Mechanical Mathematical Modelling of Two-Vehicle Collisions

Stanimir Karapetkov[1], Lubomir Dimitrov[2], Hristo Uzunov[1] and Silvia Dechkova[1(✉)]

[1] Technical University of Sofia, Faculty and College – Sliven, Bulgaria
si_yana@abv.bg
[2] Technical University of Sofia, Kliment Ohridski Blvd., Sofia 1000, Bulgaria

Abstract. Globally, statistical analysis reports show that the most serious vehicle collisions mainly constitute of two-vehicle accidents. The dynamic and kinematic model of car crashes is particularly challenging when requirements for improved accuracy of the analysis are introduced. For more sophisticated models the more accurate the analysis, the more variables in each phase of motion and impact need to be introduced. Two basic problems must be solved, defined by the final rest position of the cars: whether it is known or must be found by solving Cautchy problem. This article presents a dynamic impact analysis between two vehicles, using data for known final rest position and avoiding the uneasy task of selecting the correct coefficient of restitution. Undeniably, it represents the ratio between post impact and prior to impact relative velocity of the centers of mass of the two vehicles in projection in the direction of the crash pulse. The presented results were obtained by simulation and graphical dependencies of the proposed algorithm for reliability analysis estimation, followed by the deduced discrete positions of post-impact vehicles motion, which confirms the useful application of the proposed algorithm. A comparative study was carried out based on the known treds from first-hand inspection in the field accident.

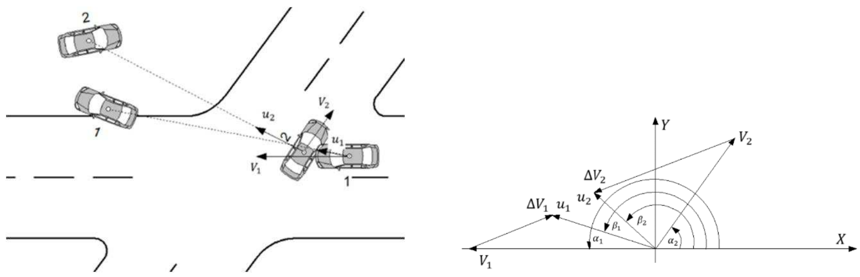
Keywords: Car, Crash, Vehicle collisions, Velocity, Mechanics.

1 Introduction

The two-car collision was reconstructed as two solids were mounted on elastic supports. The latter were connected to a platform with wheels attached onto it [1–5]. Proper selection of vehicle technical parameters such as mass characteristics, support elasticity, damping properties of shock absorbers and so on determined the accuracy of the dynamic investigation of the post-impact macro motion of the bodies. On the other hand, the problem of impact was solved using a mechanical-mathematical model developed for the purpose. The two vehicles were placed and oriented in the coordinate system as they had been at the point of impact and in position to each other, which completely corresponded to first-hand inspection in the field accident, including vehicle deformation data [6–10].

2 Impact Problem

Determine magnitude and direction of the crash pulse, pre-impact velocity of the centers of mass of the two vehicles if post-impact velocity of the centers of mass of the two vehicles are known (Fig. 1).



(a) (b)

Fig. 1. (a) Car accident diagram and (b) Vector analysis.

Boundary-value problem of impact: Determine the following initial conditions: post-impact linear and angular velocity of both vehicles if some of the initial conditions are known, such as the position of the two vehicles at the moment of impact and the vehicles final rest position.

The simplest and easiest way to solve the problem when two vehicles are impacted is by applying Momentum Conservation Principle that has the form [11–17]:

$$m_1.\vec{V}_1 + m_2.\vec{V}_2 = m_1.\vec{u}_1 + m_2.\vec{u}_2 \tag{1}$$

where m_1 and m_2 are total vehicles mass; V_1 and V_2 - velocity of the centers of mass of the vehicles right before impact; u_1 and u_2- velocity of the centers of mass of the vehicles right after impact. The application of the principle has some limitations, assuming that both cars are material points. The second equation of the conservation of momentum principle involves trigonometric function of "sin", in which a small change of the angle causes significant deviations in the final solution of the system.

