F(\xi,\eta) G^+(\xi,\eta;\xi_0,1) \, d\eta d\xi \\ = U^H(\xi_0,1) + \int_0^{\xi_0} \int_{\xi_0}^1 F(\xi,\eta) G^+(\xi,\eta;\xi_0,1) \, d\eta d\xi.$$

Then we have

$$g_3'(\xi_0) = U_{\xi_0}^H(\xi_0, 1) + \int_0^{\xi_0} \int_{\xi_0}^1 F(\xi, \eta) G_{\xi_0}^+(\xi, \eta; \xi_0, 1)\, d\eta d\xi \\ - \int_0^{\xi_0} F(\xi, \xi_0) G^+(\xi, \xi_0; \xi_0, 1)\, d\xi.$$

Applying here the estimates (2.40)-(2.41) and (2.52)-(2.53), we conclude that

$$|g_3'(\xi_0)| \le C_5 \|F\|_{C(D)} (1 - \xi_0)^{-\beta}$$

with $C_5 = \text{const} > 0$ independent of F.

The proof is complete. □

Remark 2.9.1. *For the terms*

$$\frac{d}{d\xi}\left[\lambda_k^n J_k^{n,\beta}(\xi)(1-\xi)^{2k-n}\right] = (n-2k)\lambda_k^n J_k^{n,\beta}(\xi)(1-\xi)^{2k-n-1} \\ - \lambda_k^n (1-\xi)^{2k-n} \int_0^{\xi} (\xi - \xi_1)^{2\beta} E_k^{n,\beta}(\xi_1, \xi) F(\xi_1, \xi)\, d\xi_1$$

in the expansion (2.130) *we have the following estimate on the segment* $0 \le \xi < 1$:

$$\left| \frac{d}{d\xi}\left[\lambda_k^n J_k^{n,\beta}(\xi)(1-\xi)^{2k-n}\right] \right| \le C \|F\|_{C(D)} (1-\xi)^{-\beta}$$

with $C = \text{const} > 0$ *independent of* F. *This easily follows from the estimate* (2.115) *and from* (2.116) *which implies the estimate*

$$\left| \frac{d}{d\xi} J_k^{n,\beta}(\xi) \right| \le C \|F\|_{C(D)} (1-\xi)^{n-2k-\beta}. \tag{2.132}$$

2.10. Asymptotic expansion of the function $\mathrm{U}(\xi, \eta)$

From here, using the data which we obtained for the generalized solution on the boundary segment $\eta = 1$, $0 \le \xi < 1$, we will study the behavior of $U(\xi, \eta)$ in the region D. Actually, the generalized solution of Problem P_{m2} can be obtained by solving the following Goursat problem:

Problem $\mathbf{P^G_{m2}}$. *Given a point $(\xi_0, \eta_0) \in D$, find a solution of the equation*

$$E_\beta[U] = F(\xi, \eta)$$

in the region

$$\Pi := \{(\xi, \eta) : 0 < \xi < \xi_0,\ \eta_0 < \eta < 1\},$$

satisfying the following boundary conditions:

$$\begin{aligned} &U(0, \eta) = 0, \\ &U(\xi, 1) = \sum_{k=0}^{[n/2]} \lambda_k^n \big(\mu_k^{n,\beta} - J_k^{n,\beta}(\xi)\big)(1-\xi)^{2k-n} + g(\xi). \end{aligned}$$

(*The function $U(\xi, 1)$ is the one which we found in Theorem* 2.8.1).

The solution of this problem obtained by the Riemann method is the following one:

$$\begin{aligned} U(\xi_0, \eta_0) = \int_0^{\xi_0} \left(U_\xi(\xi, 1) - \frac{\beta}{1-\xi} U(\xi, 1) \right) \mathcal{R}(\xi, 1; \xi_0, \eta_0)\, d\xi& \\ - \int_{\eta_0}^1 \int_0^{\xi_0} F(\xi, \eta) \mathcal{R}(\xi, \eta; \xi_0, \eta_0)\, d\xi d\eta,& \quad (2.133) \end{aligned}$$

where $\mathcal{R}(\xi,\eta;\xi_0,\eta_0)$ is the Riemann function for equation (2.15).

At the points $(\xi,\eta)\in\Pi$, where $|X(\xi,\eta;\xi_0,\eta_0)|<1$, the Riemann function coincides with the function $\Psi^+(\xi,\eta;\xi_0,\eta_0)$ (see (2.22)), which may be written as

$$\Psi^+(\xi,\eta;\xi_0,\eta_0)=\left(\frac{\eta-\xi}{\eta_0-\xi_0}\right)^\beta\sum_{i=0}^{n}c_iY^i\,{}_2F_1(\beta,1-\beta,i+1;X). \quad (2.134)$$

If (ξ_0,η_0) belongs to the region

$$\mathcal{T}:=\left\{(\xi,\eta):\ 0<\xi<1,\ \frac{1}{2}(1+\xi)<\eta<1\right\}\subset D, \quad (2.135)$$

then at all the points $(\xi,\eta)\in\Pi$ we have $|X|<1$ and, respectively, $\Psi^+(\xi,\eta;\xi_0,\eta_0)$ is well defined. Otherwise, if $(\xi_0,\eta_0)\notin\mathcal{T}$, in a part of the region Π we may have $X<-1$. In this case, applying (A.14) into (2.134), the series $\Psi^+(\xi,\eta;\xi_0,\eta_0)$ can be analytically continued in the whole region Π as

$$\Psi^+_A(\xi,\eta;\xi_0,\eta_0):=\frac{(\eta-\xi)^{2\beta}}{(\eta_0-\xi)^\beta(\eta-\xi_0)^\beta}\sum_{i=0}^{n}c_iY^i\,{}_2F_1(\beta,\beta+i,i+1;Z) \quad (2.136)$$

or

$$\Psi^+_B(\xi,\eta;\xi_0,\eta_0):=\frac{(\eta-\xi)(\eta_0-\xi_0)^{1-2\beta}}{(\eta_0-\xi)^{1-\beta}(\eta-\xi_0)^{1-\beta}}\times\sum_{i=0}^{n}c_iY^i\,{}_2F_1(1-\beta,1-\beta+i,i+1;Z) \quad (2.137)$$

with

$$Z = \frac{X}{X-1} = \frac{(\eta - \eta_0)(\xi_0 - \xi)}{(\eta_0 - \xi)(\eta - \xi_0)}.$$

Further, in dependence of the value of the parameter β the Riemann function can be estimated in a different way in the region Π:

A. The case $0 < \beta < 1/2$. In this case for the Riemann function it is convenient to use the representation form Ψ_A^+. Note that according to (A.8) the hypergeometric function in (2.136) is bounded for $0 < \beta < 1/2$. Respectively we have

$$|\Psi_A^+(\xi, \eta; \xi_0, \eta_0)| \leq \text{const}\, \frac{(\eta - \xi)^{2\beta}}{(\eta_0 - \xi)^{\beta}(\eta - \xi_0)^{\beta}}, \qquad (\xi, \eta) \in \Pi. \qquad (2.138)$$

B. The case $1/2 < \beta < 1$. In this case, according to (A.8), the hypergeometric function in (2.137) is bounded and, respectively, for the function Ψ_B^+ we have

$$|\Psi_B^+(\xi, \eta; \xi_0, \eta_0)| \leq \text{const}\, \frac{(\eta - \xi)(\eta_0 - \xi_0)^{1-2\beta}}{(\eta_0 - \xi)^{1-\beta}(\eta - \xi_0)^{1-\beta}}, \qquad (\xi, \eta) \in \Pi. \quad (2.139)$$

C. The case $\beta = 1/2$. Note that in this case the expressions (2.136) and (2.137) coincide and the hypergeometric function in (2.136) (or (2.137)) becomes unbounded as $Z \to 1$. According to (A.6), for each $\alpha > 0$ there exists a constant $c(\alpha) > 0$ such that

$$|\Psi_A^+(\xi, \eta; \xi_0, \eta_0)| = |\Psi_B^+(\xi, \eta; \xi_0, \eta_0)| \leq c(\alpha)\, \frac{(\eta - \xi)^{1-\alpha}(\eta_0 - \xi_0)^{-\alpha}}{(\eta_0 - \xi)^{1/2-\alpha}(\eta - \xi_0)^{1/2-\alpha}} \qquad (2.140)$$

for $(\xi, \eta) \in \Pi$.

For the solution of Problem P^G_{m2} (i.e. for the generalized solution of P_{m2}) we obtain the following main result, which we will prove in the next section:

Theorem 2.10.1. *Let $0 < \beta < 1$ and $F \in C^1(\bar{D})$. Then the unique generalized solution of Problem P_{m2} has the following asymptotic representation at the singular point $(1,1)$:*

$$U(\xi,\eta) = \sum_{k=0}^{[n/2]} \mu_k^{n,\beta} a_k^{n,\beta} G_k^{n,\beta}(\xi,\eta)(2-\xi-\eta)^{2k-n} + G^{(\beta)}(\xi,\eta), \quad (\xi,\eta) \in D,$$

where $a_k^{n,\beta} = \text{const} \neq 0$,

$$G_k^{n,\beta}(\xi,\eta) := {}_2F_1\left(n-k+\frac{1}{2}, -k, \frac{1}{2}+\beta; \frac{(\eta-\xi)^2}{(2-\xi-\eta)^2}\right)$$

and the function $G^{(\beta)}(\xi,\eta) \in C(\bar{D})$ is such that $G^{(\beta)}(1,1) = 0$. For $0 < \beta < 1/2$ $G^{(\beta)}(\xi,\eta)$ satisfies the following estimate in D:

$$|G^{(\beta)}(\xi,\eta)| \leq K\|F\|_{C(D)}(1-\xi)^{1-\beta}(1+|\ln(1-\xi)|)$$

with a constant $K > 0$ independent of F. For $1/2 \leq \beta < 1$ such an estimate holds at least in $\mathcal{T} \subset D$.

