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Cite as: AIP Conference Proceedings **2333**, 120009 (2021); https://doi.org/10.1063/5.0041745
Published Online: 08 March 2021

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Integral Form of the Generalized Solution of a Darboux-Goursat Problem with Third-Type Boundary Condition

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Abstract. In the present paper we study a Darboux-Goursat problem for a hyperbolic equation with a singular coefficient, where a third-type boundary condition on the non-characteristic segment is imposed. This problem is related to a four-dimensional Protter's problem for the wave equation and it has an unique generalized solution with possible singularity at a boundary point. We find an integral representation of this solution via Riemann-Hadamard function. The singularity of the generalized solution depends on orthogonality condition on the right-hand side of the equation in respect to nontrivial classical solutions of the corresponding adjoint homogeneous problem. Here we find an explicit formula for such nontrivial classical solutions.

STATEMENT OF THE PROBLEM

For $\alpha \in \mathrm{R}$ consider the following boundary value problem:

$$L[U] \equiv U_{\xi\eta} - \frac{2}{(2-\xi-\eta)^2} U = F(\xi,\eta) \quad \text{in} \quad D, \tag{1}$$

$$U(0,\eta) = 0, \qquad \big(U_\xi - U_\eta\big)(\xi,\xi) = \alpha U(\xi,\xi), \tag{2}$$

where

$$D := \{(\xi,\eta) : 0 < \xi < \eta < 1\}.$$

Note that equation (1) involves a coefficient with singularity at the point $(\xi,\eta) = (1,1)$.

Following the research in the papers [1, 2, 4, 9], where similar problems are treated, it is not difficult to conclude that if $F \in C^1(\bar{D})$, then there exists an unique function $U(\xi,\eta)$, belonging to $C^2(D \backslash \{(1,1)\})$, which is a classical solution of this problem in each domain $D \cap \{\xi < \delta, 0 < \delta < 1\}$, but it may become unbounded as $(\xi,\eta) \to (1,1)$. In this paper we will call this function a **generalized solution** of problem (1)-(2).

RELATION TO A PROTTER'S PROBLEM

Problem (1)-(2) may be considered as a part of the investigation of a *Protter's problem* for the four-dimensional wave equation with third-type boundary condition.

More precisely, let Ω be the domain bounded by the surfaces

$$\Sigma_0 := \{(x,t) : \ t = 0, \ |x| < 1\}, \quad \Sigma_1 := \{(x,t) : \ 0 < t < 1/2, \ |x| = 1 - t\}, \quad \Sigma_2 := \{(x,t) : \ 0 < t < 1/2, \ |x| = t\},$$

where the points in R^4 are denoted as $(x,t) := (x_1,x_2,x_3,t)$ and, respectively, $|x| := \sqrt{x_1^2 + x_2^2 + x_3^2}$.

Then the following boundary value problem

$$u_{x_1x_1} + u_{x_2x_2} + u_{x_3x_3} - u_{tt} = f(x,t) \quad \text{in} \quad \Omega, \tag{3}$$

$$u|_{\Sigma_1} = 0, \qquad [u_t + \alpha u]|_{\Sigma_0} = 0 \tag{4}$$

Applications of Mathematics in Engineering and Economics (AMEE'20)
AIP Conf. Proc. 2333, 120009-1–120009-6; https://doi.org/10.1063/5.0041745
Published by AIP Publishing. 978-0-7354-4077-7/$30.00

is one of the so called Protter's problems – these are problems for multidimensional hyperbolic or weakly hyperbolic equation, where the boundary data is prescribed on a characteristic surface and on non-characteristic one, i.e. they are multidimensional analogues of the two-dimensional Darboux problems.

