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COUNTING DISTRIBUTIONS IN RISK THEORY*

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Abstract

In this paper we introduce some significant counting distributions in risk theory. The first one is the I-Delaporte distribution. It is a generalization of the Non-central negative binomial distribution. The second distribution is the Non-central Pólya-Aeppli distribution. It is a sum of two independent random variables, one that is a Poisson and another one, a Pólya-Aeppli distributed. The Pólya-Aeppli-Lindley, the compound Pólya and compound binomial distributions are also considered. They are mixed Pólya-Aeppli distribution with Lindley mixing distribution, compound negative binomial and compound binomial distribution with geometric compounding distribution. The main application of these distributions is that they can be used as corresponding counting processes' distributions in risk models.

MSC: 60K10; 62P05.

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1 Introduction

The Inflated-parameter negative binomial distribution (INBD) was introduced in [18] as a compound negative binomial distribution (NBD) with geometric compounding distribution. We analyse a convolution of INBD

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and Pólya-Aeppli (PA) distribution. It is a Generalized Delaporte distribution, defined in [19] and it is called an I-Delaporte distribution, [14]. The I-Delaporte distribution is a generalization of the Non-central negative binomial distribution (NNBD), defined in [21] and developed in [20] and [22]. The NNBD is a convolution of independent NB distribution and PA distribution. The PA distribution was derived by Anscombe in 1950, see [1] from a model of randomly distributed colonies. In 1953 it was also studied by Evans, see [3]. Anscombe states that in 1930 the distribution was given by A. Aeppli in a thesis and then developed by G. Pólya. For this reason he called it a PA distribution. It is a compound Poisson with geometric compounding distribution. The Non-central Pólya-Aeppli distribution (NPAD), also considered in this overview paper is a sum of independent Poisson and PA distribution. It is introduced in [13] as a corresponding distribution of the Non-central Pólya-Aeppli process (NPAP) which is applied as a counting process in a risk model. In 1970 Sankaran introduced a mixed Poisson distribution with Lindley mixing distribution and called it a Poisson-Lindley distribution, [25]. In 2002 Minkova defined a mixed PA distribution with a mixing gamma distribution and called it an INBD, [18]. As the properties of the PA distribution are very close to these of the Poisson distribution, see [2] this led to the idea of introducing a PA distribution with Lindley mixing distribution, [15]. The resulting distribution is called a Pólya-Aeppli-Lindley distribution. When the random variable (r.v.) N has a compound distribution then it is interpreted as an aggregate claim amount. This fact provokes our attention on two compound distributions - the compound Pólya distribution and the compound binomial distribution. The first distribution is a compound NBD with compounding geometric distribution and the second one is a compound binomial distribution with geometric compounding distribution. Nice properties of these distributions are derived in [9] and [10]. In 1934 Fisher, [4] introduces a dispersion measure of the r.v. N, known as a Fisher index of dispersion - $FI(N)$. This index gives the ratio of the r.v.'s dispersion to its mean, see [26] and is given by the following formula

$$FI(N) = \frac{Var(N)}{E(N)}.$$

According to this measure a distribution is said to be equi-dispersed when $FI(N) = 1$, under-dispersed when $FI(N) < 1$ and over-dispersed when $FI(N) > 1$. Our main purpose in this paper is to introduce some distributions which have the over-dispersion property and are suitable for financial data. This property give the advantage to use them as counting distributions in the risk theory.

2 Counting distributions in risk theory

2.1 I-Delaporte distribution

The Delaporte distribution is given in [5] as a counting distribution in risk models. It is a sum of independent NB distribution and a Poisson distribution. In this subsection we introduce an Inflated-parameter Delaporte distribution (I-Delaporte distribution) which is a Generalized Delaporte distribution. It is a mixed PA distribution with shifted Gamma mixing distribution and is used as a corresponding distribution of the I-Delaporte process, see [11] .

Suppose that the r.v. N with a given $\lambda > 0$ has a PA distribution, i.e.

$$P(N = m \mid \lambda) = \begin{cases} e^{-\lambda}, & m = 0, \\ e^{-\lambda} \sum_{i=1}^{m} \binom{m-1}{i-1} \frac{[\lambda(1-\rho)]^i}{i!} \rho^{m-i}, & m = 1, 2, \dots, \end{cases} \tag{1}$$

where $\rho \in [0, 1)$ is a parameter. We use the notation $N \sim PA(\lambda, \rho)$.

The probability generating function (PGF) of the PA distribution with given parameter λ is given by

$$\psi_N(s|\lambda) = e^{-\lambda(1-\psi_X(s))}, \tag{2}$$

where

$$\psi_X(s) = Es^X = \frac{(1-\rho)s}{1-\rho s} \tag{3}$$

is the PGF of the compounding geometric distribution, $Ge_1(1-\rho)$.

The mean and the variance of the PA distribution are given by

$$E(N) = \frac{\lambda}{1-\rho} \quad \text{and} \quad Var(N) = \frac{\lambda(1+\rho)}{(1-\rho)^2}.$$

The related Fisher index of dispersion is $FI(N) = \frac{1+\rho}{1-\rho} > 1$, i.e. for $\rho \neq 0$ the PA distribution is over-dispersed related to the Poisson distribution.

Let the mixing distribution be a shifted Γ - distribution with density function given by

$$g(\lambda) = \frac{\beta^r}{\Gamma(r)} (\lambda - \alpha)^{r-1} e^{-\beta(\lambda-\alpha)}, \quad \lambda > \alpha, \tag{4}$$

where r and β are positive parameters. The parameter α in (4) can be interpreted as a risk parameter, see [5]. The function $\Gamma(r)$ is a Gamma function defined by $\Gamma(r) = \int_0^\infty x^{r-1} e^{-x} dx, \ r > 0$.

Mixing the parameter λ in (2) with mixing distribution (4) we obtain the following PGF of the I-Delaporte distribution

$$\psi_N(s) = \left[\frac{\pi}{1-(1-\pi)\psi_X(s)}\right]^r e^{-\alpha(1-\psi_X(s))} = \left[\frac{\pi(1-\rho s)}{1-(1-\pi(1-\rho))s}\right]^r e^{-\alpha\left(1-\frac{(1-\rho)s}{1-\rho s}\right)}, \tag{5}$$

where $\pi = \frac{\beta}{1+\beta}$.