Remark 2.10.1. *Note that the functions $G_k^{n,\beta}(\xi,\eta)$ are connected with* (2.18) *by the following relation:*

$$E_k^{n,\beta}(\xi,\eta) = \frac{(1-\xi)^{n-2k-\beta}(1-\eta)^{n-2k-\beta}}{(2-\xi-\eta)^{n-2k}} G_k^{n,\beta}(\xi,\eta)$$

in $\bar{D}\backslash(1,1)$.

2.11. Proof of Theorem 2.10.1

Since the proof of this theorem is too long, we start here with some auxiliary lemmas.

Lemma 2.11.1. *For* $p = 1, 2, \dots,\ i = 0, 1, 2, \dots$ *and* $\alpha > -1$ *define the integrals*:

$$I^{\alpha}_{p,i} := \int_0^1 (t - \omega t^2)^{p-1}(1 - 2\omega t)^{2i+1}(1 - t)^{\alpha}(1 - \sigma t)^{\alpha}\, dt,$$

where

$$\sigma = \frac{\omega}{1 - \omega}, \qquad 0 < \omega < \frac{1}{2}. \tag{2.141}$$

Then

$$I^{\alpha}_{p,i} = \frac{\Gamma(p)\, i!\, (1 - \omega)^p}{(1 + \alpha)_{i+p}} \sum_{s=0}^{i} \frac{(1 + \alpha)_{i-s}}{(i - s)!} \frac{(p)_s}{s!}(1 - 2\omega)^{2s}. \tag{2.142}$$

Proof. First, by (2.141) we have

$$1 - 2\omega t = (1 - \omega)\big\{\sigma(1 - t) + (1 - \sigma t)\big\}.$$

Then $I^{\alpha}_{p,i}$ may be written as

$$\begin{aligned} I^{\alpha}_{p,i} = (1 - \omega)\sigma &\int_0^1 (t - \omega t^2)^{p-1}(1 - 2\omega t)^{2i}(1 - t)^{\alpha+1}(1 - \sigma t)^{\alpha}\, dt \\ - \frac{1 - \omega}{\alpha + 1} &\int_0^1 (t - \omega t^2)^{p-1}(1 - 2\omega t)^{2i}\frac{d}{dt}\big\{(1 - t)^{\alpha+1}\big\}(1 - \sigma t)^{\alpha+1}\, dt. \end{aligned}$$

Next, integrating by parts in the second integral, for $I^{\alpha}_{p,i}$ we obtain four different cases in dependence of the parameters p and i.

A. The case $p = 1,\ i = 0$. In this case we have:

$$I^{\alpha}_{1,0} = -\frac{1-\omega}{\alpha+1}(1-t)^{\alpha+1}(1-\sigma t)^{\alpha+1}\Big|_{0}^{t=1} = \frac{1-\omega}{1+\alpha}. \tag{2.143}$$

B. The case $p = 1,\ i = 1, 2, \ldots$ In this case we obtain:

$$\begin{aligned} I^{\alpha}_{1,i} = &-\frac{1-\omega}{\alpha+1}(1-2\omega t)^{2i}(1-t)^{\alpha+1}(1-\sigma t)^{\alpha+1}\Big|_{0}^{t=1} \\ &- \frac{4i\omega(1-\omega)}{\alpha+1}\int_0^1 (1-2\omega t)^{2i-1}(1-t)^{\alpha+1}(1-\sigma t)^{\alpha+1}\,dt, \end{aligned}$$

which actually gives the recurrence relation:

$$I^{\alpha}_{1,i} = \frac{1-\omega}{\alpha+1} - \frac{4i\omega(1-\omega)}{\alpha+1} I^{\alpha+1}_{1,i-1} = \frac{1-\omega}{\alpha+1} - i\,\frac{\left\{1-(1-2\omega)^2\right\}}{\alpha+1} I^{\alpha+1}_{1,i-1}.$$

From here, taking into account (2.143), we find that

$$I^{\alpha}_{1,i} = \frac{i!\,(1-\omega)}{(1+\alpha)_{i+1}} \sum_{s=0}^{i} \frac{(1+\alpha)_{i-s}}{(i-s)!}(1-2\omega)^{2s}.$$

C. The case $i = 0,\ p = 2, 3, \ldots$ For $I^{\alpha}_{p,0}$ we get

$$I^{\alpha}_{p,0} = \frac{(p-1)(1-\omega)}{\alpha+1}\int_0^1 (t-\omega t^2)^{p-2}(1-2\omega t)(1-t)^{\alpha+1}(1-\sigma t)^{\alpha+1}\,dt,$$

or

$$I^{\alpha}_{p,0} = \frac{(p-1)(1-\omega)}{\alpha+1} I^{\alpha+1}_{p-1,0}.$$

From this recurrence relation, starting from $I^{\alpha}_{p,0}$, we find that

$$I^{\alpha}_{p,0} = \frac{\Gamma(p)(1-\omega)^p}{(1+\alpha)_p}.$$

D. The case $p = 2, 3, \dots,\ i = 1, 2, \dots$ For this case we have:

$$\begin{aligned} I^{\alpha}_{p,i} = {} & \frac{(p-1)(1-\omega)}{\alpha+1} \int_0^1 (t-\omega t^2)^{p-2}(1-2\omega t)^{2i+1}(1-t)^{\alpha+1}(1-\sigma t)^{\alpha+1}\,dt \\ & - \frac{4i\omega(1-\omega)}{\alpha+1} \int_0^1 (t-\omega t^2)^{p-1}(1-2\omega t)^{2i-1}(1-t)^{\alpha+1}(1-\sigma t)^{\alpha+1}\,dt. \end{aligned}$$

This gives the following relation:

$$I^{\alpha}_{p,i} = \frac{(p-1)(1-\omega)}{\alpha+1} I^{\alpha+1}_{p-1,i} - i\,\frac{\left\{1-(1-2\omega)^2\right\}}{\alpha+1} I^{\alpha+1}_{p,i-1}. \tag{2.144}$$

Now we try to find a representation of $I^{\alpha}_{p,i}$ of the form:

$$I^{\alpha}_{p,i} = \frac{\Gamma(p)\, i!\, (1-\omega)^p}{(1+\alpha)_{i+p}} \sum_{s=0}^{i} \frac{(1+\alpha)_{i-s}}{(i-s)!} C_{p,s}(1-2\omega)^{2s}, \tag{2.145}$$

where we suppose that $C_{p,s}$ do not depend on i and $C_{1,s} = 1$. Putting this expression into (2.144) and simplifying, we find that the coefficients $C_{p,s}$ must satisfy the relations:

$$C_{p,s} = C_{p-1,s} + C_{p,s-1}, \qquad C_{p,0} = C_{1,s} = 1.$$

From here we obtain

$$C_{p,s} = \frac{(s+p-1)!}{s!(p-1)!} = \frac{(p)_s}{s!}.$$

Substituting this into (2.145) gives the final result (2.142).

The proof is complete. $\square$

Lemma 2.11.2. *For $0 < \beta < 1$ and $p \in \mathbb{N} \cup 0$ define the series*

$$S_0(\xi_0, \eta_0) := \sum_{j=1}^{\infty} \frac{(\beta)_j (1-\beta)_j}{j!\,(j-1)!} \frac{(\eta_0 - 1)^j}{(\eta_0 - \xi_0)^j}$$
$$\times \int_0^{\xi_0} \frac{\phi(\xi)}{1-\xi} \, {}_2F_1\left(p - \beta + 1, 1 - j, p - \beta + 2; \frac{1-\xi_0}{1-\xi}\right) d\xi,$$

where $\phi(\xi)$ is an integrable function, bounded on the interval $[0,1]$. Then this series converges for $(\xi_0, \eta_0) \in \mathcal{T} \subset D$ and the following estimate holds:

$$|S_0(\xi_0, \eta_0)| \leq C M_\phi \frac{1-\eta_0}{1-\xi_0} \big(1 + |\ln(1-\xi_0)|\big), \qquad (\xi_0, \eta_0) \in \mathcal{T},$$

where $C = \text{const} > 0$ and

$$M_\phi := \max_{[0,1]} |\phi(\xi)|.$$

Further, $S_0(\xi_0, \eta_0)$ can be analytically continued in the whole region D, where:

(i) *if $0 < \beta < 1/2$, then*

$$|S_0(\xi_0, \eta_0)| \leq C M_\phi \frac{(1-\eta_0)(\eta_0 - \xi_0)^\beta}{(1-\xi_0)^{1+\beta}} \big(1 + |\ln(1-\xi_0)|\big);$$

(ii) *if $1/2 \leq \beta < 1$, then*

$$|S_0(\xi_0, \eta_0)| \leq C M_\phi \frac{(1-\eta_0)(\eta_0 - \xi_0)^{\beta-\varepsilon}}{(1-\xi_0)^{1+\beta-\varepsilon}} \big(1 + |\ln(1-\xi_0)|\big)$$

with some $\varepsilon \in (0, \beta)$.

Proof. First, using (A.5), we have

$$
\begin{aligned}
S_0(\xi_0, \eta_0) = K_1 \frac{\eta_0 - 1}{\eta_0 - \xi_0} \sum_{k=0}^{\infty} \frac{(1+\beta)_k (2-\beta)_k}{(2)_k \, k!} \frac{(\eta_0 - 1)^k}{(\eta_0 - \xi_0)^k} \\
\times \int_0^{\xi_0} \frac{\phi(\xi)}{1-\xi} \left\{ \int_0^1 t^{p-\beta} \left(1 - \frac{1-\xi_0}{1-\xi} \, t \right)^k dt \right\} d\xi \\
= K_1 \frac{\eta_0 - 1}{\eta_0 - \xi_0} \int_0^{\xi_0} \frac{\phi(\xi)}{1-\xi} \left\{ \int_0^1 t^{p-\beta} \, {}_2F_1(1+\beta, 2-\beta, 2; z) \, dt \right\} d\xi \quad (2.146)
\end{aligned}
$$

with $K_1 = \text{const} = \beta(1-\beta)(p-\beta+1)$ and

$$
z = \left(\frac{\eta_0 - 1}{\eta_0 - \xi_0} \right) \left(1 - \frac{1-\xi_0}{1-\xi} \, t \right). \quad (2.147)
$$

For $(\xi_0, \eta_0) \in \mathcal{T}$ we have $0 < 1 - \eta_0 < \eta_0 - \xi_0$ (see (2.135)), which implies $|z| < 1$. Then the series ${}_2F_1(1+\beta, 2-\beta, 2; z)$ is absolutely convergent.