Further, suppose that for some $n \in \mathrm{N}$ and $s \in \mathrm{N}$, $s \leq 2n+1$ the right-hand side function has the form

$$f(x,t) = g(|x|,t)\, Y_n^s(x/|x|),$$

where $Y_m^p(x)$, $m = 0,1,2,\dots,$ $p = 1,2,\dots,2n+1$ are the three-dimensional spherical functions, defined on the unit sphere $S^2 := \{x \in \mathrm{R}^3 : |x| = 1\}$, i.e. we may regard $f(x,t)$ as a single term of a Fourier expansion of some function, since the spherical functions form a complete orthogonal system in $L_2(S^2)$.

In this case problem (3)-(4) reduces to a two-dimensional problem. More precisely, looking for solution to (3)-(4) of the form

$$u(x,t) = \frac{1}{|x|}\, w(|x|,t)\, Y_n^s(x/|x|)$$

and passing to the characteristic coordinates

$$\xi = 1 - |x| - t, \qquad \eta = 1 - |x| + t,$$

one obtains that the function

$$U(\xi,\eta) := w\left(\frac{2-\xi-\eta}{2}, \frac{\eta-\xi}{2}\right)$$

should solve the equation

$$U_{\xi\eta} - \frac{n(n+1)}{(2-\xi-\eta)^2}\, U = F(\xi,\eta) \quad \text{in} \quad D, \tag{5}$$

satisfying the boundary conditions (2), where the function $F(\xi,\eta)$ is defined by

$$F(\xi,\eta) = \frac{1}{8}\,(2-\xi-\eta)\, g\left(\frac{2-\xi-\eta}{2}, \frac{\eta-\xi}{2}\right).$$

In this paper we study the particular case $\mathbf{n} = \mathbf{1}$ when equation (5) becomes equation (1). For $n > 1$ the computations lead to quite complicated forms of the solution which we will not present here.

The Protter's problems are introduced by Protter ([18, 19]) while he studied multidimensional variants of a famous planar problem from transonic fluid dynamics. However the Protter's problems are not well-posed in the frame of classical solvability, since their adjoint homogeneous problems have infinitely many linearly independent nontrivial classical solutions ([5, 16, 21]). This means that for the existence of classical solutions it is necessary infinitely many orthogonality conditions on the right-hand side of the equation to be fulfilled. For this reason (following [16]) it is suitable to study the Protter's problems in the frame of generalized solutions with possible big singularities. It is well-known that in the general case such singularities really exist. It is interesting that they are isolated at one boundary point and do not propagate along the bicharacteristics, which is quite unusually for the hyperbolic equations. For wide range of Protter's problems it was shown ([8, 10, 11, 13, 14, 15]) that the order of singularity of the generalized solutions is controlled by orthogonality conditions on the right-hand side of the equation in respect to corresponding nontrivial classical solutions of the adjoint homogeneous problems.

Protter's problems for hyperbolic equations with third-type boundary conditions were studied in [1, 2, 4, 9]. Actually, there were considered three-dimensional variants of problem (3)-(4) with more general statement, where the equation involves lower order terms and also the coefficient α is a function of $|x|$. In these papers different results on the existence and uniqueness of generalized solutions were proven. However, in the case $\alpha \not\equiv 0$ any explicit formulas for the solutions of these problems or their adjoint problems were not given. In this paper we make a first step in this direction.

For other statements of Protter's problems see for example [3, 6, 10, 11, 12, 14, 17, 20] and the references therein.

INTEGRAL REPRESENTATION OF THE GENERALIZED SOLUTION

Here we give an explicit representation of the generalized solution via Riemann-Hadamard function. There exists an unique generalized solution and it has this form even if the right-hand side function $F(\xi,\eta)$ is only continuous in $\bar{D}$.

