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Protter-Morawetz Problem for a Keldysh-Type Equation with Power-Type Degeneracy of Order m = 4/3

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Abstract. In series of our works ([6, 7, 18, 19]) we considered a four-dimensional Protter-Morawetz problem for a Keldysh-type weakly hyperbolic equation with power-type degeneracy of order m, where $0 < m < 2$. It was shown that this problem is not well posed, since it has an infinite-dimensional cokernel, but it can be studied in the frame of generalized solutions with possible big singularities. However, existence and uniqueness results, as well as results on the asymptotic behavior of the singular solutions, were established only for the case $0 < m < 4/3$. In this paper we succeed to extend these results for $m = 4/3$.

STATEMENT OF THE PROBLEM

In series of our works ([6, 7, 18, 19]) we considered a four-dimensional boundary value problem (the so-called Protter-Morawetz problem) for the Keldysh-type equation

$$u_{x_1x_1} + u_{x_2x_2} + u_{x_3x_3} - (t^m u_t)_t = f(x,t) \tag{1}$$

with $m \in \mathbb{R}$, $0 < m < 2$, where we denote the points in $\mathbb{R}^4$ as $(x,t) := (x_1, x_2, x_3, t)$. It was shown that this problem is not well posed, since it has an infinite-dimensional cokernel, but it can be studied in the frame of generalized solutions with possible big singularities. However, existence and uniqueness results, as well as results on the asymptotic behavior of the singular solutions, were established only for the case $\mathbf{0 < m < 4/3}$, since the applied calculations fail for $m \geq 4/3$.

In this paper we succeed to extend our results for $\mathbf{m = 4/3}$. More precisely, here we study the following boundary value problem:

Problem PK. *Find a solution of the equation*

$$u_{x_1x_1} + u_{x_2x_2} + u_{x_3x_3} - (t^{4/3} u_t)_t = f(x,t) \quad \text{in} \quad \Omega, \tag{2}$$

satisfying the boundary conditions

$$u|_{\Sigma_1} = 0, \qquad t^{4/3} u_t \to 0 \text{ as } t \to +0, \tag{3}$$

where the region Ω *is given by*

$$\Omega := \left\{(x,t): \quad 0 < t < \frac{1}{216}, \;\; 3t^{1/3} < |x| < 1 - 3t^{1/3}\right\}, \qquad |x| = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

and it is bounded by the ball

$$\Sigma_0 := \left\{(x,t): \;\; t = 0, \;\; |x| < 1\right\}$$

and by the following two characteristic surfaces of equation (2):

$$\Sigma_1 := \left\{(x,t): 0 < t < \frac{1}{216}, \;\; |x| = 1 - 3t^{1/3}\right\},$$

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$$\Sigma_2 := \left\{(x,t) : 0 < t < \frac{1}{216},\ |x| = 3t^{1/3}\right\}.$$

The adjoint problem of PK is the following one:

Problem PK*. *Find a solution to the self-adjoint equation* (2) *in* Ω *which satisfies the boundary conditions*

$$u|_{\Sigma_2} = 0, \qquad t^{4/3}u_t \to 0 \text{ as } t \to +0.$$

SOME REMARKS ON THE PROTTER-MORAWETZ PROBLEMS

The Protter-Morawetz problems firstly were proposed by Protter [22] for Tricomi-type equations as multidimensional analogues of the famous Guderley-Morawetz problem arising in transonic fluid dynamic (see for example [1, 14]) and later they were generalized for Keldysh-type equations. For their part, the Keldysh-type equations are known in some specific applications in plasma physics, optics and analysis on projective spaces ([13, 16, 17]).

Today it is well known that the Protter-Morawetz problems are not well posed, since they have infinite-dimensional cokernels ([12, 21, 25]). For this reason these problems, as it was proposed by Popivanov and Schneider [21], are studied in the frame of generalized solutions with possible big singularities. It is well known that in the general case such singularities really exist. It is interesting that they are isolated at one boundary point and they do not propagate along the bicharacteristics, which is not traditionally assumed for the hyperbolic equations. Results on the exact asymptotic behavior of the generalized solutions of different Protter-Morawetz problems can be found for example in [8, 15, 20].