Remark 1 *In the case $\rho = 0$, the compounding variable X degenerates at point one and the distribution of N coincides with the Delaporte distribution, given in [5].*

Definition 1 *The random variable N with PGF (5) has an Inflated-parameter Delaporte distribution (I-Delaporte distribution).*

2.1.1 The Probability Mass Function

The unconditional probability mass function (PMF) of the I-Delaporte distribution is the following

$$P(N = m) = \int_\alpha^\infty P(N = m \mid \lambda)\frac{\beta^r}{\Gamma(r)}(\lambda-\alpha)^{r-1}e^{-\beta(\lambda-\alpha)}d\lambda. \tag{6}$$

Calculating the integral in (6) leads to the PMF of the I-Delaporte distribution.

Lemma 1 *The PMF of the I-Delaporte distribution is given by*

$$P(N=m) = \begin{cases} e^{-\alpha}\left(\frac{\beta}{\beta+1}\right)^r, & m = 0, \\ e^{-\alpha}\left(\frac{\beta}{\beta+1}\right)^r \left[r(1-\rho)\frac{1}{\beta+1} + \alpha(1-\rho)\right], & m = 1, \\ e^{-\alpha}\left(\frac{\beta}{\beta+1}\right)^r \left[\sum_{i=1}^m \binom{m-1}{i-1}\frac{[\alpha(1-\rho)]^i}{i!}\rho^{m-i}\right. & \\ + \sum_{i=1}^m \binom{m-1}{i-1}\binom{r+i-1}{i}\left((1-\rho)\frac{1}{\beta+1}\right)^i \rho^{m-i} & \\ + \sum_{i=1}^{m-1}\sum_{j=1}^i \binom{i-1}{j-1}\frac{[\alpha(1-\rho)]^j}{j!}\rho^{i-j}\sum_{k=1}^{m-i}\binom{m-i-1}{k-1} & \\ \left.\times \binom{r+k-1}{k}\left((1-\rho)\frac{1}{\beta+1}\right)^k \rho^{m-i-k}\right], & m = 2, 3, \dots \end{cases}$$

The proof of Lemma 1 is given in [14]. From (5) it follows that the r.v. N is a sum $N = N_1 + N_2$ of two independent random variables. The random variable N_1 has the following PMF

$$P(N_1 = m) = \begin{cases} \left(\frac{\beta}{\beta+1}\right)^r, & m = 0, \\ \left(\frac{\beta}{\beta+1}\right)^r \sum_{i=1}^{m} \binom{m-1}{i-1}\binom{r+i-1}{i}[(1-\rho)\frac{1}{\beta+1}]^i \rho^{m-i}, & m = 1, 2, \dots \end{cases} \tag{7}$$

The random variable in (7) has an INBD with parameters $\pi = \frac{\beta}{1+\beta}$, ρ, r and is given in [18]. We shortly say I-negative binomial distribution and use the notation $N_1 \sim INB\left(\dfrac{\beta}{1+\beta}, \rho, r\right)$. The random variable N_2 is Pólya-Aeppli distributed, i.e $N_2 \sim PA(\alpha, \rho)$.

Remark 2 *The r.v. N can be represented as a compound Delaporte distribution, i.e. $N = X_1 + \ldots + X_{N_0}$, where N_0 has a Delaporte distribution and X_i, $i = 1, 2, \ldots$ are independent, geometrically distributed with success probability $1-\rho$ and PGF, given in (3), independent of the r.v. N_0.*

According to the idea of Ong and Lee [21] we can give the following interpretation. Suppose that the PGF of the r.v. N has the form

$$\Psi_N(s) = \left[\frac{\pi}{1-(1-\pi)\psi_X(s)}\right]^T,$$

where the r.v. T is represented as a sum of Poisson distributed r.v. V with parameter α and a positive constant r, i.e. $T = r + V$. Then the resulting PGF $\psi_N(s)$ is given by (5).

Let denote by $p_m = P(N = m)$, $\quad m = 0, 1, \ldots$ the PMF of the r.v. N. Then the following proposition holds.

Proposition 1 *The PMF of the I-Delaporte distribution satisfies the fol-*

lowing recursions

$$
\begin{aligned}
p_0 &= \pi^r e^{-\alpha}, && m=0,\\
p_1 &= (1-\rho)\left[r(1-\pi)+\alpha\right]p_0, && m=1,\\
p_2 &= \left[1+2\rho-\pi(1-\rho)+\tfrac{(1-\rho)(r(1-\pi)+\alpha)-2\rho-(1-\pi(1-\rho))}{2}\right]p_1\\
&\quad -\tfrac{(1-\rho)}{2}\left[r(1-\pi)\rho+\alpha(1-\pi(1-\rho))\right]p_0, && m=2,\\
p_m &= \left[1+2\rho-\pi(1-\rho)+\tfrac{(1-\rho)(r(1-\pi)+\alpha)-2\rho-(1-\pi(1-\rho))}{m}\right]p_{m-1}\\
&\quad -\left[\tfrac{(m-2)}{m}\rho(\rho+2(1-\pi(1-\rho)))+\tfrac{(1-\rho)[r(1-\pi)\rho+\alpha(1-\pi(1-\rho))]}{m}\right]p_{m-2}\\
&\quad +\rho^2\left(1-\tfrac{3}{m}\right)(1-\pi(1-\rho))p_{m-3}, && m=3,4,\ldots
\end{aligned}
$$

Proof. Upon substituting $s=0$ in the PGF $\psi_N(s)$, given in (5) we obtain the initial value p_0 of the recursion formulas. Differentiation in (5) leads to

$$\psi'(s)=\frac{1-\rho}{1-\rho s}\left[\frac{r(1-\pi)}{1-(1-\pi(1-\rho))s}+\frac{\alpha}{1-\rho s}\right]\psi(s), \tag{8}$$

where $\psi(s)=\sum_{i=0}^{\infty}p_i s^i$ and $\psi'(s)=\sum_{i=0}^{\infty}(i+1)p_{i+1}s^i$. After some mathematical steps and substituting $\psi(s)$ and $\psi'(s)$ with their expressions in (8) we obtain

$$
\begin{aligned}
&\sum_{i=0}^{\infty}(i+1)p_{i+1}s^i-\left[1+2\rho-\pi(1-\rho)\right]\sum_{i=1}^{\infty}ip_is^i+\rho\left[2+\rho-2\pi(1-\rho)\right]\\
&\times\sum_{i=2}^{\infty}(i-1)p_{i-1}s^i-\rho^2\left[1-\pi(1-\rho)\right]\sum_{i=3}^{\infty}(i-2)p_{i-2}s^i\\
&=\ (1-\rho)\left[r(1-\pi)+\alpha\right]\sum_{i=0}^{\infty}p_is^i-(1-\rho)\cdot\\
&\left[(1-\pi)(r\rho+\alpha)+\rho\pi\alpha\right]\sum_{i=1}^{\infty}p_{i-1}s^i.
\end{aligned}\tag{9}
$$

The recursions are obtained by equating the coefficients before s^i from the both sides of the equality (9) for $i=0,1,2,\ldots$

An alternative recursion formulas are given in the next corollary.