Theorem 1. *Let $F \in C(\bar{D})$. Then there exists an unique generalized solution of problem* (1)-(2) *and it has the following integral representation at a point $(\xi_0,\eta_0) \in D$:*

$$U(\xi_0,\eta_0) = \int_0^{\xi_0} \int_{\xi}^{\eta_0} \Phi(\xi,\eta;\xi_0,\eta_0) F(\xi,\eta)\, d\eta\, d\xi, \tag{6}$$

where the Riemann-Hadamard function $\Phi(\xi,\eta;\xi_0,\eta_0)$ is defined as

$$\Phi(\xi,\eta;\xi_0,\eta_0) := \begin{cases} \Phi^+(\xi,\eta;\xi_0,\eta_0), & \eta > \xi_0, \\ \Phi^-(\xi,\eta;\xi_0,\eta_0), & \eta < \xi_0, \end{cases}$$

where

$$\Phi^+(\xi,\eta;\xi_0,\eta_0) := 1 + \frac{2(\xi_0-\xi)(\eta_0-\eta)}{(2-\xi-\eta)(2-\xi_0-\eta_0)},$$

$$\Phi^-(\xi,\eta;\xi_0,\eta_0) := \Phi_0(\xi,\eta;\xi_0,\eta_0) + 2\Phi_\alpha(\xi,\eta;\xi_0,\eta_0),$$

with

$$\Phi_0(\xi,\eta;\xi_0,\eta_0) := 2 + \frac{2(\xi_0-\xi)(\eta_0-\eta)}{(2-\xi-\eta)(2-\xi_0-\eta_0)} + \frac{2(\eta_0-\xi)(\xi_0-\eta)}{(2-\xi-\eta)(2-\xi_0-\eta_0)},$$

$$\begin{aligned} \Phi_\alpha(\xi,\eta;\xi_0,\eta_0) &:= \sum_{k=1}^{\infty} \frac{\alpha^k(\xi_0-\eta)^k}{k!} \\ &+ \frac{2(\xi_0-\xi+\eta_0-\eta)}{(2-\xi-\eta)(2-\xi_0-\eta_0)} \sum_{k=1}^{\infty} \frac{\alpha^k(\xi_0-\eta)^{k+1}}{(k+1)!} - \frac{4}{(2-\xi-\eta)(2-\xi_0-\eta_0)} \sum_{k=1}^{\infty} \frac{\alpha^k(\xi_0-\eta)^{k+2}}{(k+2)!}. \end{aligned}$$

Proof. Let $(\xi_0,\eta_0) \in D$. Direct calculations show that the function $\Phi(\xi,\eta;\xi_0,\eta_0)$ has the following properties, which we need for our considerations:

(i) $\Phi_{\xi\eta} - \dfrac{2}{(2-\xi-\eta)^2}\Phi = 0, \qquad (\xi,\eta) \in D;$

(ii) $\Phi_{\xi_0\eta_0} - \dfrac{2}{(2-\xi_0-\eta_0)^2}\Phi = 0, \quad (\xi,\eta) \in D;$

(iii) $\Phi^+(\xi_0,\eta;\xi_0,\eta_0) = 1, \quad \xi_0 \le \eta \le 1$ and $\Phi^+(\xi,\eta_0;\xi_0,\eta_0) = 1, \quad 0 \le \xi \le \xi_0;$

(iv) $(\Phi^-_\xi - \Phi^-_\eta)(\xi,\xi;\xi_0,\eta_0) = \alpha\Phi^-(\xi,\xi;\xi_0,\eta_0), \quad 0 \le \xi \le \xi_0;$

(v) $(\Phi^-_{\xi_0} - \Phi^-_{\eta_0})(\xi,\eta;\xi_0,\xi_0) = \alpha\Phi^-(\xi,\eta;\xi_0,\xi_0), \quad 0 \le \xi < \eta \le \xi_0;$

(vi) $\Phi^-(\xi,\xi_0;\xi_0,\eta_0) - \Phi^+(\xi,\xi_0;\xi_0,\eta_0) = 1, \quad 0 \le \xi \le \xi_0.$

First, suppose that $U(\xi,\eta)$ is a generalized solution of problem (1)-(2). Then applying an integration by parts into the identity

$$\int_0^{\xi_0}\int_\xi^{\eta_0} L[U](\xi,\eta)\Phi(\xi,\eta;\xi_0,\eta_0)\,d\eta d\xi = \int_0^{\xi_0}\int_\xi^{\eta_0} F(\xi,\eta)\Phi(\xi,\eta;\xi_0,\eta_0)\,d\eta d\xi,$$

with use of the properties (i), (iii), (vi), (vi) of the function $\Phi(\xi,\eta;\xi_0,\eta_0)$ and the boundary conditions (2), we obtain that the function $U(\xi,\eta)$ should have the representation (6) at the point (ξ_0,η_0), which confirms the uniqueness.