In particular, as we mentioned above, similar facts we find for the Protter-Morawetz problem for equation (1). A generalization of this problem for three-dimensional equations involving lower order terms is treated in [9], where an uniqueness result was proved.

Here we mention a specific feature in the statement of the problems for Keldysh-type equations: on the degenerate boundary $\{t = 0\}$ we have no prescribed values of the derivative u_t and instead of this we have only limitation on the growth of its singularity. Actually, it is well known that the solutions of the Keldysh-type equations are not differentiable at the degenerate boundary ([2]).

Other different boundary value problems for Keldysh-type equations and some their generalizations are studied in [3, 4, 5, 10, 11, 23, 24].

ILL-POSEDNESS OF PROBLEM PK

The adjoint homogeneous Problem PK^* has infinitely many linearly independent classical solutions. Actually, in [19] we proved this fact for the general case when we have equation (1) with $m \in (0,2)$, as well as we gave an explicit representation of these solutions. Here we will interpret this result for our case $m = 4/3$.

More precisely, for $k, n \in \mathbb{N} \cup \{0\}$ define the functions

$$\mathcal{E}_k^n(|x|,t) := \frac{(|x|^2 - 9t^{2/3})^{n-2k-1}}{|x|^{n-2k+1}}\ {}_2F_1\left(n-k+\frac{1}{2}, -k, \frac{3}{2}; \frac{9t^{2/3}}{|x|^2}\right), \tag{4}$$

where ${}_2F_1(a,b,c;z)$ is the Gauss hypergeometric series. Further, denote by $Y_n^s(x)$, $n \in \mathbb{N}\cup\{0\}$, $s = 1, 2, \ldots, 2n+1$ the three-dimensional spherical functions. They are usually defined on the unit sphere $S^2 := \{x \in \mathbb{R}^3 : |x| = 1\}$, but for convenience of our discussions we extend them out of S^2 radially, keeping the same notation for the extended functions:

$$Y_n^s(x) := Y_n^s(x/|x|), \quad x \in \mathbb{R}^3 \setminus \{(0,0,0)\}.$$

Lemma 1. *For all* $k, n \in \mathbb{N}\cup\{0\}$, $n > 2k+3$ *and* $s = 1, 2, \ldots, 2n+1$, *the functions*

$$v_{k,s}^n(x,t) := \begin{cases} \mathcal{E}_k^n(|x|,t)Y_n^s(x), & (x,t) \neq O, \\ 0, & (x,t) = O, \end{cases}$$

with $O := (0,0,0,0)$*, are classical solutions from* $C^2(\Omega) \cap C(\bar{\Omega})$ *of the homogeneous Problem* PK^*.

A necessary condition for the existence of a classical solution of Problem PK is the orthogonality of the right-hand side function $f(x,t)$ to all these functions $v^n_{k,s}(x,t)$. Respectively, an infinite number of orthogonality conditions $\mu^n_{k,s} = 0$ with

$$\mu^n_{k,s} := \int_\Omega v^n_{k,s}(x,t) f(x,t)\, dxdt \tag{5}$$

must be fulfilled.

In order to study Problem PK in the frame of generalized solvability, we will also interpret for our case the generalized solution's definition given in ([18]):

Definition 1. *We call a function $u(x,t)$ a generalized solution of Problem PK in Ω* if:
(1) $u, u_{x_j} \in C(\bar{\Omega} \setminus O)$, $j = 1,2,3$, $u_t \in C(\bar{\Omega} \setminus \bar{\Sigma}_0)$;
(2) $u|_{\Sigma_1} = 0$;
(3) *For each* $\varepsilon \in (0,1)$ *there exists a constant* $C(\varepsilon) > 0$, *such that*

$$|u_t(x,t)| \le C(\varepsilon) t^{-1} \quad \text{in} \quad \Omega \cap \{|x| > \varepsilon\}\,; \tag{6}$$

(4) *The identity*

$$\int_\Omega \{t^{4/3} u_t v_t - u_{x_1} v_{x_1} - u_{x_2} v_{x_2} - u_{x_3} v_{x_3} - f v\}\, dxdt = 0$$

holds for all v from

$$V := \left\{ v(x,t) : \ v \in C^2(\bar{\Omega}),\ v|_{\Sigma_2} = 0,\ v \equiv 0 \ \text{ in a neighborhood of } \ O \right\}.$$

This definition allows the generalized solutions to have strong singularities at the point O. Note that the inequality (6) restricts the generalized solution's function space to a class which is smaller than it is allowed by the second boundary condition in (3).