Corollary 1 *The PMF of I-Delaporte distribution satisfies the following alternative recursion formulas*

$$
\begin{aligned}
p_0 &= \pi^r e^{-\alpha}, && m=0,\\
p_1 &= (1-\rho)[r(1-\pi)+\alpha]p_0, && m=1,\\
2p_2 &= (1-\rho)\left[\tfrac{1+\rho}{1-\rho}-\pi+(r(1-\pi)+\alpha)\right]p_1\\
&-\ [(1-\rho)^2(1-\pi)\alpha]p_0, && m=2,\\
mp_m &= (1-\rho)\left[\left(\tfrac{1+\rho}{1-\rho}-\pi\right)(m-1)+(r(1-\pi)+\alpha)\right]p_{m-1}\\
&+\ (1-\rho)\sum_{k=0}^{m-2}[-\alpha\rho^{m-2-k}(1-\pi(1-\rho))+\alpha\rho^{m-1-k}]p_k\\
&-\rho[1-\pi(1-\rho)](m-2)p_{m-2}, && m=3,4,\ldots
\end{aligned}
$$

Proof. Representing (8) in the form

$$
\begin{aligned}
&[1-[1+\rho-\pi(1-\rho)]s+\rho[1-\pi(1-\rho)]s^2]\psi'(s)\\
&=(1-\rho)[r(1-\pi)+\alpha-[(1-\pi)(r\rho+\alpha)+\rho\pi\alpha]s]\sum_{j=0}^{\infty}(\rho s)^j\psi(s)
\end{aligned}
$$

and substituting $\psi(s)=\sum_{i=0}^{\infty}p_is^i$ and $\psi'(s)=\sum_{i=0}^{\infty}(i+1)p_{i+1}s^i$ we obtain

$$
\begin{aligned}
&\sum_{i=0}^{\infty}(i+1)p_{i+1}s^i=[1+\rho-\pi(1-\rho)]\sum_{i=1}^{\infty}ip_is^i-\rho[1-\pi(1-\rho)]\\
&\times\sum_{i=2}^{\infty}(i-1)p_{i-1}s^i+(1-\rho)[r(1-\pi)+\alpha]\sum_{i=0}^{\infty}\sum_{j=0}^{i}\rho^{i-j}p_js^i\\
&-(1-\rho)[(1-\pi)(r\rho+\alpha)+\rho\pi\alpha]\sum_{i=1}^{\infty}\sum_{j=0}^{i-1}\rho^{i-j-1}p_js^i. \qquad (10)
\end{aligned}
$$

The alternative recursions are obtained by equating the coefficients before s^i from the both sides of the equality (10) for $i=0,1,2,\ldots$

2.1.2 Moments of I-Delaporte distribution

The mean and the variance of the I-Delaporte distribution are given by

$$E(N) = \left(\alpha + \frac{r}{\beta}\right)\frac{1}{1-\rho}$$

and

$$Var(N) = \left[\alpha(1+\rho) + \frac{r[(1+\rho)\beta+1]}{\beta^2}\right]\frac{1}{(1-\rho)^2}.$$

For the Fisher index of dispersion we obtain

$$FI(N) = \frac{1+\rho}{1-\rho} + \frac{r}{(1-\rho)\beta^2(\alpha+\frac{r}{\beta})} > \frac{1+\rho}{1-\rho},$$

i.e. the I-Delaporte distribution is over-dispersed related to the PA distribution. It means that this distribution is suitable for financial data and can be applied in risk theory.

2.2 Non-central Pólya-Aeppli distribution

The NNBD arises as a model in photon and neural counting, birth and death processes and mixture models, see [21]. Ong and Lee gave a formulation of the NNBD as a Poisson and NB mixture. In 1983 Gurland et. al., [6] considered the NNBD and referred it to as a Laguerre series distribution. Lately it had been developed in the work of Ong et. al., [20] and Ong and Shimizu [22]. The PMF and the PGF of the NNBD are given by

$$P(N=i) = e^{-\lambda p}p^i q^\nu L_i^{\nu-1}(-\lambda q)$$

and

$$\psi_N(s) = \left(\frac{q}{1-ps}\right)^\nu e^{-\left[1-\frac{q}{1-ps}\right]}, \tag{11}$$

where $\nu > 0$ and $\lambda > 0$ are parameters, $0 < p = 1-q < 1$ and $L_i^\alpha(x)$ are the Laguerre polinomials orthogonal over $(0,\infty)$ with respect to $x^{\alpha-1}e^{-x}$.

The PA distribution is a compound Poisson with geometric compounding distribution. The PMF of the compounding distribution is given by

$$P(X=i) = (1-\rho)\rho^{i-1}, \qquad i = 1, 2, \ldots$$

and the PGF by (3).

Another generalization of the NNBD is the I-Delaporte distribution, defined in [14]. It is a convolution of INBD, introduced in [17] as a compound

NB distribution with geometric compounding distribution and a PA distribution. The second counting distribution introduced in this paper is the Non-central Pólya-Aeppli distribution (NPAD). Useful properties of this distribution are given in [12].

Suppose that the first random variable N_1 with a given parameter $\lambda_1 > 0$ has a Poisson distribution, i.e.

$$P(N_1 = i) = \frac{(\lambda_1)^i}{i!} e^{-\lambda_1}, \; i = 0, 1, \dots$$

with the following PGF $\psi_{N_1}(s) = e^{-\lambda_1(1-s)}$. We use the notation $N_1 \sim P_o(\lambda_1)$. The second random variable N_2 with parameters $\lambda_2 > 0$ and $\rho \in [0, 1)$ has a PA distribution with PMF, given in (1). This distribution is also known as an Inflated-parameter Poisson distribution, see [17]. For the random variable N_2 we use the notation $N_2 \sim PA(\lambda_2, \rho)$.

Considering the fact that $N = N_1 + N_2$ where N_1 and N_2 are independent random variables we obtain that the PGF of the NPAD r.v. is given by

$$\psi_N(s) = e^{-\lambda_1(1-s)} e^{-\lambda_2[1-\psi_X(s)]}, \tag{12}$$

where $\psi_X(s)$ is the PGF of the compounding geometric distribution.

Definition 2 *The random variable N with PGF given in (12) is referred to a NPAD. We use the notation $N \sim NPAD(\lambda_1, \lambda_2, \rho)$.*

2.2.1 The Probability Mass Function

The paper of Ong and Lee, [21] motivated the name of this distribution i.e NPAD, see [12]. It is well known that the Poisson distribution is a limiting case of the NBD. If we take $\nu(1 - q) \to \lambda > 0$ in the first term of (11) then we will obtain the PGF, given in (12). Thus if the both terms in (11) have different parameters, then the NPAD could be a limiting case of the NNBD.