This ratio is actually the coefficient of restitution, which is as follows:

$$k = \frac{|\Delta u_n|}{|\Delta V_n|} = \frac{|(\vec{u}_2 - \vec{u}_1).\vec{e}|}{|(\vec{V}_1 - \vec{V}_2).\vec{e}|} \tag{2}$$

where $\vec{e}$ is a single vector of the crash pulse vector and the "k" is the coefficient of recovery. “Here Δu_n is projection of the relative velocity between the centers of mass of the vehicles after the impact on the crash pulse directrix, and ΔV_n - projection of the relative velocity between the centers of mass of the vehicles before impact on the crash pulse directrix.

It is obvious that there is a necessity to select a particular value for the coefficient of restitution in the range of $0 \leq k \leq 1$, with a possible increase in steps of 0.001.

The impact for each of the vehicles is characterized by the impulse-momentum change theorem for the time of impact, which for each of the vehicles has the form:

$$m.\vec{u} - m.\vec{V} = \vec{S} \tag{3}$$

where $\vec{u}$ is the velocity of the center of mass of the post-impact vehicle; $\vec{V}$ is the velocity of the center of mass of prior-to-impact vehicle; $\vec{S}$ – crash pulse.

The impulse-momentum change theorem applied to the mechanical system of the car related to the vehicle center of mass during its relative motion around it has the form of:

$$\frac{d\vec{K}_C^r}{dt} = \vec{M}_C^{(e)} \tag{4}$$

where $\vec{M}_C^{(e)}$ is the principle of moments of impact forces related to the vehicle center of mass.

After solving the system of equations (4), the projections of the crash pulse are determined as follows:

$$S_x = \frac{J_{1z}\,\omega_{z1}\,\rho_{2x} + J_{2z}\,\omega_{z2}\,\rho_{1x}}{\rho_{1x}\,\rho_{2y} - \rho_{2x}\,\rho_{1y}}; \quad S_y = \frac{J_{1z}\,\omega_{z1}\,\rho_{2y} + J_{2z}\,\omega_{z2}\,\rho_{1y}}{\rho_{1x}\,\rho_{2y} - \rho_{2x}\,\rho_{1y}} \tag{5}$$

where J_{1z}, J_{2z} are the mass moments of inertia of the vehicles around the central vertical axes perpendicular to the plane of motion; ω_{z1}, ω_{z2} – angular velocities of the vehicles after collision; ρ_{x_j}, ρ_{y_j}, $j = 1{,}2$ - coordinates of the point of application A_j of the impact force related to a center of mass reference frame, moving translationally.

The magnitude of the projections of the crash pulse is used to obtain the velocities of the centers of mass of the two vehicles before collision according to equations (3) after their projection on two mutually perpendicular axes:

$$V_{1x} = u_{1x} - \frac{S_x}{m_1}; \; V_{1y} = u_{1y} - \frac{S_y}{m_1}; V_{2x} = u_{2x} + \frac{S_x}{m_2}; \; V_{2y} = u_{2y} + \frac{S_y}{m_2} \tag{6}$$

In the above equations, the magnitude of ω_{z1}, ω_{z2}, i.e. the angular velocities of the cars after collision and $\vec{u}_j$ – the vehicle velocities of the centers of mass after collision should be analyzed.

The differential equations of motion for each vehicle after impact, considered as a multi-mass spatial mechanical system, have the type (Fig. 2):

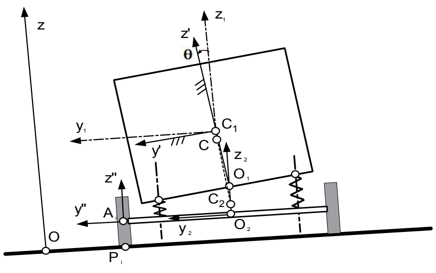


Fig. 2. Rear view of the automobile.

$$m\ddot{x}_C = \sum_{i=1}^{4}[F_{ix}] + mg\sin\xi - W_x\sqrt{\dot{x}_C^2 + \dot{y}_C^2}\dot{x}_C \quad (7)$$

$$m\ddot{y}_C = \sum_{i=1}^{4}[F_{iy}] + mg\sin\nu - W_y\sqrt{\dot{x}_C^2 + \dot{y}_C^2}\dot{y}_C \quad (8)$$

$$m\ddot{z}_C = \sum_{i=1}^{4} N_i - \frac{mg}{\sqrt{1+tg^2\xi+tg^2\nu}} \quad (9)$$

where ξ – longitudinal slope of road; ν – transverse slope of road.

