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Improved Asymptotic Representation of the Singular Solutions of a 4-D Problem for Keldysh-Type Equations

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Abstract. We study a boundary value problem for a multidimensional weakly hyperbolic Keldysh-type equation (Problem PK). This problem is analogous to a similar one proposed by M. Protter in 1952, but for Tricomi-type equations known with their applications in transonic fluid dynamics. For their part, the Keldysh-type equations are known in some specific applications in plasma physics, optics and analysis on projective spaces.

Actually, Problem PK is not well-posed in the frame of the classical solvability since it has infinite-dimensional cokernel. Given this fact, in [16] we proved the existence and uniqueness of a generalized solution of this problem at certain conditions. This solution, in the general case, has a strong singularity at one boundary point and recently we also derived an asymptotic formula for the generalized solution's behavior on a boundary surface containing the singular point ([17]).

In the present paper it is found an extension of this formula in the whole region where Problem PK is stated.

INTRODUCTION

Denoting the points in $\mathbb{R}^4$ as $(x,t) := (x_1, x_2, x_3, t)$, for $m \in \mathbb{R}$, $0 < m < 2$ we consider a boundary value problem for a weakly hyperbolic Keldysh-type equation, degenerating on the boundary $\{t = 0\}$:

Problem PK. *Find a solution of the equation*

$$L_m[u] \equiv u_{x_1x_1} + u_{x_2x_2} + u_{x_3x_3} - (t^m u_t)_t = f(x,t) \quad \text{in} \quad \Omega_m, \tag{1}$$

satisfying the boundary conditions

$$u|_{\Sigma_1^m} = 0, \qquad t^m u_t \to 0 \text{ as } t \to +0, \tag{2}$$

where the region Ω_m *is given by*

$$\Omega_m := \left\{(x,t): \quad 0 < t < t_0, \quad \frac{2}{2-m} t^{\frac{2-m}{2}} < |x| < 1 - \frac{2}{2-m} t^{\frac{2-m}{2}}\right\}, \qquad t_0 = \left(\frac{2-m}{4}\right)^{2/(2-m)}$$

and it is bounded by the ball

$$\Sigma_0 := \left\{(x,t): \ t = 0, \ \ |x| < 1\right\}$$

and by the following two characteristic surfaces of equation (1):

$$\Sigma_1^m := \left\{(x,t) : 0 < t < t_0, \ |x| = 1 - \frac{2}{2-m} t^{\frac{2-m}{2}}\right\},$$

$$\Sigma_2^m := \left\{(x,t) : 0 < t < t_0, \ |x| = \frac{2}{2-m} t^{\frac{2-m}{2}}\right\}.$$

Remark 1. *Here as usual* $|x| = \sqrt{x_1^2 + x_2^2 + x_3^2}$.

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Here we also formulate the adjoint problem of PK:

Problem PK*. *Find a solution to the self-adjoint equation* (1) *in* Ω_m *which satisfies the boundary conditions*

$$u|_{\Sigma_2^m} = 0, \qquad t^m u_t \to 0 \text{ as } t \to +0.$$

The statement of Problem PK is analogous to the statement of some boundary value problems proposed by Protter [20] for Tricomi-type weakly hyperbolic equation. Actually, Protter arrived to his problems while studied some multidimensional variants of the famous Guderley-Morawetz problem for mixed-type equations and restricted his investigation in the hyperbolic part of the domain in which these problems were stated. The Guderley-Morawetz problem is known with its important application in transonic fluid dynamics (see for example [1, 12]).

However, a specific feature of the Keldysh-type equations is that their solutions are not differentiable at the degenerate boundary $\{t = 0\}$ (see [2]). Then, in contrary to the Protter problems for Tricomi-type equations, we cannot prescribe Neumann boundary data on $\{t = 0\}$. Indeed, in our Problem PK we have no data on the ball Σ_0. Instead of this, we have only a limitation on the growth of the possible singularity of the derivative u_t, imposed by the second condition in (2).

Nevertheless, in our recent works [16] and [17] we find some essential similarities between Problem PK and the Protter problems for Tricomi-type equations: the infinite-dimensional co-kernel of these problems and the existence of generalized solutions with strong singularities isolated at one boundary point. This is considered in more detail in the next sections.

The Keldysh-type equations are known in some specific applications in plasma physics, optics and analysis on projective spaces ([11, 14, 15]). Different problems for Keldysh-type equations are studied in [5, 8, 9, 21, 22]. Boundary value problems for Keldysh-type equations with arbitrary manifold of type switch are considered in [3, 4]. A three-dimensional analogue of Problem PK involving lower order terms is studied in [7], where an uniqueness result was proved.

SOME RECENT RESULTS

Ill-Posedness of Problem PK

The adjoint homogeneous Problem PK^* has infinitely many linearly independent nontrivial classical solutions.