Next, to estimate the integrals involved in (2.146), with use of (A.13) we obtain:

$$
\begin{aligned}
S_0(\xi_0, \eta_0) = -K_1 \frac{1-\eta_0}{1-\xi_0} \int_0^{\xi_0} \phi(\xi) \\
\times \left\{ \int_0^1 \frac{t^{p-\beta}}{(1-\xi) - (1-\eta_0)\, t} \, {}_2F_1(1-\beta, \beta, 2; z) \, dt \right\} d\xi. \quad (2.148)
\end{aligned}
$$

Now, according to (A.5), (A.6) and (A.8), we have

$$
|\, {}_2F_1(1-\beta, \beta, 2; z)| \leq \text{const} \quad (2.149)
$$

and

$$\begin{aligned} 0 &< (p-\beta+1)\int_0^1 \frac{t^{p-\beta}\,dt}{(1-\xi)-(1-\eta_0)\,t} \\ &= \frac{1}{1-\xi}\,{}_2F_1\left(p-\beta+1,1,p-\beta+2;\frac{1-\eta_0}{1-\xi}\right) \le c(\alpha)\frac{(1-\xi)^{\alpha-1}}{(\eta_0-\xi)^{\alpha}}, \end{aligned} \tag{2.150}$$

where we choose $\alpha \in (0,1)$. Next, making a substitution $\xi = \xi_0 t$ and using once again (A.5), we get

$$\begin{aligned} 0 < \int_0^{\xi_0} \frac{(1-\xi)^{\alpha-1}}{(\eta_0-\xi)^{\alpha}}\,d\xi &\le \int_0^{\xi_0} \frac{(1-\xi)^{\alpha-1}}{(\xi_0-\xi)^{\alpha}}\,d\xi \\ &= \frac{\xi_0^{1-\alpha}}{1-\alpha}\,{}_2F_1(1,1-\alpha,2-\alpha;\xi_0). \end{aligned} \tag{2.151}$$

From the one hand, taking into account (A.12), we can estimate

$${}_2F_1(1,1-\alpha,2-\alpha;\xi_0) < {}_2F_1(1,1,2;\xi_0) = \frac{1}{\xi_0}|\ln(1-\xi_0)|,$$

since

$$\frac{(1-\alpha)_j}{(2-\alpha)_j} < \frac{(1)_j}{(2)_j}, \qquad j = 1,2,\ldots, \qquad 0<\alpha<1$$

and from the other hand by (A.6) we have

$${}_2F_1(1,1-\alpha,2-\alpha;\xi_0) \le c(\delta)(1-\xi_0)^{-\delta}$$

with $\delta > 0$, hence

$${}_2F_1(1,1-\alpha,2-\alpha;\xi_0) \le \mathrm{const}\,\big(1+|\ln(1-\xi_0)|\big). \tag{2.152}$$

Applying the estimates (2.149)-(2.152) into (2.148), we see that:

$$|S_0(\xi_0,\eta_0)| \le CM_\phi \, \frac{1-\eta_0}{1-\xi_0} \big(1 + |\ln(1-\xi_0)|\big), \qquad (\xi_0,\eta_0) \in \mathcal{T}.$$

Now, using (A.14) we may prolong the series $S_0(\xi_0,\eta_0)$ in the whole region D:

$$S_0(\xi_0,\eta_0) = -K_1 \, \frac{(1-\eta_0)(\eta_0-\xi_0)^\beta}{(1-\xi_0)^{1+\beta}} \int_0^{\xi_0} \phi(\xi)(1-\xi)^\beta$$
$$\times \left\{ \int_0^1 \frac{t^{p-\beta}}{\{(1-\xi)-(1-\eta_0)\,t\}^{1+\beta}} \, {}_2F_1\left(1+\beta,\beta,2;\frac{z}{z-1}\right) dt \right\} d\xi.$$

The hypergeometric series here is well defined in D, since for each point $(\xi_0,\eta_0) \in D$ we have $z<0$, which implies $0 < z/(z-1) < 1$.

A. The case $0<\beta<1/2$. In this case the series ${}_2F_1(1+\beta,\beta,2;\zeta)$ is bounded as $\zeta \to 1$. Then working in the same manner in which we estimated (2.148), we obtain

$$|S_0(\xi_0,\eta_0)| \le CM_\phi \, \frac{(1-\eta_0)(\eta_0-\xi_0)^\beta}{(1-\xi_0)^{1+\beta}} \big(1 + |\ln(1-\xi_0)|\big), \qquad (\xi_0,\eta_0) \in D.$$

B. The case $1/2 \le \beta < 1$. Taking into account that

$$|\,{}_2F_1(3/2,1/2,2;\zeta)| \le \text{const} \, \frac{(1-\xi_0)^\delta}{(\eta_0-\xi_0)^\delta} \left(1 - \frac{1-\eta_0}{1-\xi}\,t\right)^\delta$$

for some $\delta \in (0,1/2)$ and

$$|\,{}_2F_1(1+\beta,\beta,2;\zeta)| \le \text{const} \, \frac{(1-\xi_0)^{2\beta-1}}{(\eta_0-\xi_0)^{2\beta-1}} \left(1 - \frac{1-\eta_0}{1-\xi}\,t\right)^{2\beta-1}, \quad \beta > 1/2,$$

which follows from (A.7)-(A.6), with similar calculations we come to

$$|S_0(\xi_0,\eta_0)| \leq CM_\phi \frac{(1-\eta_0)(\eta_0-\xi_0)^{\beta-\varepsilon}}{(1-\xi_0)^{1+\beta-\varepsilon}} \big(1+|\ln(1-\xi_0)|\big), \qquad (\xi_0,\eta_0)\in D,$$

where

$$\varepsilon = \begin{cases} \delta, & \beta = 1/2, \\ 2\beta-1, & 1/2<\beta<1. \end{cases}$$

The proof is complete. □

Lemma 2.11.3. *For* $0<\beta<1$, $i\in\mathbb{N}$ *and* $p\in\mathbb{N}\cup 0$ *define the series*

$$S_i(\xi_0,\eta_0) := \sum_{j=0}^{\infty} \frac{(\beta)_j(1-\beta)_j}{j!\,(i+j-1)!} \frac{(\eta_0-1)^j}{(\eta_0-\xi_0)^j} \times \int_0^{\xi_0} \frac{\phi(\xi)}{1-\xi}\, {}_2F_1\left(p-\beta+1, 1-i-j, p-\beta+2; \frac{1-\xi_0}{1-\xi}\right) d\xi,$$

where $\phi(\xi)$ *is an integrable function, bounded on the interval* $[0,1]$ *with* $|\phi(\xi)|\leq M_\phi = \text{const}>0$. *Then this series converges for* $(\xi_0,\eta_0)\in\mathcal{T}\subset D$ *and the following estimate holds*:

$$|S_i(\xi_0,\eta_0)| \leq CM_\phi\big(1+|\ln(1-\xi_0)|\big), \qquad (\xi_0,\eta_0)\in\mathcal{T}, \tag{2.153}$$

where $C=\text{const}>0$.

Further, $S_i(\xi_0,\eta_0)$ *can be analytically continued in the whole region* D*, where*:

(i) *if* $0 < \beta < 1/2$, *then*

$$|S_i(\xi_0, \eta_0)| \le CM_\phi \frac{(\eta_0 - \xi_0)^\beta}{(1-\xi_0)^\beta} \big(1 + |\ln(1-\xi_0)|\big); \tag{2.154}$$

(ii) *if* $1/2 \le \beta < 1$, *then*

$$|S_i(\xi_0, \eta_0)| \le CM_\phi \frac{(\eta_0 - \xi_0)^{\beta-\varepsilon}}{(1-\xi_0)^{\beta-\varepsilon}} \big(1 + |\ln(1-\xi_0)|\big) \tag{2.155}$$

with some $\varepsilon \in (0, \beta)$.

Proof. The proof is very similar to the proof of the previous lemma. First, in analogous way we come to

$$S_i(\xi_0, \eta_0) = K_2 \int_0^{\xi_0} \frac{\phi(\xi)}{1-\xi} \left\{ \int_0^1 t^{p-\beta} h(\xi, \xi_0, t)\, {}_2F_1(\beta, 1-\beta, i; z)\, dt \right\} d\xi$$

with

$$K_2 = \text{const} = \frac{p-\beta+1}{(i-1)!}, \qquad h(\xi, \xi_0, t) = \left(1 - \frac{1-\xi_0}{1-\xi}\, t\right)^{i-1}$$

and z defined by (2.147). Clearly, for $0 < \xi < \xi_0 < 1$ and $0 < t < 1$ we have $|h(\xi, \xi_0, t)| < 1$ and working in a similar manner as in Lemma 2.11.2 we come to the estimate (2.153).