After some transformations and projection of equation (5) on the axes permanently connected to the unsprung mass, the system of differential equations in matrix is obtained:

$$\{[J_{C_1}] + [J_{C_3}]\}\,[\dot{\omega}] = [M_{C_\omega}] + [M_{C_{3}\omega}] + [M_{C_a1}] + [M_{C_a2}] + [M_{C_a3}] + [M_{C_k}] + [M_{C_m}] \quad (10)$$

where $[\dot{\omega}] = [\dot{\omega}_{x'}\ \ \dot{\omega}_{y'}\ \ \dot{\omega}_{z'}]^T$ is a matrix-column of the derivatives of the angular velocity projections on the coordinate axes permanently connected to the unsprung mass:

$$[M_{C_\omega}] = \begin{bmatrix} M_{F_{x'}} + M_{N_{x'}} + J_{3x'z'}\omega_{3x'}\omega_{3y'} - J_{3x'y'}\omega_{3x'}\omega_{3z'} + \\ (J_{3y'} - J_{3z'})\omega_{3y'}\omega_{3z'} + J_{3y'z'}\left(\omega_{3y'}^2 - \omega_{3z'}^2\right) \\ M_{F_{y'}} + M_{N_{y'}} + J_{3x'y'}\omega_{3y'}\omega_{3z'} - J_{3y'z'}\omega_{3x'}\omega_{3y'} + \\ (J_{3z'} - J_{3x'})\omega_{3x'}\omega_{3z'} + J_{3x'z'}\left(\omega_{3z'}^2 - \omega_{3x'}^2\right) \\ M_{F_{z'}} + M_{N_{z'}} + J_{3y'z'}\omega_{3x'}\omega_{3z'} - J_{3x'z'}\omega_{3y'}\omega_{3z'} + \\ +(J_{3x'} - J_{3y'})\omega_{3x'}\omega_{3y'} + J_{3x'y'}\left(\omega_{3x'}^2 - \omega_{3y'}^2\right) \end{bmatrix} \quad (11)$$

$$[M_{Ck}] = -\begin{bmatrix} \begin{bmatrix} \sum_{i=1}^{4} J_{ky''i}\,\ddot{\gamma}_i\,\vec{J}_{2i}'' + \vec{\omega}_2 \times \sum_{i=1}^{4} J_{k2i}\,\ddot{\gamma}_i\,\vec{J}_{2i}'' + \sum_{i=1}^{4}[J_{kz'i}(\dot{\omega}_2 + \ddot{\vartheta}_{ki}) + \\ +m_{ki}\left|C_2\vec{A}_i\right|^2\dot{\omega}_2]\vec{k}_2 \end{bmatrix}_{x'} \\ \begin{bmatrix} \sum_{i=1}^{4} J_{ky''i}\,\ddot{\gamma}_i\,\vec{J}_{2i}'' + \vec{\omega}_2 \times \sum_{i=1}^{4} J_{k2i}\,\ddot{\gamma}_i\,\vec{J}_{2i}'' + \sum_{i=1}^{4}[J_{kz'i}(\dot{\omega}_2 + \ddot{\vartheta}_{ki}) + \\ +m_{ki}\left|C_2\vec{A}_i\right|^2\dot{\omega}_2]\vec{k}_2 \end{bmatrix}_{y'} \\ \begin{bmatrix} \sum_{i=1}^{4} J_{ky''i}\,\ddot{\gamma}_i\,\vec{J}_{2i}'' + \vec{\omega}_2 \times \sum_{i=1}^{4} J_{k2i}\,\ddot{\gamma}_i\,\vec{J}_{2i}'' + \sum_{i=1}^{4}[J_{kz'i}(\dot{\omega}_2 + \ddot{\vartheta}_{ki}) + \\ +m_{ki}\left|C_2\vec{A}_i\right|^2\dot{\omega}_2]\vec{k}_2 \end{bmatrix}_{z'} \end{bmatrix} \quad (12)$$