Next, using the properties (ii), (iii), (v), (vi), a direct calculation shows that in $\bar{D}\backslash(1,1)$ the function $U(\xi,\eta)$, defined by (6), really satisfies the differential equation (1) and the boundary conditions (2). This confirms the existence. □

Remark 1. *Note that the functions Φ^+ and Φ_0 do not depend on α and also $\Phi_\alpha \equiv 0$ for $\alpha = 0$. In the special case when $\alpha = 0$ problem* (1)-(2) *is very well studied* ([7, 8, 13, 15]) *and the Riemann-Hadamard function is known. This means that the functions Φ^+ and Φ_0 are known from before and the contribution of this theorem consists in constructing the component Φ_α.*

DERIVATION OF THE SOLUTION

Here we briefly describe a way to obtain the integral representation (6).

As we mentioned in Remark 1, the functions Φ^+ and Φ_0 are known. Then the function $U(\xi,\eta)$ can be expressed at a point $(\xi_0,\eta_0)\in D$ as

$$U(\xi_0,\eta_0) = \frac{1}{2}\int_0^{\xi_0}(U_\xi - U_\eta)(\xi,\xi)\,\Phi_0(\xi,\xi;\xi_0,\eta_0)\,d\xi + \int_0^{\xi_0}\int_0^{\eta} F(\xi,\eta)\,\Phi_0(\xi,\eta;\xi_0,\eta_0)\,d\xi d\eta + \int_{\xi_0}^{\eta_0}\int_0^{\xi_0} F(\xi,\eta)\,\Phi^+(\xi,\eta;\xi_0,\eta_0)\,d\xi d\eta. \quad (7)$$

Here the first integral at the right-hand side obviously contains unknown boundary data of $U(\xi,\xi)$. In the case $\alpha = 0$ this integral vanishes and then (7) gives the integral representation of $U(\xi,\eta)$ via Riemann-Hadamard function.

Denote $\tau(\xi) := U(\xi,\xi)$.

Setting $\eta_0 = \xi_0$ and using the second of the boundary conditions (2), we come to the following integral equation:

$$\tau(\xi_0) = \alpha\int_0^{\xi_0}\tau(\xi)\,\Phi^+(\xi,\xi;\xi_0,\xi_0)\,d\xi + \int_0^{\xi_0}\int_0^{\eta} F(\xi,\eta)\,\Phi_0(\xi,\eta;\xi_0,\xi_0)\,d\xi d\eta. \quad (8)$$

Multiplying both sides of the equation by $1-\xi_0$ and differentiating three times in respect to ξ_0, we obtain the following ordinary differential equation

$$\omega'' - \alpha\omega' + \frac{\alpha}{1-\xi_0}\omega = G'''(\xi_0) \quad (9)$$

for the unknown function

$$\omega(\xi_0) := \frac{d}{d\xi_0}\big[(1-\xi_0)\tau(\xi_0)\big], \quad (10)$$

where

$$G(\xi_0) := (1-\xi_0)\int_0^{\xi_0}\int_0^{\eta} F(\xi,\eta)\,\Phi_0(\xi,\eta;\xi_0,\xi_0)\,d\xi d\eta.$$

Here we temporarily suppose that $F\in C^1(\bar{D})$ in order to provide the continuity of G''', but we showed that the final result of Theorem 1 is valid for $F\in C(\bar{D})$ as well.