Next, with use of (A.14), we continue the series $S_i(\xi_0, \eta_0)$ in the whole region D and, in a similar manner as in Lemma 2.11.2, we get the estimates (2.154) and (2.155) in D. □

Lemma 2.11.4. *Let* $(\xi_0, \eta_0) \in D$, $0 < \beta < 1$, $p \in \mathbb{N}$, $p \le n$ *and* p *be of* *the same parity as* n*. Then*

$$\begin{aligned} \mathcal{I}_{p,n}(\xi_0,\eta_0) &:= (p-\beta)\int_0^{\xi_0}(1-\xi)^{-p-1}\mathcal{R}(\xi,1;\xi_0,\eta_0)\,d\xi \\ &= \frac{K_{p,n}}{(2-\xi_0-\eta_0)^p}\,{}_2F_1\left(\frac{p+n+1}{2},\frac{p-n}{2},\beta+\frac{1}{2};\frac{(\eta_0-\xi_0)^2}{(2-\xi_0-\eta_0)^2}\right) \\ &\qquad - H_{p,n}(\xi_0,\eta_0), \quad (2.156) \end{aligned}$$

where $K_{p,n} = \text{const} \neq 0$ *and* $H_{p,n}(\xi_0,\eta_0)$ *is a function with the following series representation in* $\mathcal{T} \subset D$:

$$\begin{aligned} H_{p,n}(\xi_0,\eta_0) &:= \sum_{i=0}^{n}\sum_{j=0}^{\infty}\frac{(-n)_i(n+1)_i(\beta)_j(1-\beta)_j}{i!\,j!\,(i+j)!\,(-1)^j} \\ &\times \frac{(1-\eta_0)^{i+j}}{(2-\xi_0-\eta_0)^i(\eta_0-\xi_0)^{j+\beta}}\,{}_2F_1(p-\beta,-i-j,p+1-\beta;1-\xi_0). \end{aligned} \quad (2.157)$$

Proof. First, suppose that $(\xi_0,\eta_0)\in\mathcal{T}$. Then we may represent the function $\mathcal{R}(\xi,\eta;\xi_0,\eta_0)$ by $\Psi^+(\xi,\eta;\xi_0,\eta_0)$ and according to (2.134) we have

$$\begin{aligned} \mathcal{I}_{p,n}(\xi_0,\eta_0) &= (p-\beta)\sum_{i=0}^{n}\sum_{j=0}^{\infty}\frac{(-n)_i(n+1)_i(\beta)_j(1-\beta)_j}{i!\,j!\,(i+j)!\,(-1)^j} \\ &\times \int_0^{\xi_0}\frac{(\xi_0-\xi)^{i+j}(1-\eta_0)^{i+j}}{(2-\xi_0-\eta_0)^i(1-\xi)^{i+j+p+1-\beta}(\eta_0-\xi_0)^{j+\beta}}\,d\xi. \end{aligned}$$

For fixed indexes i,j consider a single term with

$$S_{i,j}(\xi_0) := \int_0^{\xi_0}(1-\xi)^{-i-j-p-1+\beta}(\xi_0-\xi)^{i+j}\,d\xi.$$

Making a substitution $\xi = t\,\xi_0$, with use of (A.5) we obtain:

$$\begin{aligned} S_{i,j}(\xi_0) &= \xi_0^{i+j+1} \int_0^1 (1-t)^{i+j}(1-\xi_0 t)^{-p-i-j-1+\beta}\, dt \\ &= \xi_0^{i+j+1} \frac{\Gamma(i+j+1)}{\Gamma(i+j+2)}\, {}_2F_1(1, p+i+j+1-\beta, i+j+2; \xi_0). \end{aligned}$$

Next, we transform the hypergeometric function in the above expression by (A.15):

$$\begin{aligned} S_{i,j}(\xi_0) &= \xi_0^{i+j+1} \\ &\times \left\{ \frac{\Gamma(\beta-p)}{\Gamma(1+\beta-p)}\, {}_2F_1(1, p+i+j+1-\beta, p+1-\beta; 1-\xi_0) \right. \\ &\qquad + \frac{\Gamma(i+j+1)\Gamma(p-\beta)}{\Gamma(p+i+j+1-\beta)}(1-\xi_0)^{-p+\beta} \\ &\qquad\qquad \left. \times\, {}_2F_1(i+j+1, 1-p+\beta, 1-p+\beta; 1-\xi_0) \right\} \\ &\qquad\qquad\qquad := S_{i,j}^{(1)}(\xi_0) + S_{i,j}^{(2)}(\xi_0). \end{aligned}$$

For $S_{i,j}^{(1)}(\xi_0)$ by (A.13) we have

$$S_{i,j}^{(1)}(\xi_0) = \frac{1}{\beta-p}\, {}_2F_1(p-\beta, -i-j, p+1-\beta; 1-\xi_0)$$

and for $S_{i,j}^{(2)}(\xi_0)$ by (A.11) we have

$$S_{i,j}^{(2)}(\xi_0) = \frac{\Gamma(i+j+1)\Gamma(p-\beta)}{\Gamma(p+i+j+1-\beta)}(1-\xi_0)^{-p+\beta}.$$

Returning back to $\mathcal{I}_{p,n}(\xi_0, \eta_0)$ we obtain that

$$\mathcal{I}_{p,n}(\xi_0, \eta_0) = \mathcal{J}_{p,n}(\xi_0, \eta_0) - H_{p,n}(\xi_0, \eta_0), \tag{2.158}$$

where $H_{p,n}(\xi_0,\eta_0)$ is the series (2.157) and

$$\mathcal{J}_{p,n}(\xi_0,\eta_0) := \frac{(p-\beta)}{(1-\xi_0)^{p-\beta}} \sum_{i=0}^{n}\sum_{j=0}^{\infty} \frac{\Gamma(i+j+1)\Gamma(p-\beta)}{\Gamma(p+i+j+1-\beta)} \times \frac{(-n)_i(n+1)_i(\beta)_j(1-\beta)_j}{i!\,j!\,(i+j)!\,(-1)^j} \frac{(1-\eta_0)^{i+j}}{(\eta_0-\xi_0)^{j+\beta}(2-\xi_0-\eta_0)^i}. \quad (2.159)$$

The series (2.157) converges in $\mathcal{T}$, because it is a superposition of the convergent series $\mathcal{J}_{p,n}(\xi_0,\eta_0)$ and $-\mathcal{I}_{p,n}(\xi_0,\eta_0)$.

Simplifying the coefficients in the series (2.159), $\mathcal{J}_{p,n}(\xi_0,\eta_0)$ becomes

$$\mathcal{J}_{p,n}(\xi_0,\eta_0) = (1-\xi_0)^{-p+\beta} \sum_{i=0}^{n}\sum_{j=0}^{\infty} \frac{(-n)_i(n+1)_i(\beta)_j(1-\beta)_j}{(p+1-\beta)_{i+j}\,i!\,j!\,(-1)^j} \times \frac{(1-\eta_0)^{i+j}}{(\eta_0-\xi_0)^{j+\beta}(2-\xi_0-\eta_0)^i} \quad (2.160)$$

or

$$\mathcal{J}_{p,n}(\xi_0,\eta_0) = \frac{(1-\xi_0)^{-p+\beta}}{(\eta_0-\xi_0)^{\beta}} \sum_{i=0}^{n} \frac{(-n)_i(n+1)_i}{(p+1-\beta)_i\,i!} \frac{(1-\eta_0)^i}{(2-\xi_0-\eta_0)^i} \times {}_2F_1\left(\beta, 1-\beta, p+i+1-\beta; \frac{\eta_0-1}{\eta_0-\xi_0}\right).$$

Now, applying (A.14), we derive

$$\mathcal{J}_{p,n}(\xi_0,\eta_0) = (1-\xi_0)^{-p} \sum_{i=0}^{n} \frac{(-n)_i(n+1)_i}{(p+1-\beta)_i\,i!} \frac{(1-\eta_0)^i}{(2-\xi_0-\eta_0)^i} \times {}_2F_1\left(\beta, p+i, p+i+1-\beta; \frac{1-\eta_0}{1-\xi_0}\right), \quad (2.161)$$

which actually analytically continues the function $\mathcal{J}_{p,n}(\xi_0,\eta_0)$ at the points $(\xi_0,\eta_0)\in D\backslash\mathcal{T}$.

Denoting for shortness

$$\omega := \frac{1-\eta_0}{2-\xi_0-\eta_0}, \qquad \sigma := \frac{1-\eta_0}{1-\xi_0} \tag{2.162}$$

and applying (A.5), we have:

$$\mathcal{J}_{p,n}(\xi_0,\eta_0) = \frac{(1-\beta)_p(1-\xi_0)^{-p}}{\Gamma(p)} \times \sum_{i=0}^{n} \frac{(-n)_i(n+1)_i}{(p)_i\, i!}\omega^i \int_0^1 t^{p+i-1}(1-t)^{-\beta}(1-\sigma t)^{-\beta}\, dt \tag{2.163}$$

and, consequently,

$$\mathcal{J}_{p,n}(\xi_0,\eta_0) = \frac{(1-\beta)_p(1-\xi_0)^{-p}}{\Gamma(p)} \times \int_0^1 {}_2F_1(n+1,-n,p;\omega t)\, t^{p-1}(1-t)^{-\beta}(1-\sigma t)^{-\beta}\, dt.$$

Applying the auto transformation formula (A.13), we obtain

$$\mathcal{J}_{p,n}(\xi_0,\eta_0) = \frac{(1-\beta)_p(1-\xi_0)^{-p}}{\Gamma(p)} \times \int_0^1 {}_2F_1(p-n-1,p+n,p;\omega t)\,(t-\omega t^2)^{p-1}(1-t)^{-\beta}(1-\sigma t)^{-\beta}\, dt.$$

This makes possible to use the quadratic transformation (A.17)-(A.19) to derive

$$\mathcal{J}_{p,n}(\xi_0,\eta_0) = \frac{h_2(1-\beta)_p}{\Gamma(p)(1-\xi_0)^p} \sum_{i=0}^{(n-p)/2} \frac{(\frac{p+n+1}{2})_i(\frac{p-n}{2})_i}{(3/2)_i\, i!} \\ \times \int_0^1 (t-\omega t^2)^{p-1}(1-2\omega t)^{2i+1}(1-t)^{-\beta}(1-\sigma t)^{-\beta}\, dt$$

with

$$h_2 := \frac{\Gamma(-1/2)\Gamma(p)}{\Gamma(\frac{p-n-1}{2})\Gamma(\frac{p+n}{2})},$$

where we took into account that $n-p$ is an even number.