$$M_{Ca1} = \begin{bmatrix} [(\vec{r}_C - \vec{r}_{C_1}) \times (\vec{a}_{C_1} - \vec{a}_C)]_{x'} \\ [(\vec{r}_C - \vec{r}_{C_1}) \times (\vec{a}_{C_1} - \vec{a}_C)]_{y'} \\ [(\vec{r}_C - \vec{r}_{C_1}) \times (\vec{a}_{C_1} - \vec{a}_C)]_{z'} \end{bmatrix} \quad (13)$$

$$M_{Ca2} = \begin{bmatrix} [(\vec{r}_C - \vec{r}_{C_2}) \times (\vec{a}_{C_2} - \vec{a}_C)]_{x'} \\ [(\vec{r}_C - \vec{r}_{C_2}) \times (\vec{a}_{C_2} - \vec{a}_C)]_{y'} \\ [(\vec{r}_C - \vec{r}_{C_2}) \times (\vec{a}_{C_2} - \vec{a}_C)]_{z'} \end{bmatrix} \quad (14)$$

$$M_{Ca3} = \begin{bmatrix} \left[(\vec{r}_C - \vec{r}_{C_3}) \times (\vec{a}_{C_3} - \vec{a}_C)\right]_{x'} \\ \left[(\vec{r}_C - \vec{r}_{C_3}) \times (\vec{a}_{C_3} - \vec{a}_C)\right]_{y'} \\ \left[(\vec{r}_C - \vec{r}_{C_3}) \times (\vec{a}_{C_3} - \vec{a}_C)\right]_{z'} \end{bmatrix} \tag{15}$$

$$[M_{Cm}] = -\begin{bmatrix} \sum_{i=1}^{2}\left[J_{mi} + m_{mi}\left|C_2\vec{C}_{mi}\right|^2\right]\dot{\omega}_2\vec{k}_{2x'} \\ \sum_{i=1}^{2}\left[J_{mi} + m_{mi}\left|C_2\vec{C}_{mi}\right|^2\right]\dot{\omega}_2\vec{k}_{2y'} \\ \sum_{i=1}^{2}\left[J_{mi} + m_{mi}\left|C_2\vec{C}_{mi}\right|^2\right]\dot{\omega}_2\vec{k}_{2z'} \end{bmatrix} \tag{16}$$

Here, m is the total mass of the vehicle; m_{ki} $/i = 1 \div 4/$ – mass of each of the wheels; m_{mi} $/i = 1 \div 2/$ - mass of each of the drives; $x_c,\ y_c, z_c$ - coordinates of the vehicle center of mass in relation to the fixed coordinate system; ϑ_k- average angle of rotation of the steering wheels around their axles; $\ddot{\vartheta}_k$ - angular acceleration; γ_i $/i = 1 \div 4/$ - angles of rotation of the wheels on their own rotary axis; $\dot{\gamma}_i$ $/i = 1 \div 4/$ - wheel angular velocity; $\ddot{\gamma}$ - angular acceleration of the wheels; $\vec{F}_i,$ $/i = 1 \div 4/$ - friction forces in the wheels; N_i $/i = 1 \div 4/$ - normal reactions in the wheels; w – drag coefficient; $\vec{\omega}$ - angular velocity of the movable coordinate system $C_1x'y'z'$ permanently connected to the unsprung mass; $\vec{\omega}_2$ - angular velocity of the sprung mass and the constantly connected to it movable coordinate system $O_2x_2y_2z_2$;; $M_{F,N_{x'}}$, $M_{F,N_{y'}}$, $M_{F,N_{z'}}$ - moments of the frictional forces on the wheels and the normal reactions to the permanently connected to the vehicle coordinate axes; $[J_{C1}]$ - matrix of the mass inertia of the bodywork related to the coordinate axes, permanently connected to it; $[\omega]$ - matrix column of the projections of angular velocity on the same axes determined by Euler's formula; $J_{ky''i}$,$J_{kz''i}$ $/i = 1 \div 4/$ - mass inertia of each wheel relative to its own axis of rotation and its radial axis; J_{m_i} - $/i = 1 \div 2/$ - intrinsic mass moment of inertia of each of the drives relative to its central axis parallel to z_2.