The function ω satisfies the following initial conditions:

$$\omega(0) = 0, \qquad \omega'(0) = G''(0). \quad (11)$$

The functions $\psi(\xi_0)$ and $\psi(\xi_0)E(\xi_0)$, where

$$\psi(\xi_0) := (1-\xi_0)e^{\alpha\xi_0} \quad \text{and} \quad E(\xi_0) := \int_0^{\xi_0} \frac{e^{-\alpha\xi}}{(1-\xi)^2}\,d\xi,$$

are two linearly independent solutions of equation (9) with $G \equiv 0$. Then, applying the Lagrange method, we may write the solution of problem (9), (11) as

$$\omega(\xi_0) = \psi(\xi_0)E(\xi_0)\int_0^{\xi_0}(1-\xi)\,G'''(\xi)\,d\xi - \psi(\xi_0)\int_0^{\xi_0}(1-\xi)\,G'''(\xi)\,E(\xi)\,d\xi + G''(0)\,\psi(\xi_0)E(\xi_0),$$

which, using integration by parts, may be simplified as

$$\omega(\xi_0) = G'(\xi_0) + \alpha\,G(\xi_0) + \alpha\psi(\xi_0)\int_0^{\xi_0}\frac{\psi'(\xi)}{\psi^2(\xi)}G(\xi)\,d\xi. \tag{12}$$

Taking into account (10), we have

$$\tau(\xi_0) = \frac{1}{1-\xi_0}\int_0^{\xi_0}\omega(\xi)\,d\xi.$$

Setting (12) into this expression, after some transformations which we omit here, we find the representation

$$\tau(\xi_0) = \int_0^{\xi_0}\int_0^{\eta} F(\xi,\eta)\,\Phi^{-}(\xi,\eta;\xi_0,\xi_0)\,d\xi d\eta.$$

This actually is formula (6) for the particular case $\eta_0 = \xi_0$.

Once we know the function $\tau(\xi) = U(\xi,\xi) = \alpha^{-1}(U_\xi - U_\eta)(\xi,\xi)$, with use of (7) we obtain the integral representation (6) for $\eta_0 \neq \xi_0$.

NONTRIVIAL CLASSICAL SOLUTION OF THE ADJOINT HOMOGENEOUS PROBLEM

The adjoint homogeneous problem of (1)-(2) is the following one:

$$V_{\xi\eta} - \frac{2}{(2-\xi-\eta)^2}V = 0 \quad \text{in} \quad D, \tag{13}$$

$$V(\xi,1) = 0, \qquad \big(V_\xi - V_\eta\big)(\xi,\xi) = \alpha V(\xi,\xi). \tag{14}$$

Theorem 2. *The function*

$$V(\xi,\eta) := \sum_{k=0}^{\infty}\frac{\alpha^k(1-\eta)^{k+1}}{(k+1)!} - \frac{2}{2-\xi-\eta}\sum_{k=0}^{\infty}\frac{\alpha^k(1-\eta)^{k+2}}{(k+2)!}$$

is a classical solution of problem (13)-(14).

Clearly, all the functions of the form $kV(\xi,\eta)$, $k = \text{const}$, are classical solutions of this problem as well.

It can be shown that the generalized solution of problem (1)-(2) is bounded in $\bar{D}$ if $V(\xi,\eta)$ is orthogonal to the function $V(\xi,\eta)$, i.e. if

$$\int_D V(\xi,\eta)F(\xi,\eta)\,d\xi d\eta = 0.$$

Otherwise the function $U(\xi,\eta)$ has singularity at the point $(\xi,\eta) = (1,1)$.

This fact is well known for $\alpha = 0$ and it is expected in view of the known results on the Protter's problems and the planar problems related to them.

ACKNOWLEDGMENTS

This work was supported by the Bulgarian Ministry of Education and Science under the National Program for Research "Young Scientists and Postdoctoral Students" (approved with RMS No. 577/ 17.08.2018).

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