Note that σ and ω defined by (2.162) are connected by the relation

$$\sigma = \frac{\omega}{1-\omega}$$

and, also, $0 < \omega < 1/2$. Therefore we may apply Lemma 2.11.1 with $\alpha = -\beta$ to obtain

$$\mathcal{J}_{p,n}(\xi_0,\eta_0) = \frac{h_2(1-\beta)_p}{(1-\xi_0)^p} \sum_{i=0}^{(n-p)/2} \frac{(\frac{p+n+1}{2})_i(\frac{p-n}{2})_i}{(3/2)_i} \\ \times \frac{(1-\omega)^p}{(1-\beta)_{i+p}} \sum_{s=0}^{i} \frac{(1-\beta)_{i-s}}{(i-s)!}\frac{(p)_s}{s!}(1-2\omega)^{2s}.$$

Noting that

$$\frac{1-\omega}{1-\xi_0} = \frac{1}{2-\xi_0-\eta_0},$$

we come to

$$\mathcal{J}_{p,n}(\xi_0,\eta_0) = \frac{h_2}{(2-\xi_0-\eta_0)^p} \sum_{s=0}^{(n-p)/2} \frac{(p)_s}{s!}(1-2\omega)^{2s} \times \sum_{i=s}^{(n-p)/2} \frac{(1-\beta)_{i-s}}{(i-s)!} \frac{(\frac{p+n+1}{2})_i(\frac{p-n}{2})_i}{(3/2)_i(1-\beta+p)_i}. \quad (2.164)$$

Next, using (A.1), we have:

$$\sum_{i=s}^{(n-p)/2} \frac{(1-\beta)_{i-s}}{(i-s)!} \frac{(\frac{p+n+1}{2})_i(\frac{p-n}{2})_i}{(3/2)_i(1-\beta+p)_i} = \frac{(\frac{p+n+1}{2})_s(\frac{p-n}{2})_s}{(3/2)_s(1-\beta+p)_s} \sum_{j=0}^{\frac{n-p-2s}{2}} \frac{(1-\beta)_j}{(j)!} \frac{(\frac{2s+p+n+1}{2})_j(\frac{2s+p-n}{2})_j}{(s+3/2)_j(s+1-\beta+p)_j}. \quad (2.165)$$

For the generalized hypergeometric series

$$\sum_{j=0}^{\frac{n-p-2s}{2}} \frac{(1-\beta)_j}{(j)!} \frac{(\frac{2s+p+n+1}{2})_j(\frac{2s+p-n}{2})_j}{(s+3/2)_j(s+1-\beta+p)_j} = {}_3F_2\left(1-\beta, \frac{2s+p+n+1}{2}, \frac{2s+p-n}{2}; s+1-\beta+p, s+\frac{3}{2}; 1\right) \quad (2.166)$$

we can apply the Saalschutz's Theorem A.0.1 (see page 146) to obtain

$${}_3F_2\left(1-\beta, \frac{2s+p+n+1}{2}, \frac{2s+p-n}{2}; s+1-\beta+p, s+\frac{3}{2}; 1\right) = \frac{\Gamma(1-\beta+p+s)\Gamma(\frac{1}{2}-\beta-s)\Gamma(\frac{p-n-1}{2})\Gamma(\frac{p+n}{2})}{\Gamma(1-\beta+\frac{p+n}{2})\Gamma(\frac{1}{2}-\beta+\frac{p-n}{2})\Gamma(-s-\frac{1}{2})\Gamma(p+s)}. \quad (2.167)$$

Substituting (2.165)-(2.167) into (2.164) and simplifying, we come to

$$\mathcal{J}_{p,n}(\xi_0,\eta_0) = \frac{K_{p,n}}{(2-\xi_0-\eta_0)^p}\sum_{s=0}^{(n-p)/2}\frac{(\frac{p+n+1}{2})_s(\frac{p-n}{2})_s}{(\beta+1/2)_s\, s!}(1-2\omega)^{2s}$$
$$= \frac{K_{p,n}}{(2-\xi_0-\eta_0)^p}\,{}_2F_1\left(\frac{p+n+1}{2},\frac{p-n}{2},\beta+\frac{1}{2};\frac{(\eta_0-\xi_0)^2}{(2-\xi_0-\eta_0)^2}\right) \quad (2.168)$$

with

$$K_{p,n} = \frac{\Gamma(1-\beta+p)\Gamma(\frac{1}{2}-\beta)}{\Gamma(1-\beta+\frac{p+n}{2})\Gamma(\frac{1}{2}-\beta+\frac{p-n}{2})} \neq 0. \quad (2.169)$$

Finally, substituting (2.168) into (2.158), we complete the proof. □

Lemma 2.11.5. *Let* $(\xi_0,\eta_0)\in D$*,* $0<\beta<1$*,* $p=0$ *and* n *be an even number. Then formula* (2.156) *is still valid, i.e.*

$$\mathcal{I}_{0,n}(\xi_0,\eta_0) = -\beta\int_0^{\xi_0}(1-\xi)^{-1}\mathcal{R}(\xi,1;\xi_0,\eta_0)\,d\xi$$
$$= K_{0,n}\,{}_2F_1\left(\frac{n+1}{2},\frac{-n}{2},\beta+\frac{1}{2};\frac{(\eta_0-\xi_0)^2}{(2-\xi_0-\eta_0)^2}\right) - H_{0,n}(\xi_0,\eta_0), \quad (2.170)$$

where $H_{p,n}(\xi_0,\eta_0)$ *is the function from Lemma 2.11.4 and* $K_{p,n}$ *are the coefficients* (2.169).

Proof. We can repeat the calculations in the proof of Lemma 2.11.4 with $p=0$ up to formula (2.161). Formula (2.163), obviously, is not well defined for $p=0$. The repeated calculations with $p=0$ up to formula (2.161) give

$$\mathcal{I}_{0,n}(\xi_0,\eta_0) = \Upsilon_n(\xi_0,\eta_0) - H_{0,n}(\xi_0,\eta_0),$$

where

$$\Upsilon_n(\xi_0,\eta_0) := \sum_{i=0}^{n} \frac{(-n)_i (n+1)_i}{(1-\beta)_i \, i!} \frac{(1-\eta_0)^i}{(2-\xi_0-\eta_0)^i} \, {}_2F_1\left(\beta, i, i+1-\beta; \frac{1-\eta_0}{1-\xi_0}\right).$$

We have to prove that $\Upsilon_n(\xi_0,\eta_0) = \mathcal{J}_{0,n}(\xi_0,\eta_0)$, where $\mathcal{J}_{p,n}(\xi_0,\eta_0)$ is the function (2.168).

For $n = 0$ we have obviously that $\Upsilon_0(\xi_0,\eta_0) = \mathcal{J}_{0,0}(\xi_0,\eta_0) \equiv 1$. For $n = 2, 4, \ldots$, using (A.5), we may write:

$$\begin{aligned}
\Upsilon_n(\xi_0,\eta_0) &= 1 - n(n+1)\omega \\
&\quad \times \sum_{i=0}^{n-1} \frac{(-n+1)_i (n+2)_i}{(2)_i \, i!} \int_0^1 \omega^i t^i (1-t)^{-\beta} (1-\sigma t)^{-\beta} \, dt \\
&= 1 - n(n+1)\omega \int_0^1 {}_2F_1(-n+1, n+2, 2; \omega t) \, (1-t)^{-\beta} (1-\sigma t)^{-\beta} \, dt,
\end{aligned}$$

where σ and ω are given by (2.162). Next, we use the quadratic transformation (A.17)-(A.19) to obtain:

$$\begin{aligned}
\Upsilon_n(\xi_0,\eta_0) &= 1 - h^{(2)} n(n+1)\omega \\
&\quad \times \sum_{i=0}^{(n-2)/2} \frac{(\frac{n+3}{2})_i (\frac{2-n}{2})_i}{(3/2)_i \, i!} \int_0^1 (1-2\omega t)^{2i+1} (1-t)^{-\beta} (1-\sigma t)^{-\beta} \, dt
\end{aligned}$$

with

$$h^{(2)} := \frac{\Gamma(-1/2)}{\Gamma(\frac{1-n}{2})\Gamma(\frac{n+2}{2})}.$$

Now we apply Lemma 2.11.1 with $\alpha = -\beta$ and $p = 1$:

$$\begin{aligned}\Upsilon_n(\xi_0,\eta_0) &= 1 - h^{(2)} n(n+1)\omega(1-\omega) \\ &\times \sum_{i=0}^{(n-2)/2} \frac{(\frac{n+3}{2})_i(\frac{2-n}{2})_i}{(3/2)_i(1-\beta)_{i+1}} \sum_{s=0}^{i} \frac{(1-\beta)_{i-s}}{(i-s)!}(1-2\omega)^{2s} \\ &= 1 - h^{(2)}\frac{n(n+1)}{4}\Big\{1-(1-2\omega)^2\Big\} \\ &\times \sum_{i=0}^{(n-2)/2} \frac{(\frac{n+3}{2})_i(\frac{2-n}{2})_i}{(3/2)_i(1-\beta)_{i+1}} \sum_{s=0}^{i} \frac{(1-\beta)_{i-s}}{(i-s)!}(1-2\omega)^{2s}.\end{aligned}$$

Next, rearranging the above expression, we find that

$$\Upsilon_n(\xi_0,\eta_0) = \sum_{s=0}^{n/2} a_s(1-2\omega)^{2s} \tag{2.171}$$

with

$$a_0 := 1 - h^{(2)}\frac{n(n+1)}{4}\sum_{i=0}^{(n-2)/2} \frac{(\frac{n+3}{2})_i(\frac{2-n}{2})_i}{(3/2)_i(1-\beta)_{i+1}}\frac{(1-\beta)_i}{i!},$$

$$a_{n/2} := h^{(2)}\frac{n(n+1)}{4}\frac{(\frac{n+3}{2})_{(n-2)/2}(\frac{2-n}{2})_{(n-2)/2}}{(3/2)_{(n-2)/2}(1-\beta)_{n/2}} \tag{2.172}$$

and

$$\begin{aligned}a_s := -h^{(2)}\frac{n(n+1)}{4}&\Bigg\{\sum_{i=s}^{(n-2)/2} \frac{(\frac{n+3}{2})_i(\frac{2-n}{2})_i}{(3/2)_i(1-\beta)_{i+1}} \\ &\times\left(\frac{(1-\beta)_{i-s}}{(i-s)!} - \frac{(1-\beta)_{i-s+1}}{(i-s+1)!}\right) - \frac{(\frac{n+3}{2})_{s-1}(\frac{2-n}{2})_{s-1}}{(3/2)_{s-1}(1-\beta)_s}\Bigg\}\end{aligned}$$

for $s = 1, \dots, (n-2)/2$.