Lemma 2 *The PMF of the NPAD is given by*

$$P(N = m) =$$
$$= \begin{cases} e^{-(\lambda_1+\lambda_2)}, & m = 0, \\ e^{-(\lambda_1+\lambda_2)} \left[\frac{(\lambda_1)^m}{m!} + \sum_{j=1}^{m} \frac{(\lambda_1)^{m-j}}{(m-j)!} \sum_{k=1}^{j} \binom{j-1}{k-1} \frac{[\lambda_2(1-\rho)]^k}{k!} \rho^{j-k} \right], & m = 1, 2, \dots \end{cases}$$

In the next proposition we obtain recursion formulas for the PMF of the defined distribution.

Proposition 2 *The PMF of the NPAD satisfies the following recursions*

$$
\begin{aligned}
&p_0 = e^{-(\lambda_1+\lambda_2)}, && m=0,\\
&p_1 = [\lambda_1+\lambda_2(1-\rho)]p_0, && m=1,\\
&p_2 = \left[\rho+\tfrac{\lambda_1+\lambda_2(1-\rho)}{2}\right]p_1 - \lambda_1\rho p_0, && m=2,\\
&p_m = [2\rho+\tfrac{[\lambda_1+\lambda_2(1-\rho)-2\rho]}{m}]p_{m-1} - \rho[\rho+2\tfrac{\lambda_1-\rho}{m}]p_{m-2} + \tfrac{\lambda_1\rho^2}{m}p_{m-3}, &&\\
&\quad m=3,4,\ldots
\end{aligned}
$$

Proof. Upon substituting $s=0$ in the PGF $\psi_N(s)$, given in (12) we obtain the initial value p_0. Differentiation in (12) leads to

$$
\psi'(s) = \left[\frac{\lambda_1(1-\rho s)^2+\lambda_2(1-\rho)}{(1-\rho s)^2}\right]\psi(s) = \left[\lambda_1+\frac{\lambda_2(1-\rho)}{(1-\rho s)^2}\right]\psi(s), \quad (13)
$$

where $\psi(s)=\sum_{i=0}^{\infty} p_i s^i$ and $\psi'(s)=\sum_{i=0}^{\infty}(i+1)p_{i+1}s^i$. After some mathematical steps and substituting $\psi(s)$ and $\psi'(s)$ with their expressions in equation (13) we obtain

$$
\begin{aligned}
&\sum_{i=0}^{\infty}(i+1)p_{i+1}s^i = 2\rho\sum_{i=1}^{\infty} ip_i s^i - \rho^2\sum_{i=2}^{\infty}(i-1)p_{i-1}s^i \\
&+ [\lambda_1+\lambda_2(1-\rho)]\sum_{i=0}^{\infty}p_i s^i - 2\lambda_1\rho\sum_{i=1}^{\infty}p_{i-1}s^i + \lambda_1\rho^2\sum_{i=2}^{\infty}p_{i-2}s^i.
\end{aligned} \quad (14)
$$

The recursions are obtained by equating the coefficients before s^i from the both sides of the equality (14) for $i=0,1,2,\ldots$

An alternative recursion formulas are given in the next corollary.

Corollary 2 *The PMF of the NPAD satisfies the following alternative recursion formulas*

$$
\begin{aligned}
&p_0 = e^{-(\lambda_1+\lambda_2)}, \qquad m=0,\\
&p_1 = [\lambda_1+\lambda_2(1-\rho)]p_0, \qquad m=1,\\
&p_m = \frac{[\lambda_1+\lambda_2(1-\rho)]}{m}p_{m-1} + \lambda_2(1-\rho)\sum_{j=1}^{m-1}\left(1-\frac{j-1}{m}\right)\rho^{m-1-j}p_{j-1},\\
&\qquad m=2,3,\ldots
\end{aligned}
$$

Proof. Representing (13) in the form

$$
\psi'(s) = [\lambda_1+\lambda_2(1-\rho)\sum_{j=0}^{\infty}(j+1)\rho^j s^j]\psi(s)
$$

and after substituting $\psi(s) = \sum_{i=0}^{\infty} p_i s^i$ and $\psi'(s) = \sum_{i=0}^{\infty}(i+1)p_{i+1}s^i$ in the above equality we obtain

$$\sum_{i=0}^{\infty}(i+1)p_{i+1}s^i = [\lambda_1 + \lambda_2(1-\rho)\sum_{j=0}^{\infty}(j+1)\rho^j s^j]\sum_{i=0}^{\infty} p_i s^i$$

$$= \lambda_1 \sum_{i=0}^{\infty} p_i s^i + \lambda_2(1-\rho)\sum_{i=0}^{\infty} p_i s^i \sum_{j=0}^{\infty}(j+1)\rho^j s^j \tag{15}$$

$$= \lambda_1 \sum_{i=0}^{\infty} p_i s^i + \lambda_2(1-\rho)\sum_{i=0}^{\infty}\sum_{j=0}^{i}(i-j+1)\rho^{i-j}p_j s^i$$

The alternative recursions are obtained by equating the coefficients before s^i from the both sides of the equality (15) for $i = 0, 1, 2, \ldots$

2.2.2 Moments of Non-central Pólya-Aeppli distribution

The mean and the variance of the NPAD are given by

$$E(N) = \left(\lambda_1 + \frac{\lambda_2}{1-\rho}\right) \quad \text{and} \quad Var(N) = \left[\lambda_1 + \lambda_2\frac{1+\rho}{(1-\rho)^2}\right].$$

For the Fisher index we obtain

$$FI(N) = \frac{\lambda_1(1-\rho)^2 + \lambda_2(1+\rho)}{(1-\rho)[\lambda_1(1-\rho)+\lambda_2]}.$$

It is easy to check that

$$FI(N) = 1 + \frac{2\lambda_2\rho}{(1-\rho)[\lambda_1(1-\rho)+\lambda_2]},$$

i.e. NPAD is over-dispersed related to Poisson distribution and

$$FI(N) = \frac{1+\rho}{1-\rho} - \frac{2\lambda_1\rho}{\lambda_1(1-\rho)+\lambda_2} < \frac{1+\rho}{1-\rho},$$

i.e. NPAD is under-dispersed related to Pólya-Aeppli distribution.