Indeed, for $k, n \in \mathbb{N} \cup \{0\}$ define the functions

$$E_k^{n,m}(|x|, t) := \sum_{i=0}^{k} A_i^{k,m} |x|^{-n+2i-1} \left(|x|^2 - \frac{4}{(2-m)^2} t^{2-m} \right)^{n-k-i-\frac{m}{2(2-m)}}$$

with

$$A_i^{k,m} := (-1)^i \frac{(k-i+1)_i (n-k-i+(4-3m)/(4-2m))_i}{i!(n+1/2-i)_i}.$$

Further, denote by $Y_n^s(x)$, $n \in \mathbb{N} \cup \{0\}$, $s = 1, 2, \ldots, 2n+1$ the three-dimensional spherical functions. They are usually defined on the unit sphere $S^2 := \{x \in \mathbb{R}^3 : |x| = 1\}$, but for convenience of our discussions we extend them out of S^2 radially, keeping the same notation for the extended functions:

$$Y_n^s(x) := Y_n^s(x/|x|), \quad x \in \mathbb{R}^3 \setminus \{(0,0,0)\}.$$

Then the following lemma holds:

Lemma 1 ([17])**.** *For all* $m \in \mathbb{R}$, $0 < m < 2$, $k, n \in \mathbb{N} \cup \{0\}$, $n > N(m,k) := 2k + 1 + m/(2-m)$ *and* $s = 1, 2, \ldots, 2n+1$, *the functions*

$$v_{k,s}^{n,m}(x,t) := \begin{cases} E_k^{n,m}(|x|, t) Y_n^s(x), & (x,t) \neq O, \\ 0, & (x,t) = O, \end{cases}$$

with $O := (0,0,0,0)$*, are classical solutions of the homogeneous Problem* PK^*.

Consequently, a necessary condition for the existence of a classical solution of Problem PK is the orthogonality of the right-hand side function $f(x,t)$ to all these functions $v_{k,s}^{n,m}(x,t)$. Respectively, an infinite number of orthogonality conditions $\mu_{k,s}^{n,m} = 0$ with

$$\mu_{k,s}^{n,m} := \int_{\Omega_m} v_{k,s}^{n,m}(x,t) f(x,t)\, dxdt \tag{3}$$

must be fulfilled.

Similar facts firstly were found out for the Protter problems for the multidimensional wave equation ([23]) and for the Tricomi-type equations ([19], [10]).

Generalized solvability of Problem PK

According to this situation, it is necessary to consider solutions of this problem in a generalized sense. In the case $0 < m < 4/3$ we succeed to obtain existence and uniqueness results for a generalized solution of Problem PK defined in the following way:

Definition 1 ([16])**.** *We call a function $u(x,t)$ a generalized solution of Problem PK in Ω_m, $0 < m < 4/3$, for equation* (1) if:

(1) $u, u_{x_j} \in C(\bar{\Omega}_m \setminus O)$, $j = 1,2,3$, $u_t \in C(\bar{\Omega}_m \setminus \bar{\Sigma}_0)$;

(2) $u|_{\Sigma_1^m} = 0$;

(3) *For each $\varepsilon \in (0,1)$ there exists a constant $C(\varepsilon) > 0$, such that*

$$|u_t(x,t)| \le C(\varepsilon) t^{-\frac{3m}{4}} \quad \text{in} \quad \Omega_m \cap \{|x| > \varepsilon\}\,; \tag{4}$$

(4) *The identity*

$$\int_{\Omega_m} \{t^m u_t v_t - u_{x_1} v_{x_1} - u_{x_2} v_{x_2} - u_{x_3} v_{x_3} - f v\}\, dxdt = 0$$

holds for all v from

$$V_m := \left\{ v(x,t) :\ v \in C^2(\bar{\Omega}_m),\ v|_{\Sigma_2^m} = 0,\ v \equiv 0 \ \text{ in a neighborhood of } \ O \right\}.$$

We mention that the inequality (4) restricts the generalized solution's function space to a class which is smaller than it is allowed by the second boundary condition in (2).

Theorem 1([16])**.** *If $m \in (0, 4/3)$, then there exists at most one generalized solution of Problem PK in Ω_m.*

Theorem 2 ([16])**.** *Let $m \in (0, 4/3)$. Suppose that the right-hand side function $f(x,t)$ is of the form*

$$f(x,t) = \sum_{n=0}^{l} \sum_{s=1}^{2n+1} f_n^s(|x|,t) Y_n^s(x), \tag{5}$$

and $f \in C^1(\bar{\Omega}_m)$. Then there exists an unique generalized solution $u(x,t)$ of Problem P_m in Ω_m and it has the form

$$u(x,t) = \sum_{n=0}^{l} \sum_{s=1}^{2n+1} u_n^s(|x|,t) Y_n^s(x).$$

Singular solutions of Problem PK

Note that Definition 1 allows the generalized solutions to have some singularity at the point O. Indeed, in [17] it was found that there exist such singular solutions of this problem and it was derived their asymptotic expansion on the cone Σ_2^m in negative powers of $|x|$.