Taking into account that

$$\frac{(1-\beta)_{i-s}}{(i-s)!} - \frac{(1-\beta)_{i-s+1}}{(i-s+1)!} = \beta\,\frac{(1-\beta)_{i-s}}{(i-s+1)!}, \tag{2.173}$$

for $s = 1, \ldots, (n-2)/2$ we derive

$$\begin{aligned} a_s = {} & h^{(2)}\frac{n(n+1)}{4}\frac{(\frac{n+3}{2})_{s-1}(\frac{2-n}{2})_{s-1}}{(3/2)_{s-1}(1-\beta)_s} \\ & \times {}_3F_2\left(-\beta, \frac{2s+n+1}{2}, \frac{2s-n}{2}; s+1-\beta, s+\frac{1}{2}; 1\right). \end{aligned}$$

Applying here the Saalschutz's Theorem A.0.1 and simplifying, we obtain:

$$a_s = \frac{\Gamma(1-\beta)\Gamma(\frac{1}{2}-\beta)}{\Gamma(\frac{1}{2}-\beta-\frac{n}{2})\Gamma(1-\beta+\frac{n}{2})}\,\frac{(\frac{n+1}{2})_s(-\frac{n}{2})_s}{(\frac{1}{2}+\beta)_s\,s!}. \tag{2.174}$$

Next, using (2.173) with $s = 0$ we have

$$\begin{aligned} a_0 &= 1 - h^{(2)}\frac{n(n+1)}{4}\sum_{i=0}^{(n-2)/2}\frac{(\frac{n+3}{2})_i(\frac{2-n}{2})_i}{(3/2)_i(1-\beta)_{i+1}}\left(\frac{(1-\beta)_{i+1}}{(i+1)!} + \beta\,\frac{(1-\beta)_i}{(i+1)!}\right) \\ &= 1 + \frac{h^{(2)}}{2}\left[{}_2F_1\left(\frac{n+1}{2}, -\frac{n}{2}, \frac{1}{2}; 1\right) - 1\right] \\ &\qquad - \frac{h^{(2)}}{2}\left[{}_3F_2\left(-\beta, \frac{n+1}{2}, -\frac{n}{2}; 1-\beta\frac{1}{2}; 1\right) - 1\right]. \end{aligned}$$

For $\varepsilon \in (0,1)$ with use of (A.9) we calculate

$${}_2F_1\left(\frac{n+1}{2}, -\frac{n}{2}, \frac{1}{2}+\varepsilon; 1\right) = \frac{(\varepsilon-\frac{n}{2})_{n/2}}{(\varepsilon+\frac{1}{2})_{n/2}}$$

and letting $\varepsilon \to +0$ we find that

$$ {}_2F_1\left(\frac{n+1}{2}, -\frac{n}{2}, \frac{1}{2}; 1\right) = \frac{-2}{h^{(2)}}. $$

Then, using once again the Saalschutz's Theorem and simplifying, for a_0 we obtain the following value:

$$ a_0 = \frac{\Gamma(1-\beta)\Gamma(\frac{1}{2}-\beta)}{\Gamma(\frac{1}{2}-\beta-\frac{n}{2})\Gamma(1-\beta+\frac{n}{2})}. \tag{2.175} $$

For the coefficient $a_{n/2}$ from (2.172) we get:

$$ a_{n/2} = \frac{\Gamma(1-\beta)\Gamma(\frac{1}{2}-\beta)}{\Gamma(\frac{1}{2}-\beta-\frac{n}{2})\Gamma(1-\beta+\frac{n}{2})} \frac{(\frac{n+1}{2})_{n/2}(-\frac{n}{2})_{n/2}}{(\frac{1}{2}+\beta)_{n/2}\,(\frac{n}{2})!}. \tag{2.176} $$

Finally, putting (2.174), (2.175) and (2.176) into (2.171), we see that

$$ \begin{aligned} \Upsilon_n(\xi_0,\eta_0) = {}& \frac{\Gamma(1-\beta)\Gamma(\frac{1}{2}-\beta)}{\Gamma(\frac{1}{2}-\beta-\frac{n}{2})\Gamma(1-\beta+\frac{n}{2})} \\ & \times {}_2F_1\left(\frac{n+1}{2}, -\frac{n}{2}, \frac{1}{2}+\beta; \frac{(\eta_0-\xi_0)^2}{(2-\xi_0-\eta_0)^2}\right) = \mathcal{J}_{0,n}(\xi_0,\eta_0), \end{aligned} $$

which completes the proof. $\square$

Lemma 2.11.6. *Let $(\xi_0,\eta_0) \in D$, $0<\beta<1$, $p \in \mathbb{N}\cup\{0\}$, $p \le n$ and p be of the same parity as n. Next, define the function*

$$ \begin{aligned} I^0_{p,n}(\xi_0,\eta_0) := (p-\beta) &\int_0^{\xi_0} \varphi_p(\xi)(1-\xi)^{-p-1}\mathcal{R}(\xi,1;\xi_0,\eta_0)\,d\xi \\ &+ \int_0^{\xi_0} \varphi_p'(\xi)(1-\xi)^{-p}\,\mathcal{R}(\xi,1;\xi_0,\eta_0)\,d\xi, \end{aligned} $$

where $\varphi_p \in C^1([0,1))$ *and for* $\xi \in [0,1)$

$$|\varphi_p(\xi)| \leq M_\varphi (1-\xi)^{p+1-\beta}, \qquad |\varphi_p'(\xi)| \leq M_\varphi (1-\xi)^{p-\beta} \tag{2.177}$$

with a positive constant M_φ. *Then*

$$I^0_{p,n}(\xi_0,\eta_0) = J^0_{p,n}(\xi_0,\eta_0) - \varphi_p(0)\, H_{p,n}(\xi_0,\eta_0)$$

where $H_{p,n}(\xi_0,\eta_0)$ *is the function from Lemmas 2.11.4-2.11.5 and* $J^0_{p,n}(\xi_0,\eta_0)$ *satisfies the following estimate*:

$$|J^0_{p,n}(\xi_0,\eta_0)| \leq C M_\varphi (1-\xi_0)^{1-\beta} \big(1+|\ln(1-\xi_0)|\big), \qquad (\xi_0,\eta_0) \in \mathcal{T}, \tag{2.178}$$

with a positive constant C. *Further*:

(i) *if* $0 < \beta < 1/2$, *then*

$$|J^0_{p,n}(\xi_0,\eta_0)| \leq C M_\varphi (1-\xi_0)^{1-\beta} \big(1+|\ln(1-\xi_0)|\big), \qquad (\xi_0,\eta_0) \in D; \tag{2.179}$$

(ii) *if* $1/2 \leq \beta < 1$, *then*

$$|J^0_{p,n}(\xi_0,\eta_0)| \leq C M_\varphi \frac{(1-\xi_0)^{1-\beta+\varepsilon}}{(\eta_0-\xi_0)^{\varepsilon}} \big(1+|\ln(1-\xi_0)|\big), \quad (\xi_0,\eta_0) \in D \tag{2.180}$$

with some $\varepsilon \in (0,\beta)$.

Additionally, this result holds for $p = 0$ *also in the case when* n *is an odd number.*

Proof. First, consider the integral

$$S^0_{i,j}(\xi_0) := \int_0^{\xi_0} \varphi_p(\xi)(1-\xi)^{-i-j-p-1+\beta}(\xi_0-\xi)^{i+j}\, d\xi.$$

According to (A.11) we have

$$(\xi_0-\xi)^{i+j} = (1-\xi)^{i+j}\sum_{s=0}^{i+j}\frac{(-i-j)_s}{s!}\frac{(1-\xi_0)^s}{(1-\xi)^s} \tag{2.181}$$

and hence we may write:

$$S^0_{i,j}(\xi_0) = \sum_{s=0}^{i+j}\frac{(-i-j)_s}{s!}\frac{(1-\xi_0)^s}{p+s-\beta}\int_0^{\xi_0}\varphi_p(\xi)\frac{d}{d\xi}(1-\xi)^{-p-s+\beta}\, d\xi.$$

An integration by parts gives:

$$\begin{aligned} S^0_{i,j}(\xi_0) = &-\sum_{s=0}^{i+j}\frac{(-i-j)_s}{s!}\frac{(1-\xi_0)^s}{p+s-\beta}\int_0^{\xi_0}\varphi'_p(\xi)(1-\xi)^{-p-s+\beta}\, d\xi \\ &+\varphi_p(\xi_0)\sum_{s=0}^{i+j}\frac{(-i-j)_s}{s!}\frac{(1-\xi_0)^{-p+\beta}}{p+s-\beta} - \varphi_p(0)\sum_{s=0}^{i+j}\frac{(-i-j)_s}{s!}\frac{(1-\xi_0)^s}{p+s-\beta}. \end{aligned}$$

Taking into account that $(p-\beta+s)(p-\beta)_s = (p-\beta)(p-\beta+1)_s$

and using (A.9) we come to

$$S^0_{i,j}(\xi_0) = \frac{-1}{p-\beta}\left\{\int_0^{\xi_0} \varphi_p'(\xi)(1-\xi)^{-p+\beta} \right.$$
$$\times\ {}_2F_1\left(p-\beta, -i-j, p-\beta+1; \frac{1-\xi_0}{1-\xi}\right) d\xi$$
$$- \varphi_p(\xi_0)(1-\xi_0)^{-p+\beta}\frac{(i+j)!}{(p-\beta+1)_{i+j}}$$
$$\left. + \varphi_p(0)\,{}_2F_1\left(p-\beta, -i-j, p-\beta+1; 1-\xi_0\right)\right\}. \quad (2.182)$$