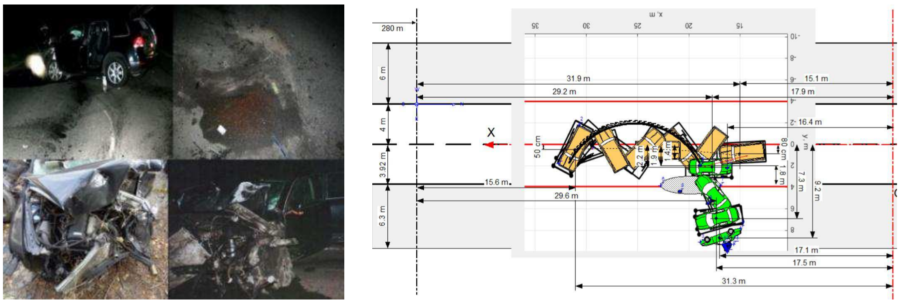
The relative movement of the wheels, differential/s / and the engine is characterized by a system of four differential equations obtained by Lagrange method, which has the type:

$$[I_\gamma][\ddot{\gamma}] = [M_{\gamma i}];\ M_{\gamma i} = \{F_{i\tau}r_i + sign\,(\dot{\gamma}_i)[M_{di} - f_iN_i - M_{si}]\} \tag{17}$$

Where μ is friction coefficient depending on slipping speed on the contact spot; $\vec{r}_i$ – radius of the wheel; f_i – coefficient of rolling friction; $\vec{F}_{i\tau}$ is tangential component of the tire-road friction force, the positive direction of which is taken backwards, in the more frequent cases of braking or loss of stiffness; M_{di}, M_{si} - corresponding engine and brake torque applied to each wheel.

Example: A head-on collision between Volkswagen Tuareg and Opel Vectra with known geometric dimensions, masses and mass moment of inertia was analysed. Their final rest positions, location impact and the position of treds left on the roadway were known.

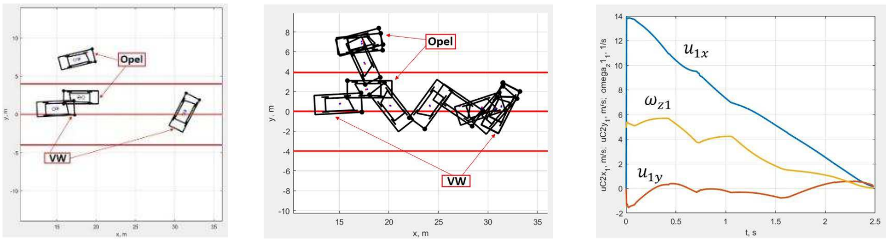
Data from the accident was the available treds on the roadway from the vehicles motion at the time of collision and after collision. They are shown in the photographic material in Fig. 3a. The two-car collision diagram is shown in Fig. 3b.



(a) (b)

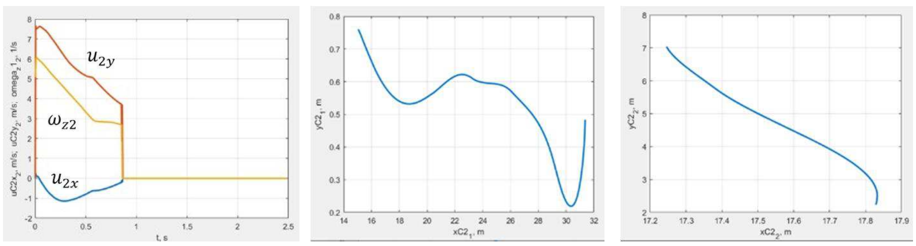
Fig. 3. (a) Photographic material, (b) Diagram of the two-vehicle crash (comparative analysis).

The developed dynamic deformation model determined the location of the two vehicles at the moment of impact and their final rest position (Figs. 4–7 and Table 1).



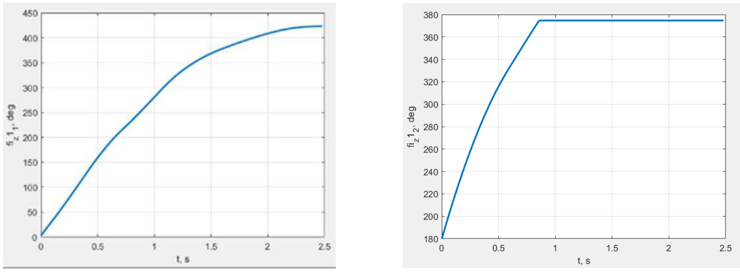
(a) (b) (c)

Fig. 4. (a) Location of vehicles at the point of impact and final rest positions, (b) Discrete positions of the two vehicles motion after collision and (c) Projections of velocity centre of mass after impact and angular velocity about Oz axis for Volkswagen Touareg.