2.3 Pólya-Aeppli-Lindley distribution

The Lindley distribution was introduced by Lindley in 1958, [16] as a mixture of $Gamma(1,\beta)$ and $Gamma(2,\beta)$ distributions with the following probability density function

$$g(\lambda) = \frac{\beta}{1+\beta}\beta e^{-\beta\lambda} + \frac{1}{1+\beta}\beta^2\lambda e^{-\beta\lambda}, \ \lambda > 0. \tag{16}$$

Definition 3 *The random variable Λ has a Lindley distribution with parameter $\beta > 0$, if its density function is given by*

$$g(\lambda) = \frac{\beta^2}{1+\beta}(1+\lambda)e^{-\beta\lambda}, \ \lambda > 0.$$

In 1970 Sankaran, see [25] had introduced the Poisson-Lindley distribution as a mixture of Poisson distribution with Lindley mixing distribution with the following PMF

$$P(N = m) = \frac{\beta^2}{(1+\beta)^{m+3}}[2 + \beta + m], \quad m = 0, 1, \dots \tag{17}$$

Let us suppose that the parameter λ in the PA distribution, given in (1) has a Lindley distribution with density function of the form (16). Then from the PGF of the PA distribution we obtain that the unconditional PGF of the r.v. N has the form

$$\Psi(s) = \frac{\beta}{1+\beta}\frac{\beta}{\beta + (1-\psi_X(s))} + \frac{1}{1+\beta}\left[\frac{\beta}{\beta + (1-\psi_X(s))}\right]^2, \tag{18}$$

where $\psi_X(s)$ is the PGF of the geometric distribution given in (3).

In (18) we denote by $\theta = \frac{\beta}{1+\beta}$ the new parameter of the distribution. This parametrization was used for the Poisson-Lindley distribution.

Then the PGF of the Pólya-Aeppli-Lindley distribution has the form

$$\Psi_N(s) = \frac{\theta}{1-(1-\theta)\psi_X(s)}\left[\theta + (1-\theta)\frac{\theta}{1-(1-\theta)\psi_X(s)}\right] \tag{19}$$

or

$$\Psi_N(s) = \frac{\theta(1-\rho s)}{1-(1-\theta(1-\rho))s}\left[\theta + (1-\theta)\frac{\theta(1-\rho s)}{1-(1-\theta(1-\rho))s}\right].$$

Definition 4 *The distribution of the random variable N with PGF given in (19) is called Pólya-Aeppli-Lindley distribution with parameters θ and ρ. We use the notation $N \sim PAL(\theta, \rho)$.*

2.3.1 The Probability Mass Function

From the PGF in (19) it follows that the r.v. N can be represented as a sum of two independent variables $N = N_1 + N_2$, where the r.v. N_1 has a compound geometric distribution and the r.v. N_2 has a compound modified geometric distribution with the same geometric compounding distribution.

The corresponding PGFs are given by

$$\psi_{N_1}(s) = \frac{\theta}{1-(1-\theta)\psi_X(s)} \quad \text{and} \quad \psi_{N_2}(s) = \theta + (1-\theta)\frac{\theta}{1-(1-\theta)\psi_X(s)}.$$

Taking into account that the probability density function of the mixing distribution $g(\lambda)$, given in (16) and the PMF of the PA distribution we obtain the following result.

Lemma 3 *The PMF of the Pólya-Aeppli-Lindley distribution is given by*

$$P(N=m) = \begin{cases} \frac{\beta^2}{(1+\beta)^3}(2+\beta), & m=0, \\ \frac{(1-\rho)\beta^2}{(1+\beta)^4}(3+\beta), & m=1, \\ \frac{(1-\rho)\beta^2(1+\beta\rho)^{m-2}}{(1+\beta)^{m+3}}[(1+\beta\rho)(3+\beta)+(m-1)(1-\rho)], & m=2,3,\ldots \end{cases} \tag{20}$$

Remark 3 *In the case of $\rho = 0$ the PMF given in (20) coincides with the PMF of the Poisson-Lindley distribution given in (17).*

From the PGFs of the r.v.'s N_1 and N_2 in the terms of $\theta = \frac{\beta}{1+\beta}$ we obtain the following PMFs

$$P(N_1=m) = \begin{cases} \theta, & m=0, \\ \theta(1-\theta)(1-\rho)[1-\theta(1-\rho)]^{m-1}, & m=1,2,\ldots \end{cases}$$

and

$$P(N_2=m) = \begin{cases} \theta(2-\theta), & m=0, \\ \theta(1-\theta)^2(1-\rho)[1-\theta(1-\rho)]^{m-1}, & m=1,2,\ldots \end{cases}$$

Taking into account that $N = N_1 + N_2$ in the terms of θ we obtain the following PMF of N :

$$P(N=m)$$
$$=\begin{cases}\theta^2(2-\theta), & m=0,\\ \theta^2(1-\theta)(1-\rho)(3-2\theta), & m=1,\\ \theta^2(1-\theta)(1-\rho)[1-\theta(1-\rho)]^{m-2} & \\ \times[(1-\theta(1-\rho))(3-2\theta+(m-1)(1-\theta)^2(1-\rho)], & m=2,3,\dots\end{cases}$$

Proposition 3 *The PMF of the Pólya-Aeppli-Lindley distribution satisfies the following recursion formulas*

$$\begin{aligned} p_0 &= \theta^2(2-\theta)\\ (2-\theta)p_1 &= (1-\rho)(1-\theta)(3-2\theta)p_0,\\ (2-\theta)2p_2 &= \big[3(2-\theta).(1-\theta(1-\rho))\big]p_1\\ &\quad -(1-\rho)(1-\theta)\big[1-\theta+(2-\theta)\rho\big]p_0,\end{aligned}$$

and for $m=3,4\dots$

$$\begin{aligned}&(2-\theta)mp_m\\ &=\big[[3-4\theta+(3+\theta)\rho+\theta^2(1-\rho)](m-1)+(1-\rho)(1-\theta)(3-2\theta)\big]p_{m-1}\\ &-\big[[(1-\theta)^2+4\rho(1-\theta)-(\theta^2-3\theta-1)\rho^2](m-2)\\ &+(1-\rho)(1-\theta)[1-\theta+(2-\theta)\rho]\big]p_{m-2}\\ &+\rho\big[(1-\theta)^2+\rho(1-\theta^2+\theta\rho)](m-3)p_{m-3}.\end{aligned}$$