Now, suppose that $(\xi_0, \eta_0) \in \mathcal{T}$. Then, representing the function $\mathcal{R}(\xi, \eta; \xi_0, \eta_0)$ as $\Psi^+(\xi, \eta; \xi_0, \eta_0)$, according to (2.134) we write

$$I^0_{p,n}(\xi_0, \eta_0) =$$
$$\sum_{i=0}^{n}\sum_{j=0}^{\infty} \frac{(-n)_i(n+1)_i(\beta)_j(1-\beta)_j}{i!\,j!\,(i+j)!\,(-1)^j}\frac{(1-\eta_0)^{i+j}}{(2-\xi_0-\eta_0)^i(\eta_0-\xi_0)^{j+\beta}}$$
$$\times\left\{(p-\beta)\,S^0_{i,j}(\xi_0) + \int_0^{\xi_0} \varphi_p'(\xi)(1-\xi)^{-i-j-p+\beta}(\xi_0-\xi)^{i+j}\,d\xi\right\}.$$

Substituting here (2.182) and using once again (2.181) we derive:

$$I^0_{p,n}(\xi_0, \eta_0) = Q^0_{p,n}(\xi_0, \eta_0) + \varphi_p(\xi_0)\,\mathcal{J}_{p,n}(\xi_0, \eta_0) - \varphi_p(0)\,H_{p,n}(\xi_0, \eta_0), \quad (2.183)$$

where $\mathcal{J}_{p,n}(\xi_0,\eta_0)$ is the series given by (2.160) and

$$
\begin{aligned}
&Q^0_{p,n}(\xi_0,\eta_0) := \\
&\quad \sum_{i=0}^{n}\sum_{j=0}^{\infty} \frac{(-n)_i(n+1)_i(\beta)_j(1-\beta)_j}{i!\,j!\,(i+j)!\,(-1)^j} \frac{(1-\eta_0)^{i+j}}{(2-\xi_0-\eta_0)^i(\eta_0-\xi_0)^{j+\beta}} \\
&\quad \times \int_0^{\xi_0} \varphi_p'(\xi)(1-\xi)^{-p+\beta}\left[{}_2F_1\left(p-\beta+1,-i-j,p-\beta+1;\frac{1-\xi_0}{1-\xi}\right)\right. \\
&\qquad\qquad\qquad \left. - {}_2F_1\left(p-\beta,-i-j,p-\beta+1;\frac{1-\xi_0}{1-\xi}\right)\right] d\xi.
\end{aligned}
$$

(a) In the case when p is of the same parity as n, according to Lemmas 2.11.4-2.11.5 the function $\mathcal{J}_{p,n}(\xi_0,\eta_0)$ has the representation (2.168) in D and obviously it satisfies the estimate

$$
\left|\mathcal{J}_{p,n}(\xi_0,\eta_0)\right| \le \text{const}\,(2-\xi_0-\eta_0)^{-p}, \qquad (\xi_0,\eta_0)\in D.
$$

Then, in view of (2.177), we have

$$
\left|\varphi_p(\xi_0)\,\mathcal{J}_{p,n}(\xi_0,\eta_0)\right| \le CM_\varphi\,(1-\xi_0)^{1-\beta}, \qquad (\xi_0,\eta_0)\in D
$$

and, clearly, $\varphi_p(\xi_0)\,\mathcal{J}_{p,n}(\xi_0,\eta_0)$ satisfies the same estimates (2.178)-(2.180) as the function $J^0_{p,n}(\xi_0,\eta_0)$ should satisfy.

(b) In the case when $p=0$ and n is an odd number we transform the series $\mathcal{J}_{p,n}(\xi_0,\eta_0)$ into (2.161) and with use of (A.6)-(A.8) we conclude that $\varphi_0(\xi_0)\,\mathcal{J}_{0,n}(\xi_0,\eta_0)$ satisfies the same estimates (2.178)-(2.180) as the function $J^0_{p,n}(\xi_0,\eta_0)$ should satisfy.

Next, to estimate $Q^0_{p,n}(\xi_0,\eta_0)$, subtracting term by term we derive:

$$
\begin{aligned}
{}_2F_1\left(p-\beta+1,-i-j,p-\beta+1;z\right)-{}_2F_1\left(p-\beta,-i-j,p-\beta+1;z\right)\\
=\frac{(i+j)\,z}{\beta-p-1}\,{}_2F_1\left(p-\beta+1,1-i-j,p-\beta+2;z\right)
\end{aligned}
$$

and therefore $Q^0_{p,n}(\xi_0,\eta_0)$ becomes

$$
\begin{aligned}
Q^0_{p,n}(\xi_0,\eta_0) &= \frac{1-\xi_0}{(\eta_0-\xi_0)^\beta}\\
&\times\left(S_0(\xi_0,\eta_0)+\sum_{i=1}^{n}\frac{(-n)_i(n+1)_i}{i!}\frac{(1-\eta_0)^i}{(2-\xi_0-\eta_0)^i}S_i(\xi_0,\eta_0)\right),
\end{aligned}
$$

where $S_0(\xi_0,\eta_0)$ and $S_i(\xi_0,\eta_0)$ are the series from Lemmas 2.11.2-2.11.3 with

$$
\phi(\xi):=\frac{\varphi_p'(\xi)}{\beta-p-1}(1-\xi)^{-p+\beta}\in C([0,1)).
$$

The function $\phi(\xi)$ according to (2.177) is bounded on the segment $[0,1]$.

Using the results of Lemmas 2.11.2-2.11.3 we conclude that the function

$$
J^0_{p,n}(\xi_0,\eta_0):=Q^0_{p,n}(\xi_0,\eta_0)+\varphi_p(\xi_0)\,\mathcal{J}_{p,n}(\xi_0,\eta_0)
$$

satisfies the estimates (2.178)-(2.180). Here we take into account that for $(\xi_0,\eta_0)\in\mathcal{T}$ we have $1-\eta_0<\eta_0-\xi_0$.

Then, in view of (2.183), the proof is complete. $\square$

Proof of Theorem 2.10.1. First, recall that the solution of Problem

P^G_{m2} is given by (2.133).

Taking into account the estimates (2.109), (2.115), (2.131) and (2.132) we apply here:

(a) Lemma 2.11.4 with $p = n - 2k$;

(b) Lemma 2.11.5 if n is an even number;

(c) Lemma 2.11.6 with

$$p = n - 2k, \quad \varphi_{n-2k}(\xi) = J_k^{n,\beta}(\xi), \qquad k = 0, \dots, [(n-1)/2]$$

and

$$\varphi_0(\xi) = \begin{cases} g(\xi) + J_{n/2}^{n,\beta}(\xi), & n \text{ even}, \\ g(\xi), & n \text{ odd}. \end{cases}$$

Then, noting that

$$J_k^{n,\beta}(0) = \mu_k^{n,\beta}, \qquad g(0) = 0,$$

we obtain:

$$\int_0^{\xi_0} \left(U_\xi(\xi,1) - \frac{\beta}{1-\xi} U(\xi,1) \right) \mathcal{R}(\xi,1;\xi_0,\eta_0)\, d\xi$$
$$= \sum_{k=0}^{[n/2]} \frac{\mu_k^{n,\beta} a_k^{n,\beta}}{(2-\xi_0-\eta_0)^{n-2k}}\, {}_2F_1\left(n-k+\frac{1}{2}, -k, \frac{1}{2}+\beta; \frac{(\eta_0-\xi_0)^2}{(2-\xi_0-\eta_0)^2} \right)$$
$$+ G_1^{(\beta)}(\xi_0,\eta_0), \quad (2.184)$$

where $a_k^{n,\beta} = \text{const} \neq 0$ and the function $G_1^{(\beta)}(\xi,\eta)$ satisfies the estimates

$$|G_1^{(\beta)}(\xi_0,\eta_0)| \leq K\|F\|_{C(D)}\,(1-\xi_0)^{1-\beta}\left(1+|\ln(1-\xi_0)|\right), \qquad (\xi_0,\eta_0)\in\mathcal{T}, \tag{2.185}$$

$$|G_1^{(\beta)}(\xi_0,\eta_0)| \leq K\|F\|_{C(D)}\,(1-\xi_0)^{1-\beta}\left(1+|\ln(1-\xi_0)|\right), \qquad (\xi_0,\eta_0)\in D$$

in the case $0<\beta<1/2$; (2.186)

$$|G_1^{(\beta)}(\xi_0,\eta_0)| \leq K\|F\|_{C(D)}\,\frac{(1-\xi_0)^{1-\beta+\varepsilon}}{(\eta_0-\xi_0)^{\varepsilon}}\left(1+|\ln(1-\xi_0)|\right), \quad (\xi_0,\eta_0)\in D$$

in the case $1/2\leq\beta<1$ (2.187)

with a positive constant K and some $\varepsilon\in(0,\beta)$.

Next, using the estimates for the Riemann function (2.138)-(2.140), we find that the function

$$G_2^{(\beta)}(\xi_0,\eta_0) := -\int_{\eta_0}^{1}\int_0^{\xi_0} F(\xi,\eta)\mathcal{R}(\xi,\eta;\xi_0,\eta_0)\,d\xi d\eta$$

satisfies the same estimates (2.185)-(2.187) as the function $G_1^{(\beta)}(\xi_0,\eta_0)$. Respectively, for the function

$$G^{(\beta)}(\xi,\eta) := G_1^{(\beta)}(\xi,\eta) + G_2^{(\beta)}(\xi,\eta)$$

such estimates hold as well.

To complete the proof we have to confirm that we have $G^{(\beta)}\in C(\bar{D})$ and $G^{(\beta)}(1,1)=0$ even in the case $1/2\leq\beta<1$.