(a) (b) (c)

Fig. 5. (a) Projections of velocity centre of mass after impact and angular velocity about Oz axis for Opel Vectra, (b) Trajectory of the Volkswagen Touareg center of mass and (c) Trajectory of the Opel Vectra center of mass.



(a) (b)

Fig. 6. (a) Change in angle of rotation about Oz axis for the Volkswagen Touareg and (b) Change in angle of rotation about Oz axis for the Opel Vectra.

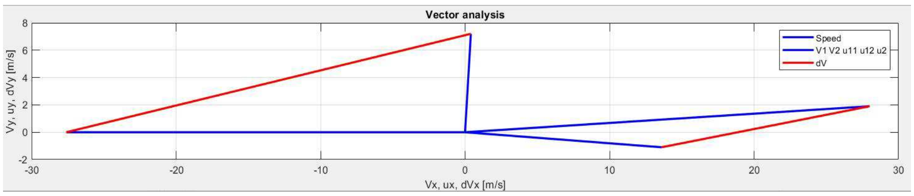


Fig. 7. Vector analysis.

Table 1. Results for the solved impact problem.

Kinematic quantities	Results		
Initial position of the Volkswagen Touareg $x_{c1}; y_{c1}; \varphi_{z1}$	15.1 m	0.8 m	3°
Initial position of the Opel Vectra $x_{c2}; y_{c2}; \varphi_{z2}$	17.7 m	2.2 m	180°
Final position of the Volkswagen Touareg $x_{c1}; y_{c1}; \varphi_{z1}$	31.4 m	0.5 m	63.1°
Final position of the Opel Vectra $x_{c2}; y_{c2}; \varphi_{z2}$	17.2 m	7.0 m	14.7°
Initial velocity prior to impact Volkswagen Touareg $\dot{x}_{c1}; \dot{y}_{c1};$	28.0 ms^{-1}	1.9 ms^{-1}	
Initial velocity prior to impact of the Opel Vectra $\dot{x}_{c2}; \dot{y}_{c2};$	–27.6 ms^{-1}	0 ms^{-1}	
Initial velocity after impact for the Volkswagen Touareg $\dot{x}_{c1}; \dot{y}_{c1}; \dot{\varphi}_{z1}$	13.6 ms^{-1}	–1.1 ms^{-1}	4.99 s^{-1}
Initial velocity after impact for the Opel Vectra $\dot{x}_{c2}; \dot{y}_{c2}; \dot{\varphi}_{z2}$	0.4 ms^{-1}	7.2 ms^{-1}	4.98 s^{-1}
Crash pulse magnitude $S_x; S_y$	–33040 Ns	–6801 Ns	
Coefficient of restitution	k = 0.207		

3 Conclusion

A mathematical model was developed to assist computer simulation of two-vehicle collisions. It has the advantage to determine with great accuracy the place of impact, the position of the two vehicles at the time of impact and the velocities of the centers of mass right before impact, based on the following reliability criteria:

- The trajectories of the vehicles centers of the wheels exactly correspond to the configuration and the position of the tire tracks on the lane.
- The direction of the velocity vectors of the centers of mass of the two vehicles prior to impact correspond completely to the position of the two vehicles at that moment
- The simulation results demonstrate that the crash pulse vector and its directix correspond completely to the deformations of the vehicles and their yaw rotation after impact.
- The absolute values of the velocity change $|\Delta\bar{V}|$ for each car correlate well with the strain energy.
- It is not necessary to select coefficient of restitution according to the theory of impact, but it is determined on the basis of the presented scientific approach.

Acknowledgements. The author/s would like to thank the Research and Development Sector at the Technical University of Sofia for the financial support.