Proof. The initial value p_0 is obtained upon substituting $s=0$ in the PGF $\psi_N(s)$, given in formula (19). Differentiation in (19) leads to

$$\begin{aligned}&[1-(1-\theta(1-\rho))s]\,[2-\theta-(1-\theta+\rho)s]\,(1-\rho s)\psi'(s)\\ &\qquad=(1-\rho)(1-\theta)[3-2\theta-(\rho+(1+\rho)(1-\theta))s]\psi(s),\end{aligned} \tag{21}$$

where $\psi(s)=\sum_{i=0}^{\infty}p_i s^i$ and $\psi'(s)=\sum_{i=0}^{\infty}(i+1)p_{i+1}s^i$. After substituting $\psi(s)$ and $\psi'(s)$ in (21) we obtain

$$\begin{aligned}
&(2-\theta)\sum_{i=0}^{\infty}(i+1)p_{i+1}s^i-[3-4\theta+(3+\theta)\rho+\theta^2(1-\rho)]\sum_{i=1}^{\infty}ip_i s^i\\
&+[(1-\theta)^2+4\rho(1-\theta)-\rho^2(\theta^2-3\theta-1)]\sum_{i=2}^{\infty}(i-1)p_{i-1}s^i\\
&-\rho[(1-\theta)^2+\rho(1-\theta^2+\theta\rho)]\sum_{i=3}^{\infty}(i-2)p_{i-2}s^i \qquad (22)\\
&=(1-\rho)(1-\theta)(3-2\theta)\sum_{i=0}^{\infty}p_i s^i\\
&-(1-\rho)(1-\theta)(1-\theta+(2-\theta)\rho)\textstyle\sum_{i=1}^{\infty}p_{i-1}s^i.
\end{aligned}$$

The recursions are obtained by equating the coefficients before s^i from the both sides of the equality (22) for $i=0,1,2,\ldots$

Corollary 3 *The PMF of the Pólya-Aeppli-Lindley distribution satisfies the following alternative recursions*

$$\begin{aligned}
p_0 &= \theta^2(2-\theta)\\
(2-\theta)p_1 &= (1-\rho)(1-\theta)(3-2\theta)p_0,\\
(2-\theta)2p_2 &= \big[(3-2\theta)(1+(1-\rho)(1-\theta))+\rho+(1-\rho)(\theta-2)\theta\big]p_1\\
&\quad -(1-\rho)^2(1-\theta)^2p_0,\\
(2-\theta)mp_m &= [3-2\theta-(1-\rho)(2-\theta)\theta+\rho](m-1)p_{m-1}\\
&\quad -(1-\theta(1-\rho))(1-\theta+\rho)(m-2)p_{m-2}\\
&\quad +(1-\rho)(1-\theta)(3-2\theta)\sum_{j=0}^{m-1}\rho^{m-1-j}p_j\\
&\quad -(1-\rho)(1-\theta)(1-\theta+(2-\theta)\rho)\sum_{j=0}^{m-2}\rho^{m-2-j}p_j,\ m=3,4,\ldots
\end{aligned}$$

Proof. Representing the PGF (21) in the form

$$
\begin{aligned}
&\big[(2-\theta)-[(1-\theta+\rho)+(1-\theta(1-\rho))(2-\theta)]s\\
&+[(1-\theta(1-\rho))(1-\theta+\rho)]s^2\big]\psi'(s)\\
&= (1-\rho)\big[(1-\theta)(3-2\theta)+[(1-\theta)(1-\theta+(2-\theta)\rho)]s\big]\sum_{j=0}^{\infty}(\rho s)^j\psi(s)
\end{aligned}
$$

and after substituting $\psi(s) = \sum_{i=0}^{\infty} p_i s^i$ and $\psi'(s) = \sum_{i=0}^{\infty}(i+1)p_{i+1}s^i$ in the above equality we obtain

$$
\begin{aligned}
&(2-\theta)\sum_{i=0}^{\infty}(i+1)p_{i+1}s^i = [3-2\theta+(1-\rho)(\theta-2)\theta+\rho]\sum_{i=1}^{\infty} ip_i s^i\\
&-(1-\theta(1-\rho))(1-\theta+\rho)\sum_{i=2}^{\infty}(i-1)p_{i-1}s^i + (1-\rho)(1-\theta)(3-2\theta)\\
&\times \textstyle\sum_{i=0}^{\infty}\sum_{j=0}^{i}\rho^{i-j}p_j s^i - (1-\rho)(1-\theta)(1-\theta+(2-\theta)\rho)\displaystyle\sum_{i=1}^{\infty}\sum_{j=0}^{i-1}\rho^{i-j-1}p_j s^i.
\end{aligned}
\tag{23}
$$

The alternative recursions are obtained by equating the coefficients before s^i from the both sides of the equality (23) for $i = 0, 1, 2, \ldots$

2.3.2 Moments of Pólya-Aeppli-Lindley distribution

The mean and the variance of the PAL distribution in terms of θ are given by

$$
E(N) = \frac{(1-\theta)(2-\theta)}{\theta(1-\rho)} \quad \text{and} \quad Var(N) = \frac{(1-\theta)[2(1+\theta\rho)-\theta^2(\theta+\rho)]}{\theta^2(1-\rho)^2}.
$$

For the Fisher index of dispersion we obtain

$$
FI(N) = \frac{Var(N)}{E(N)} > \frac{1+\rho}{1-\rho},
$$

which shows that the PAL distribution is over-dispersed related to the PA distribution. This means that this distribution is suitable for financial data.

2.4 Compound Pólya distribution

Let us consider the following random sum

$$N = X_1 + X_2 + \ldots + X_Z, \tag{24}$$

where the r.v.'s X_i are independent and identically distributed (iid) as the r.v. X. The r.v. Z belongs to the family of the Generalized Powers Series Distributions (GPSDs). The binomial, negative binomial, Poisson and logarithmic series distributions belong to this family, see [24]. Compound GPSDs are defined in [8], where the compounding r.v. X has a shifted geometric distribution with parameter $1-\gamma,\ \gamma \in [0,1)$. In this paper, let the r.v. X has a geometric distribution with parameter $\gamma \in (0,1)$, denoted by $X \sim Ge(\gamma)$ and the r.v. Z is independent of the r.v. X.

We suppose that the r.v. Z has a NBD with parameters $r \in \mathbf{N}$ and $\theta \in (0,1)$. We use the notation $Z \sim NB(r,\theta)$. In this case the r.v. N has a compound NBD with compounding geometric distribution.

The PMF and the PGF of X are given by

$$q_i = P(X = i) = \gamma(1-\gamma)^i, \quad i = 0, 1, \ldots \tag{25}$$

and

$$\psi_1(s) = \frac{\gamma}{1-(1-\gamma)s}, \ |s| < \frac{1}{1-\gamma}. \tag{26}$$

The PMF and PGF of Z are given by

$$P(Z = i) = \binom{r+i-1}{i}(1-\theta)^r\theta^i, \quad i = 0, 1, \ldots$$

and

$$\psi_Z(s) = \left(\frac{1-\theta}{1-\theta s}\right)^r, \ |s| < \frac{1}{\theta}.$$

Then the PGF of the r.v. N is given by

$$\psi_N(s) = \left(\frac{1-\theta}{1-\theta\psi_1(s)}\right)^r, \tag{27}$$

where $\psi_1(s)$ is the PGF of the compounding distribution, given by (26).