TWO-DIMENSIONAL PROBLEM CORRESPONDING TO PROBLEM PK

In the case when the right-side function $f(x,t)$ is of the form

$$f(x,t) = \sum_{n=0}^{l} \sum_{s=1}^{2n+1} f^s_n(|x|,t) Y^s_n(x) \tag{7}$$

Problem PK reduces to a two-dimensional problem. To do this, let us look for solutions of the form

$$u(x,t) = \sum_{n=0}^{l} \sum_{s=1}^{2n+1} u^s_n(|x|,t) Y^s_n(x). \tag{8}$$

Passing to the spherical coordinates $(r,\theta,\varphi,t) \in \mathbb{R}^4$, $r > 0$, $0 \le \theta < \pi$, $0 \le \varphi < 2\pi$ with

$$x_1 = r \sin\theta \cos\varphi, \quad x_2 = r \sin\theta \sin\varphi, \quad x_3 = r\cos\theta$$

and after that using the characteristic coordinates

$$\xi = 1 - r - \frac{2}{2-m} t^{\frac{2-m}{2}}, \qquad \eta = 1 - r + \frac{2}{2-m} t^{\frac{2-m}{2}},$$

for the functions

$$U(\xi,\eta) := r(\xi,\eta)\, u^s_n\big(r(\xi,\eta), t(\xi,\eta)\big)$$

the following Darboux-Goursat problem is obtained:

Problem PK$_2$. *Find a solution of the equation*

$$U_{\xi\eta} + \frac{1}{\eta-\xi}(U_\xi - U_\eta) - \frac{n(n+1)}{(2-\xi-\eta)^2}U = F(\xi,\eta) \quad \text{in} \quad D, \tag{9}$$

satisfying the following boundary conditions

$$U(0,\eta) = 0, \qquad \lim_{\eta-\xi\to+0}(\eta-\xi)^2\left(U_\xi - U_\eta\right) = 0,$$

where

$$D := \{(\xi,\eta): \ 0<\xi<\eta<1\}$$

and

$$F(\xi,\eta) := \frac{1}{8}(2-\xi-\eta)\, f_n^s\Big(r(\xi,\eta), t(\xi,\eta)\Big).$$

In conformity with [18] and with Definition 1, we define a generalized solution of Problem PK_2 in the following way:

Definition 2. *We call a function $U(\xi,\eta)$ a generalized solution of Problem PK_2 in D if*:
(1) U, $U_\xi + U_\eta \in C(\bar{D}\setminus(1,1))$, $U_\xi - U_\eta \in C(\bar{D}\setminus\{\eta=\xi\})$;
(2) $U(0,\eta) = 0$;
(3) *for each* $\varepsilon \in (0,1)$ *there exists a constant* $C(\varepsilon) > 0$, *such that*

$$|(U_\xi - U_\eta)(\xi,\eta)| \le C(\varepsilon)(\eta-\xi)^{-1} \quad \text{in} \quad D\cap\{\xi < 1-\varepsilon\};$$

(4) *the identity*

$$\int_D (\eta-\xi)^2\left\{U_\xi V_\eta + U_\eta V_\xi + \frac{2n(n+1)}{(2-\xi-\eta)^2}UV + 2FV\right\} d\xi\, d\eta = 0$$

holds for all

$$V \in V^{(2)} := \{V(\xi,\eta): \ V\in C^2(\bar{D}),\ V(\xi,1)=0, V\equiv 0 \ \text{ in a neighborhood of } \ (1,1)\}.$$

EXISTENCE AND UNIQUENESS RESULT FOR PROBLEM PK$_2$ AND ASYMPTOTIC EXPANSION OF ITS SINGULAR SOLUTIONS

To solve this problem, we find an appropriate Riemann-Hadamard function. Then we state:

Theorem 1. *Let $F \in C^1(\bar{D})$. Then there exists an unique generalized solution of Problem PK_2 and it has the following integral representation at a point $(\xi_0,\eta_0)\in D$*:

$$U(\xi_0,\eta_0) = \int_0^{\xi_0}\int_\xi^{\eta_0} \Phi(\xi,\eta;\xi_0,\eta_0)F(\xi,\eta)\, d\eta d\xi, \tag{10}$$

where the Riemann-Hadamard function $\Phi(\xi,\eta;\xi_0,\eta_0)$ is defined as

$$\Phi(\xi,\eta;\xi_0,\eta_0) := \begin{cases} \Phi^+(\xi,\eta;\xi_0,\eta_0), & \eta > \xi_0, \\ \\ \Phi^-(\xi,\eta;\xi_0,\eta_0), & \eta < \xi_0 \end{cases} \tag{11}$$

with

$$\Phi^+(\xi,\eta;\xi_0,\eta_0) := \frac{\eta-\xi}{\eta_0-\xi_0}\, {}_2F_1(n+1,-n,1;Y), \tag{12}$$

$$\Phi^-(\xi,\eta;\xi_0,\eta_0) := \frac{\eta-\xi}{\eta_0-\xi_0}\left\{ {}_2F_1(n+1,-n,1;Y) - {}_2F_1(n+1,-n,1;Y^*)\right\} \tag{13}$$

and

$$Y = Y(\xi,\eta;\xi_0,\eta_0) := \frac{-(\xi_0-\xi)(\eta_0-\eta)}{(2-\xi-\eta)(2-\xi_0-\eta_0)}, \qquad Y^* = Y(\xi,\eta;\eta_0,\xi_0) := \frac{-(\eta_0-\xi)(\xi_0-\eta)}{(2-\xi-\eta)(2-\xi_0-\eta_0)}. \tag{14}$$

Next, to investigate the asymptotic behavior of the generalized solution $U(\xi,\eta)$, we derive a special decomposition of the Riemann-Hadamard function. To give this representation, it is convenient to introduce the following functions:

$$\tilde{E}^n_k(\xi,\eta) := \frac{1}{(2-\xi-\eta)^{n-2k}}\, {}_2F_1\left(n-k+\frac{1}{2},\ -k, \frac{3}{2};\ \frac{(\eta-\xi)^2}{(2-\xi-\eta)^2}\right),$$

$$E^n_k(\xi,\eta) := (1-\xi)^{n-2k-1}(1-\eta)^{n-2k-1}\,\tilde{E}^n_k(\xi,\eta),$$

$$\tilde{Q}^n_k(\xi,\eta) := \frac{1}{(2-\xi-\eta)^{n-2k}}\, {}_2F_1\left(n-k+\frac{1}{2},\ -k, \frac{1}{2};\ \frac{(\eta-\xi)^2}{(2-\xi-\eta)^2}\right),$$

$$Q^n_k(\xi,\eta) := (1-\xi)^{n-2k}(1-\eta)^{n-2k}\,\tilde{Q}^n_k(\xi,\eta),$$

The functions $E^n_k\Big(\xi(r,t),\eta(r,t)\Big)$ obviously are proportional to the functions $r\,\mathcal{E}^n_k(r,t)$ (see (4)) and for $k = 0,1,\dots,[n/2]-1$ they solve in D equation (9) with $F(\xi,\eta)\equiv 0$, as well as they satisfy the adjoint boundary conditions

$$U(\xi,1)=0, \qquad \lim_{\eta-\xi\to+0}(\eta-\xi)^2\Big(U_\xi - U_\eta\Big) = 0.$$

Theorem 2. *The Riemann-Hadamard function* $\Phi^-(\xi,\eta;\xi_0,\eta_0)$ *given by* (11)-(14) *can be decomposed in the following way*:

$$\Phi^-(\xi,\eta;\xi_0,\eta_0) := (\eta-\xi)^2\left(\sum_{k=0}^{[(n-1)/2]} a^n_k\tilde{E}^n_k(\xi_0,\eta_0)E^n_k(\xi,\eta) + \sum_{k=0}^{[n/2]-1} a^n_k E^n_k(\xi_0,\eta_0)\tilde{E}^n_k(\xi,\eta)\right),$$

$$\Phi^+(\xi,\eta;\xi_0,\eta_0) := \frac{1}{2}\Phi^-(\xi,\eta;\xi_0,\eta_0) + \frac{\eta-\xi}{\eta_0-\xi_0}\left(\sum_{k=0}^{[n/2]} b^n_k\tilde{Q}^n_k(\xi_0,\eta_0)Q^n_k(\xi,\eta) + \sum_{k=0}^{[(n-1)/2]} b^n_k Q^n_k(\xi_0,\eta_0)\tilde{Q}^n_k(\xi,\eta)\right),$$

where a^k_n *and* b^k_n *are non-zero constants.*

Applying this expansion into (10) we come to the following theorem:

Theorem 3. *Let* $F\in C^1(\bar{D})$. *Then the unique generalized solution of Problem* PK_2 *has the following asymptotic representation at the singular point* $(1,1)$*:*

$$U(\xi,\eta) = \sum_{k=0}^{[(n-1)/2]} \mu^n_k a^n_k G^n_k(\xi,\eta)(2-\xi-\eta)^{2k-n} + G(\xi,\eta), \quad (\xi,\eta)\in D,$$

where

$$\mu^n_k := \int_D (\eta-\xi)^2 E^n_k(\xi,\eta)F(\xi,\eta)\,d\xi d\eta,$$

$$G^n_k(\xi,\eta) := {}_2F_1\left(n-k+\frac{1}{2}, -k, \frac{3}{2};\ \frac{(\eta-\xi)^2}{(2-\xi-\eta)^2}\right)$$

and $G(\xi,\eta)$ *is a bounded in* D *function.*

THE MAIN RESULTS

From the results obtained for the 2-D Problem PK_2 we obtain the following theorems for the 4-D problem PK:

Theorem 4. *There exists at most one generalized solution of Problem PK in* Ω.

Theorem 5. *Let the right-hand side function* $f(x,t)$ *be of the form* (7) *and* $f \in C^1(\bar{\Omega})$. *Then there exists an unique generalized solution* $u(x,t)$ *of Problem PK in* Ω *and it has the form* (8).

Next, we note that the coefficients μ_k^n in Theorem 3 are proportional to the scalar products $\mu_{k,s}^n$ defined by (5). Then we obtain the following expansion of $u(x,t)$ in negative powers of $|x|$:

Theorem 6. *Let the right-hand side function* $f \in C^1(\bar{\Omega})$ *has the form* (7). *Then the unique generalized solution* $u(x,t)$ *of problem PK has the following expansion at the point O*:

$$u(x,t) = \frac{1}{|x|}\left(\sum_{p=1}^{l} F_p(x,t)\,|x|^{-p} + F_0(x,t)\right),$$

where:

(i) $F_0(x,t)$ *is a bounded in* Ω *function*;

(ii) *The functions* $F_p(x,t),\ p = 1,\dots,l$ *have the following structure*:

$$F_p(x,t) = \sum_{k=0}^{[(l-p)/2]} \sum_{s=1}^{2p+4k+1} c_k^{p+2k} \mu_{k,s}^{p+2k} H_{k,s}^{p+2k}(x,t), \tag{15}$$

where $c_k^{p+2k,m} \neq 0$ *are constants independent of* $f(x,t)$ *and*

$$H_{k,s}^n(x,t) := {}_2F_1\left(n-k+\frac{1}{2}, -k, \frac{3}{2}; \frac{9t^{2/3}}{|x|^2}\right) Y_n^s(x);$$

(iii) *If at least one of the constants* $\mu_{k,s}^{p+2k}$ *in* (15) *is different from zero, then for the corresponding function* $F_p(x,t)$ *there exists a vector* $\alpha \in \mathbb{R}^3$, $|\alpha| = 1$, *such that*

$$\lim_{t\to+0} F_p(\sigma(t),t) = \text{const} \neq 0,$$

where

$$(\sigma(t),t) := \left(3\alpha t^{1/3}, t\right) \in \Sigma_2, \quad t > 0.$$

This means that in this case the order of singularity of $u(x,t)$ *is no smaller than* $p+1$.

The assertion (iii) in this theorem follows from the linear independence of the spherical functions and from the fact that $H_{k,s}^n(\sigma(t),t) = \text{const} \neq 0$.

The main conclusion from Theorem 6 is that the order of singularity of $u(x,t)$ can be strictly fixed by the coefficients $\mu_{k,s}^n$ defined by (5), i.e. by choosing the right-hand side $f(x,t)$ to be orthogonal to the appropriate functions $v_{k,s}^n(x,t)$ from Lemma 1.

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