References

1. Tomasch, E., Sinz, W., Hoschopf, H., Kolk, H., Steffan, H.: Prospektive Bewertung der Kollisionsschwere von Le Fahrzeugen unter Berьcksichtigung eines Kollisionsminderungssystems. In: Proceedings of the 10. VDI-Tagung Fahrzeugsicherheit, Sicherheit 2.0, pp. 407–418. Berlin (2015).
2. Tomasch, E., Steffan, H.: ZEDATU (Zentrale Datenbank tödlicher Unfälle in Österreich). A central database of fatalities in Austria. In: Proceedings of the International Conference Expert Symposium on Accident Research, ESAR 2006, pp. 183–185. Hannover (2006).
3. Winner, H., Hakuli, S., Lotz, F., Singer C.: Handbuch Fahrerassistenzsysteme, Grundlagen, Komponenten und Systeme für aktive Sicherheit und Komfort. Springer Vieweg, Wiesbaden (2015).
4. Zauner, C., Tomasch, E., Sinz, W., Ellersdorfer, C., Steffan, H.: Assessment of the effectiveness of intersection assistance systems at urban and rural accident sites. In: Proceedings of the International Conference Expert Symposium on Accident Research, ESAR 2014, pp. 1–10. Fachverlag NW in der Carl Schuenemann Verlag GmbH (2015).
5. Zhou, J., Lu, J., Peng, H.: Vehicle stabilization in response to exogenous impulsive disturbances to the vehicle body. Int. J. Veh. Auton. Syst. 8(2/3/4), 242–262 (2010).
6. Tomasch, E., Sinz, W., Hoschopf, H., Kolk, H., Steffan, H.: Bewertungsmethodik von integralen Sicherheitssystemen durch Kombination von Test und Simulation am Beispiel von Fugngerunfllen. In: Proceedings of the 10. VDI-Tagung Fahrzeugsicherheit, Sicherheit 2.0, pp. 157–169. Berlin (2015).
7. Billicsich, S., Tomasch, E., Sinz, W., Eichberger, A., Markovic, G., Magosi, Z.: Potentieller Einfluss von C2X auf die Vermeidung von Motorradunfllen bzw. Reduktion der Verletzungsschwere. In: Proceedings of the 10. VDI-Tagung Fahrzeugsicherheit, Sicherheit 2.0, pp. 383–392. Berlin (2015).
8. Burg, H., Moser, A.: Handbuch Verkehrsunfallrekonstruktion: Unfallaufnahme, Fahrdynamik, Simulation. Vieweg+Teubner Verlag, Wiesbaden (2009).
9. Jung, H.G., Cho, Y.H., Kim, J. Isrss: Integrated side/rear safety system. Int. J. Automot. Technol. 11, 541–553 (2010).

10. Tomasch, E., Steffan, H., Darok, M.: Retrospective accident investigation using information from court. In: Proceedings of the European Road Transport Research Arena Conference, TRA 2008, poster. Ljubljana (2008).
11. Eidehall, A., Pohl, J., Gustafsson, F., Ekmark, J.: Toward autonomous collision avoidance by steering. IEEE T. Intell. Transp. 8(1), 84–94 (2007).
12. Karapetkov, S.: Mechanics of solids and impact between bodies in identifying road accidents (2006).
13. Karapetkov, S.: Mechanical and mathematical modelling of vehicle motions in identifying road accidents (2012).
14. Karapetkov, S., Dimitrov, L., Uzunov, Hr., Dechkova, S.: Identifying vehicle and collision impact by applying the principle of conservation of mechanical energy. Transp. Telecommun 20(3), 191–204 (2019).
15. Karapetkov, S., Dimitrov, L., Uzunov, Hr., Dechkova, S.: Examination of vehicle impact against stationary roadside objects. IOP Conf. Ser.: Mater. Sci. Eng. 659, 012063-1–012063-13 (2019).
16. Sander, U., Mroz, K., Bostroem, O., Fredriksson, R.: The effect of pre-pretensioning in multiple impact crashes. In: Proceedings of the 21st International Technical Conference on the Enhanced Safety of Vehicles, ESV 2009, pp. 1–11. Stuttgart (2009).
17. Togawa, A., Murakami, D., Saeki, H., Pal, C., Okabe, T.: An insight into multiple impact crash statistics to search for future directions of counter-approaches. In: Proceedings of the 22nd International Technical Conference on the Enhanced Safety of Vehicles, ESV 2011, pp. 1–5. Washington (2011).