Definition 5 *The probability distribution of N, defined by the PGF (27) and compounding distribution, given by (25) and (26) is called a compound Pólya distribution.*

2.4.1 The Probability Mass Function

The probability function of the r.v. N is given by the expanding of the PGF $\psi(s)$ in powers of s. Denote by $f(i) = P(N = i),\ i = 0, 1, 2, \ldots,$ the PMF of the r.v. N.

Rewriting the PGF of (27) leads to

$$\psi(s) = (1-\theta)^r \sum_{m=0}^{\infty} \binom{r+m-1}{m} \left(\frac{\theta\gamma}{1-(1-\gamma)s} \right)^m . \tag{28}$$

Denote by $\psi^{(i)}(s) = \frac{\partial^{(i)}\psi(s)}{\partial s^i}$, for $i = 0, 1, \ldots,$ the derivatives of $\psi(s)$.
From (28) we get the following

$$\psi^{(i)}(s) = (1-\gamma)^i (1-\theta)^r \sum_{m=1}^{\infty} \binom{r+m-1}{m} (\theta\gamma)^m \frac{m(m+1)\ldots(m+i-1)}{(1-(1-\gamma)s)^{m+i}}.$$

From [7], it is known that

$$f(i) = \left. \frac{\psi^{(i)}(s)}{i!} \right|_{s=0} . \tag{29}$$

Lemma 4 *The PMF of the compound Pólya distribution is given by*

$$\begin{aligned} f(0) &= \left(\tfrac{1-\theta}{1-\theta\gamma}\right)^r, \\ f(i) &= (1-\gamma)^i (1-\theta)^r \textstyle\sum_{m=1}^{\infty} \binom{r+m-1}{m} \binom{m+i-1}{i} (\theta\gamma)^m, \quad i = 1, 2, \ldots \end{aligned}$$

In the next proposition we obtain recursion formulas for the PMF of the defined distribution. This proposition gives an extension of the Panjer recursion formulas (see [23]).

Proposition 4 *The PMF of the compound Pólya distribution satisfies the following recursions*

$$(1-\theta\gamma) i f(i) = (1-\gamma) \left[(i-1) f(i-1) + \theta r \sum_{j=0}^{i-1} q_j f(i-j-1) \right], \ i = 2, 3, \ldots \tag{30}$$

and $f(1) = \frac{1-\gamma}{1-\theta\gamma} \theta r q_0 f(0)$ *with* $f(0) = \left(\frac{1-\theta}{1-\theta\gamma}\right)^r$.

Proof. Differentiation in (27) leads to

$$\psi'(s) = \frac{(1-\gamma)\theta r}{1-(1-\gamma)s-\theta\gamma}\psi_1(s)\psi(s), \tag{31}$$

where $\psi(s) = \sum\limits_{i=0}^{\infty} f(i)s^i$, $\psi'(s) = \sum\limits_{i=0}^{\infty}(i+1)f(i+1)s^i$, and $\psi_1(s) = \sum\limits_{j=0}^{\infty} q_j s^j$. After substituting $\psi(s)$, $\psi'(s)$, $\psi_1(s)$, changing the variable from $i+j = l \Rightarrow i = l-j$ and after equivalent transformations in (31) yields to

$$[1-\theta\gamma]\sum_{i=0}^{\infty}(i+1)f(i+1)s^i = (1-\gamma)\sum_{i=1}^{\infty} if(i)s^i + (1-\gamma)\theta r\sum_{i=0}^{\infty}[\sum_{j=0}^{i} q_j f(i-j)]s^i.$$

The recursion formulas (30) are obtained by equating the coefficients of s^i on both sides for fixed $i = 0, 1, 2, \ldots$

Corollary 4 *The PMF of the compound Pólya distribution satisfies the recursions*

$$\begin{aligned}(1-\theta\gamma)if(i) &= (1-\gamma)[(i-1)(2-\theta\gamma)+r\theta\gamma]f(i-1)\\ &\quad -(1-\gamma)^2(i-2)f(i-2),\ i = 2, 3, \ldots\end{aligned}$$

and $f(1) = \frac{1-\gamma}{1-\theta\gamma}\theta r\gamma f(0)$ *with* $f(0) = \left(\frac{1-\theta}{1-\theta\gamma}\right)^r$.

Proof. Differentiation in (27) leads to

$$\psi'(s) = \frac{\theta r}{1-\theta\psi_1(s)}\psi_1'(s)\psi(s), \tag{32}$$

where $\psi(s) = \sum\limits_{i=0}^{\infty} f(i)s^i$, $\psi'(s) = \sum\limits_{i=0}^{\infty}(i+1)f(i+1)s^i$, and

$$\psi_1'(s) = \frac{(1-\gamma)\gamma}{(1-(1-\gamma)s)^2} \tag{33}$$

is the derivative of (26). So, the equation (32) has the form

$$\begin{aligned}(1-\theta\gamma)\sum_{i=0}^{\infty}(i+1)f(i+1)s^i &= (2-\theta\gamma)(1-\gamma)\sum_{i=1}^{\infty} if(i)s^i\\ &-(1-\gamma)^2\sum_{i=2}^{\infty}(i-1)f(i-1)s^i + r\theta\gamma(1-\gamma)\sum_{i=0}^{\infty} f(i)s^i.\end{aligned}$$

The alternative recursions are obtained by equating the coefficients in front of s^i on both sides for fixed $i = 0, 1, 2, \ldots$

2.4.2 Moments of compound Pólya distribution

The mean and the variance of the compound Pólya distribution are given by

$$E(N) = \frac{(1-\gamma)\theta r}{(1-\theta)\gamma} \quad \text{and} \quad Var(N) = \frac{(1-\gamma)((1-\gamma)(2-\theta)+(1-\theta)\gamma)}{((1-\theta)\gamma)^2}\theta r.$$

For the Fisher index of dispersion we obtain

$$FI(N) = \frac{Var(N)}{E(N)} = 1 + \frac{(1-\gamma)(2-\theta)}{(1-\theta)\gamma} > 1,$$

i.e. the compound Pólya distribution is over-dispersed related to the Poisson distribution. This makes the compound Pólya distribution suitable for financial data.