On the one hand the estimate (2.187) allows the function $G^{(\beta)}(\xi,\eta)$ to have singularities on the line $\{\eta = \xi\}$, but on the other hand from Theorem 2.6.2 we know that $U(\xi,\eta) \in C(\bar{D})\backslash(1,1)$. Therefore $G^{(\beta)}(\xi,\eta) \in C(\bar{D})\backslash(1,1)$ as well. The lines

$$l_\delta := \{(\xi,\eta) : \ \eta - \xi = \delta\,(1-\xi)\}, \quad \delta \in (0,1]$$

pass through the point $(\xi,\eta) = (1,1)$ and on each of them, according to the estimate (2.187) (applied to $G^{(\beta)}$), we have

$$|G^{(\beta)}(\xi, \delta + (1-\delta)\xi)| \leq \frac{1}{\delta^{\varepsilon}}\, K\|F\|_{C(D)}\,(1-\xi)^{1-\beta}\left(1 + |\ln(1-\xi)|\right),$$

implying

$$\lim_{(\xi,\eta)\to(1,1)} G^{(\beta)}(\xi,\eta) = 0, \qquad (\xi,\eta) \in \bar{D}\backslash\{\eta = \xi\}.$$

By continuity we have that this equality holds for $(\xi,\eta) \in \bar{D}$ as well.

The proof is complete. □

The assertions in Theorem 2.2.3 and Remark 2.2.1 follow from Theorem 2.10.1 after the inverse transformation from Problem P_{m2} to Problem P_m. To obtain this, take into account that the coefficients $\mu_k^{n,\beta}$ are proportional to the coefficients $\mu_{k,s}^{n,m}$ with non-zero constants, which can be proved in analogous way as Lemma 1.7.1 in Chapter 1.

Appendix A.

Some formulas for the hypergeometric function

In the present research we widely use various well known formulas (see for example [3], [5], [48]), concerning the hypergeometric function and some of its generalizations.

A. The Pochhammer symbol. In order to operate with the hypergeometric series we need to use some basic relations for the Pochhammer symbol

$$(a)_i := \frac{\Gamma(a+i)}{\Gamma(a)}, \quad a, a+i \neq 0, -1, -2, \dots,$$

which for each $a \in \mathbb{C}$ and nonnegative integer i is also defined as

$$(a)_i = a(a+1)\dots(a+i-1), \quad i \in \mathbb{N}, \quad (a)_0 = 1.$$

Such basic relations are:

$$(a)_{i+j} = (a)_i (a+i)_j, \tag{A.1}$$

$$(a)_{i-j} = \frac{(-1)^j \, (a)_i}{(1-a-i)_j}, \tag{A.2}$$

$$(a)_{2i} = 2^{2i} \left(\frac{a}{2}\right)_i \left(\frac{a+1}{2}\right)_i, \tag{A.3}$$

$$(a)_j = (-1)^j (1-a-j)_j. \tag{A.4}$$

B. The Gauss hypergeometric series. The Gauss hypergeometric series is defined as:

$$_2F_1(a,b,c;\zeta) := \sum_{i=0}^{\infty} \frac{(a)_i (b)_i}{(c)_i \, i!} \zeta^i, \quad a,b,c \in \mathbb{C}, \quad c \neq 0,-1,-2,\ldots.$$

For $|\zeta| < 1$ this series is absolutely convergent. In the case when a or b is a nonpositive integer, $_2F_1(a,b,c;\zeta)$ becomes a polynomial and then it is well defined for each $\zeta \in \mathbb{C}$.

- In the case when $0 < \operatorname{Re} a < \operatorname{Re} c$ the hypergeometric series has the following integral representation:

$$_2F_1(a,b,c;\zeta) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}(1-\zeta t)^{-b} \, dt. \tag{A.5}$$

- The hypergeometric series may become unbounded or it may be bounded as $\zeta \to 1$ and it satisfies the following estimates:

(i) If $c-a-b = 0$, then for each $\alpha > 0$ there exists a constant $c(\alpha) > 0$

such that

$$|\,{}_2F_1(a,b,c;\zeta)| \le c(\alpha)(1-\zeta)^{-\alpha}. \tag{A.6}$$

(ii) In the case $c-a-b<0$ we have a constant $K>0$ such that:

$$|\,{}_2F_1(a,b,c;\zeta)| \le K(1-\zeta)^{c-a-b}. \tag{A.7}$$

(iii) In the case $c-a-b>0$, as well as in the case when ${}_2F_1(a,b,c;\zeta)$ is a polynomial of a bounded argument ζ, we have a constant $K>0$ such that:

$$|\,{}_2F_1(a,b,c;\zeta)| \le K. \tag{A.8}$$

In the last case the hypergeometric series can be evaluated at $\zeta=1$ as

$${}_2F_1(a,b,c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \tag{A.9}$$

provided the Gamma functions in (A.9) are well defined.

- The derivatives of ${}_2F_1(a,b,c;\zeta)$ are given by

$$\frac{d^s}{d\zeta^s}\,{}_2F_1(a,b,c;\zeta) = \frac{(a)_s(b)_s}{(c)_s}\,{}_2F_1(a+s,b+s,c+s;\zeta), \quad s=0,1,2,\ldots. \tag{A.10}$$

- Some of the important simple particular cases of the hypergeometric series are the following ones:

(i) The binomial series:

$${}_2F_1(a,b,a;\zeta) = (1-\zeta)^{-b}; \tag{A.11}$$

(ii) The function $-\zeta^{-1}\ln(1-\zeta)$:

$$ {}_2F_1(1,1,2;\zeta) = -\frac{1}{\zeta}\ln(1-\zeta). \tag{A.12}$$

C. Transformations of the hypergeometric series. In the present work we use various formulas transforming the hypergeometric series into other hypergeometric series. Some of them give an analytical continuation of ${}_2F_1(a,b,c;\zeta)$ for values of ζ with $|\zeta| \geq 1$. The hypergeometric series with its maximal possible analytical continuation outside the circle $|\zeta| < 1$ represents the ***hypergeometric function***.

- For the hypergeometric series the so called auto transformation formula is valid:

$$ {}_2F_1(a,b,c;\zeta) = (1-\zeta)^{c-a-b}\,{}_2F_1(c-a,c-b,c;\zeta). \tag{A.13}$$

- We use the next formulas, changing the argument ζ in the hypergeometric function:

$$\begin{aligned} {}_2F_1\left(a,b,c;\zeta\right) &= (1-\zeta)^{-a}\,{}_2F_1\left(a,c-b,c;\frac{\zeta}{\zeta-1}\right) \\ &= (1-\zeta)^{-b}\,{}_2F_1\left(c-a,b,c;\frac{\zeta}{\zeta-1}\right), \end{aligned} \tag{A.14}$$

$$
{}_2F_1(a,b,c;\zeta) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\,{}_2F_1(a,b,a+b-c+1;1-\zeta)
+ \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-\zeta)^{c-a-b}\,{}_2F_1(c-a,c-b,c-a-b+1;1-\zeta). \tag{A.15}
$$

• In the case when $b = 0,-1,-2,\ldots$ and, simultaneously, $a \neq 0,-1,-2,\ldots$ the changing from ζ to $(1-\zeta)^{-1}$, $\zeta \neq 1$ is given by

$$
{}_2F_1(a,b,c;\zeta) = \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(1-\zeta)^{-b}\,{}_2F_1\left(c-a,b,b-a+1;\frac{1}{1-\zeta}\right). \tag{A.16}
$$

• In the special case when $c = (a+b+1)/2$ it can be applied the so called quadratic transformation:

$$
{}_2F_1(a,b,\frac{a+b+1}{2};\zeta) = h_1\,{}_2F_1\left(\frac{a}{2},\frac{b}{2},\frac{1}{2};(1-2\zeta)^2\right)
+ h_2(1-2\zeta)\,{}_2F_1\left(\frac{a+1}{2},\frac{b+1}{2},\frac{3}{2};(1-2\zeta)^2\right), \tag{A.17}
$$

with

$$
h_1 = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{a+b+1}{2})}{\Gamma(\frac{a+1}{2})\Gamma(\frac{b+1}{2})}, \quad h_2 = \frac{\Gamma(-\frac{1}{2})\Gamma(\frac{a+b+1}{2})}{\Gamma(\frac{a}{2})\Gamma(\frac{b}{2})}, \tag{A.18}
$$

where we additionally define:

$$
\frac{1}{\Gamma(k)} := \lim_{k\to\varepsilon}\frac{1}{\Gamma(k+\varepsilon)} = 0, \qquad \text{for} \quad k = 0,-1,-2,\ldots. \tag{A.19}
$$

D. Some generalizations of the hypergeometric series. In the present research we also use:

(i) The generalized hypergeometric series

$$ {}_3F_2(a,b,c;d,e;\zeta) := \sum_{i=0}^{\infty} \frac{(a)_i(b)_i(c)_i}{(d)_i(e)_i\, i!}\zeta^i $$

with $a,b,c,d,e \in \mathbb{C}$ and $d,e \neq 0,-1,-2,\ldots$;

(ii) The hypergeometric series of two variables

$$ F_1(a,b_1,b_2,c;x,y) := \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(a)_{i+j}(b_1)_j(b_2)_i}{(c)_{i+j}\, i!\, j!} x^j y^i, \tag{A.20} $$

$$ F_3(a_1,a_2,b_1,b_2,c;x,y) := \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(a_1)_j(a_2)_i(b_1)_j(b_2)_i}{(c)_{i+j}\, i\,!j!} x^j y^i, \tag{A.21} $$

$$ H_2(a_1,a_2,b_1,b_2,c;x,y) := \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(a_1)_{j-i}(a_2)_j(b_1)_i(b_2)_i}{(c)_j\, i!\, j!} x^j y^i \tag{A.22} $$

with $a,a_1,a_2,b_1,b_2,c \in \mathbb{C}$ and $c \neq 0,-1,-2,\ldots$.

- The Saalschutz's Theorem asserts:

Theorem A.0.1. *If a, or b, or c is a nonpositive integer and $a+b+c+1 = d+e$, then:*

$$ {}_3F_2(a,b,c;d,e;1) = \frac{\Gamma(d)\Gamma(1+a-e)\Gamma(1+b-e)\Gamma(1+c-e)}{\Gamma(1-e)\Gamma(d-a)\Gamma(d-b)\Gamma(d-c)}. $$

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