THE NEW RESULT

Here we improve the derived asymptotic expansion of the generalized solutions:

Theorem 3. *Let $m \in (0, 4/3)$ and the right-hand side function $f \in C^1(\bar{\Omega}_m)$ has the form* (5)*. Then the unique generalized solution $u(x,t)$ of problem PK has the following expansion at the point O*:

$$u(x,t) = \sum_{p=0}^{l} F_p^m(x,t)|x|^{-p-1} + F^{(m)}(x,t)|x|^{-1},$$

where:

(i) *The function $F^{(m)}(x,t) \in C^1(\Omega) \cap C(\bar{\Omega}_m)$ and $F^{(m)}(0,0,0,0) = 0$;*

(ii) *In the case $0 < m < 1$ the function $F^{(m)}(x,t)$ satisfies in Ω_m the a priori estimate*

$$|F^{(m)}(x,t)| \le C\|f\|_{C(\Omega_m)}|x|^{1-\beta}\left(1 + \left|\ln|x|\right|\right), \qquad \beta = \frac{m}{2(2-m)}$$

with a constant $C > 0$ independent of f, and in the case $1 \le m < 4/3$ such an estimate holds at least in the subset

$$\Omega_m \cap \left\{ |x| < \frac{6}{2-m} t^{\frac{2-m}{2}} \right\};$$

(iii) *The functions $F_p^m(x,t),\ p = 0, \dots, l$ have the following structure*:

$$F_p^m(x,t) = \sum_{k=0}^{[(l-p)/2]} \sum_{s=1}^{2p+4k+1} c_k^{p+2k,m} \mu_{k,s}^{p+2k,m} H_{k,s}^{p+2k,m}(x,t), \tag{6}$$

where $c_k^{p+2k,m} \neq 0$ are constants independent of $f(x,t)$ and

$$H_{k,s}^{n,m}(x,t) := {}_2F_1\left(n - k + \frac{1}{2}, -k, \frac{1}{2-m}; \frac{4t^{2-m}}{(2-m)^2\,|x|^2}\right) Y_n^s(x);$$

(iv) *If at least one of the constants $\mu_{k,s}^{p+2k,m}$ in* (6) *is different from zero, then for the corresponding function $F_p^m(x,t)$ there exists a vector $\alpha \in \mathbb{R}^3$, $|\alpha| = 1$, such that*

$$\lim_{t \to +0} F_p^m(\sigma(t), t) = \text{const} \neq 0,$$

where

$$(\sigma(t), t) := \left(\frac{2}{2-m} \alpha t^{\frac{2-m}{2}}, t \right) \in \Sigma_2^m, \quad 0 < t < t_0.$$

This means that in this case the order of singularity of $u(x,t)$ is no smaller than $p + 1$.

According to this theorem, the order of singularity of $u(x,t)$ can be strictly fixed by the coefficients (3), i.e. by choosing the right-hand side $f(x,t)$ to be orthogonal to the appropriate functions $v_{k,s}^{n,m}(x,t)$ from Lemma 1. Then the situation with this problem is not typical at all: firstly, increasing the smoothness of $f(x,t)$ does not reduce the order of singularity, since it depends only on the orthogonality conditions, and, secondly, the singularity at the point O does not propagate along the bicharacteristics, which is not traditionally assumed for the hyperbolic equations.

Analogous results firstly were established for the Protter problems for Tricomi-type equations (see for example [6, 13, 18]).

Remark 2. *Actually, the expansion found in* [17] *is a restriction on Σ_2^m of the expansion from Theorem 3. On Σ_2^m we have*:

$$H_{k,s}^{n,m}\Big|_{\Sigma_2^m} = \gamma_k^{n,m}\, Y_n^s(x), \qquad \gamma_k^{n,m} = \text{const} \neq 0.$$

Remark 3. *It is interesting that the functions* $H_{k,s}^{n,m}(x,t)$ *are connected with the functions* $v_{k,s}^{n,m}(x,t)$ *by the relation*

$$v_{k,s}^{n,m}(x,t) = K_m\, H_{k,s}^{n,m}(x,t)\, |x|^{2k-n-1} \left(|x|^2 - \frac{4}{(2-m)^2} t^{2-m} \right)^{n-2k-\frac{m}{2(2-m)}}, \qquad (x,t) \neq O,$$

where

$$K_m = \text{const} = \frac{(-1)^k \left(\frac{1}{2-m}\right)_k}{(1/2-n)_k} \neq 0.$$

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