2.5 Compound binomial distribution

Let us consider again the random sum (24), where the r.v.'s X_i are iid as the r.v. X and $X \sim Ge(\gamma)$. The PMF and the PGF of the r.v. X are given by (25) and (26). The r.v. Z is independent of the r.v.'s X_i, $i = 1, 2, \ldots$ The r.v. Z has a binomial distribution with parameters $n \in \mathbf{N}$ and $\theta \in (0, 1)$, with notation $Z \sim Bi(n, \theta)$. Then the r.v. N has a compound binomial distribution with compounding r.v. X.

The PMF and PGF of the r.v. Z are given by

$$P(Z = i) = \binom{n}{i}\theta^i(1-\theta)^{n-i}, \quad i = 0, 1, \ldots, n$$

and

$$\psi_Z(s) = (1-\theta+\theta s)^n.$$

Then the PGF of N is given by

$$\psi(s) = \psi_N(s) = (1-\theta+\theta\psi_1(s))^n, \tag{34}$$

where $\psi_1(s)$ is the PGF of the compounding distribution, given in (26).

Definition 6 *The probability distribution of the r.v. N, defined by the PGF (34) and compounding distribution, given by (25) and (26) is called a compound binomial distribution.*

2.5.1 The Probability Mass Function

Rewriting the PGF in (34) leads to

$$\psi(s) = \sum_{m=0}^{n} \binom{n}{m} \left(\frac{\theta\gamma}{1-(1-\gamma)s}\right)^{m} (1-\theta)^{n-m}. \tag{35}$$

From (35) follows

$$\psi^{(i)}(s) = (1-\gamma)^{i} \sum_{m=1}^{n} \binom{n}{m} \frac{m(m+1)\dots(m+i-1)}{(1-(1-\gamma)s)^{m+i}} (\theta\gamma)^{m}(1-\theta)^{n-m}.$$

Using (29) is obtained the following Lemma.

Lemma 5 *The PMF of the compound binomial distribution is given by*

$$\begin{aligned} f(0) &= (1-\theta+\theta\gamma)^{n}, \\ f(i) &= (1-\gamma)^{i} \sum_{m=1}^{n} \tbinom{n}{m}\tbinom{m+i-1}{i}(\theta\gamma)^{m}(1-\theta)^{n-m}, \quad i=1,2,\dots \end{aligned}$$

In the next proposition are given an extension of the Panjer recursion formulas (see [23]).

Proposition 5 *The PMF of the compound binomial distribution satisfies the following recursion formulas*

$$\begin{aligned} &(1-\theta+\theta\gamma)if(i) \\ &= (1-\gamma)\left[(1-\theta)(i-1)f(i-1) + n\theta\sum_{j=0}^{i-1} q_j f(i-j-1)\right], \ i=2,3,\dots \end{aligned} \tag{36}$$

and $f(1) = \frac{n\theta(1-\gamma)}{1-\theta+\theta\gamma} q_0 f(0)$ *with* $f(0) = (1-\theta+\theta\gamma)^{n}$.

Proof. Differentiation in (34) leads to

$$\psi'(s) = \frac{n\theta(1-\gamma)}{(1-\theta)(1-(1-\gamma)s)+\theta\gamma}\psi_1(s)\psi(s), \tag{37}$$

where $\psi(s) = \sum_{i=0}^{n} f(i)s^{i}$, $\psi'(s) = \sum_{i=0}^{n}(i+1)f(i+1)s^{i}$, and $\psi_1(s) = \sum_{j=0}^{\infty} q_j s^{j}$. After substituting $\psi(s)$, $\psi'(s)$, $\psi_1(s)$, changing the variable from $i+j = l \Rightarrow i = l-j$ and after equivalent transformations in (37) yields to

$$[1-\theta+\theta\gamma]\sum_{i=0}^{n}(i+1)f(i+1)s^i = (1-\theta)(1-\gamma)\sum_{i=1}^{n+1} if(i)s^i$$
$$+n\theta(1-\gamma)\left[\sum_{i=0}^{n}\sum_{j=0}^{i} q_j f(i-j) + \sum_{i=n}^{\infty}\sum_{j=i-n}^{i} q_j f(i-j)\right] s^i.$$

The recursions (36) are obtained by equating the coefficients of s^i in front of the both sides of the previous equation for fixed $i = 0, 1, 2, \ldots$

Corollary 5 *The PMF of the compound binomial distribution satisfies the recursions*

$$\begin{aligned}(1-\theta+\theta\gamma)if(i) &= (1-\gamma)\left[(i-1)(2-2\theta+\theta\gamma)+n\theta\gamma\right] f(i-1)\\ &\quad -(1-\gamma)^2(1-\theta)(i-2)f(i-2),\ i=2,3,\ldots\end{aligned}$$

and $f(1) = \frac{n\theta\gamma(1-\gamma)}{1-\theta+\theta\gamma} f(0)$ *with* $f(0) = (1-\theta+\theta\gamma)^n$.

Proof. Differentiation in (34) leads to

$$\psi'(s) = \frac{n\theta}{1-\theta+\theta\psi_1(s)}\psi_1'(s)\psi(s), \tag{38}$$

where $\psi(s) = \sum_{i=0}^{n} f(i)s^i$, $\psi'(s) = \sum_{i=0}^{n}(i+1)f(i+1)s^i$, and $\psi_1'(s)$ is given by (33). So, the equation (38) has the form

$$(1-\theta+\theta\gamma)\sum_{i=0}^{n}(i+1)f(i+1)s^i = (2-2\theta+\theta\gamma)(1-\gamma)\sum_{i=1}^{n+1} if(i)s^i$$
$$-(1-\gamma)^2(1-\theta)\sum_{i=2}^{n+2}(i-1)f(i-1)s^i + n\theta\gamma(1-\gamma)\sum_{i=0}^{n} f(i)s^i.$$

The recursions are obtained by equating the coefficients of s^i on both sides for fixed $i = 0, 1, 2, \ldots$

2.5.2 Moments of compound binomial distribution

The mean and the variance of the compound binomial distribution are given by

$$E(N) = \frac{n\theta(1-\gamma)}{\gamma} \quad \text{and} \quad Var(N) = \frac{(1-\gamma)((1-\gamma)(2-\theta)+\gamma)}{\gamma^2} n\theta.$$

For the Fisher index of dispersion we obtain

$$FI(N) = \frac{Var(N)}{E(N)} = 1 + \frac{(1-\gamma)(2-\theta)}{\gamma} > 1,$$

i.e. the compound binomial distribution is over-dispersed related to the Poisson distribution. This makes the compound binomial distribution suitable for financial data.

3 Concluding remarks

In this paper we introduced five essential counting distributions with their applications in risk theory as corresponding distributions of counting processes in risk models. The probability generating functions, the probability mass functions, moments, recursion formulas and alternative recursion formulas are obtained. Some nice properties of these distributions are derived.

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