RESEARCH OF VEHICLES DIRECTIONAL STABILITY

Hr. Lzpoldzhiyska Ph.D., Prof. L. Kunchev,
Faculty of Transport – technical University of Sofia, Bulgaria
hgeorgieva@tu-sofia.bg, lkuchev@tu-sofia.bg

Abstract: In this work for the research of vehicles directional stability was used three different mechanical-mathematical models. The behavior of car stability has been simulated with MATLAB software.

Keywords: DIRECTIONAL STABILITY, MECHANICAL-MATHEMATICAL MODEL, LATERAL ELASTICITY, SLIP ANGLE

1. Introduction

Stability of a vehicle concerns itself with the tendency of a vehicle to return to its original direction when disturbed (rotated) away from that original direction.

The goal of this work is research of vehicles directional stability. The situation is described by three different mechanical-mathematical models around to the axes $O_x$ and $O_z$.

During the research we make the following considerations:

- The car and his models are symmetrical to the axe $O_x$
- The dynamical process (displacement around to the axes $O_z$) isn’t examined.
- The main factor of cars stability is the wheels slip angle.
- Working in the zone of pure rolling motion (the slip angle for the wheels was zero). Figure 1 shows the general relationship between the lateral force a tire $F_y$ and the slip angle of the tire $\delta_y$.

![Fig. 1 Relationship between the lateral force a tire $F_y$ and the slip angle of the tire $\delta_y$.](Image)

Mechanical-mathematical model examine only the lateral elasticity

The behavior of the car is described using mechanical-mathematical model. Scheme of the model is shown to Figure 2 and the suspended masses include the masses of the elements of the car, passengers and load. If we consider that the slip angle for the wheels was zero we could examine the car as a mass witch point 1, 2, 3 and 4 around to the axe $O_x$ are bending.

The motion of system is exarninated as function to the displacements are around the axes $O_x$ and the angular displacement around the axes $O_z$.

In the center of gravity is fixed local coordinate system attached $O_x0y0z0$. All displacements of local coordinate systems are given to the absolute coordinate system $O_xA_yA_zA$. In the equilibrium position the axis of all coordinate systems are parallel.

![Fig. 2 Mechanical-mathematical model, examined the zone of directional stability](Image)

To find the lows motion to the absolute coordinate system $O_xA_yA_zA$ is necessary to define the transition matrices of each local coordinate systems to the absolute.

For generalized coordinate systems are assumed:

- $y_0$ - linear displacement of the local coordinate system $O_x0y0z0$ to absolute $O_xA_yA_zA$ around axis $O_z$;
- $\psi_0$ - angular displacement of the coordinate system $O_x0y0z0$ to absolute $O_xA_yA_zA$ around axis $O_z$.

Matrix of transition from $O_x0y0z0$ to $O_xA_yA_zA$ to the system is:

$$T^{O_xA_yA_zA}_{O_x0y0z0} = \begin{bmatrix}
\cos \psi_0 & -\sin \psi_0 & 0 & 0 \\
\sin \psi_0 & \cos \psi_0 & 0 & y_0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}$$

$x_0$ and $z_0$ are zero (we consider only linear displacement around axis $O_x$, i.e. only laterally).

The matrix of transition $T^{O_xA_yA_zA}_{O_x0y0z0}$ can be simplified with the assuming that $\psi_0$ is small witch leads to the linearization of trigonometrically functions ($\sin \psi_0 \approx \psi_0$ and $\cos \psi_0 \approx 1$):

$$T^{O_xA_yA_zA}_{O_x0y0z0} = \begin{bmatrix}
1 & -\psi_0 & 0 & 0 \\
\psi_0 & 1 & 0 & y_0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}$$

Radius vector of the point gravity: $\rho_{mo} = [0, y_0, 0, 1]^T$

Radius vector of the point 1: $\rho^0 = [a_1, -b, 0, 1]^T$

$$\rho^1 = T^{O_xA_yA_zA}_{O_x0y0z0} \rho^0 \rightarrow \begin{bmatrix}
a_1 + b y_0 \\
y_0 + a_1 \psi_0 - b \\
0 \\
1
\end{bmatrix}$$
Radius vector of the point 2: \( \rho_2^0 = [a_2, b, 0, 1]^T \)

\[
\rho_2^4 = T_0^4 \cdot \rho_2^0 \rightarrow \begin{pmatrix} a_1 - b \psi_0 \\ y_0 + a \psi_0 + b \\ 0 \\ 1 \end{pmatrix}
\]

Radius vector of the point 3: \( \rho_3^0 = [-a_2, -b, 0, 1]^T \)

\[
\rho_3^4 = T_0^4 \cdot \rho_3^0 \rightarrow \begin{pmatrix} b \psi_0 - a_2 \\ y_0 - a \psi_0 - b \\ 0 \\ 1 \end{pmatrix}
\]

Radius vector of the point 4: \( \rho_4^0 = [-a_2, b, 0, 1]^T \)

\[
\rho_4^4 = T_0^4 \cdot \rho_4^0 \rightarrow \begin{pmatrix} -a_2 - b \psi_0 \\ y_0 - a \psi_0 + b \\ 0 \\ 1 \end{pmatrix}
\]

The components of the angular velocity of the system are set in advance:

\[
\dot{\omega}_{yx}^0 = 0
\]
\[
\dot{\omega}_{xy}^0 = 0
\]
\[
\dot{\omega}_{xz}^0 = \psi
\]

Kinetic energy of the system is

\[
T = \frac{1}{2} m_0 \dot{y}^2 + \frac{1}{2} J_{xy} \dot{\psi}^2
\]

Potential energy of the system is

\[
U = \frac{1}{2} \left[ c_{11} \left( y_0 + a \psi_0 + b \right)^2 + c_{12} \left( y_0 - a \psi_0 - b \right)^2 + c_{13} \left( y_0 - a \psi_0 + b \right)^2 + c_{14} \left( y_0 - a \psi_0 - b \right)^2 \right]
\]

Applying Lagrange’s equation of 2nd kind

\[
\frac{d}{dt} \left( \frac{\partial \mathbf{L}}{\partial \dot{\mathbf{q}}} \right) = \frac{\partial \mathbf{L}}{\partial \mathbf{q}} + \mathbf{M} \mathbf{\ddot{q}} - \mathbf{F}
\]

The differentials equations of the system are

\[
m_0 \ddot{y} + \left[ c_{11} \left( y_0 + a \psi_0 + b \right) + c_{12} \left( y_0 - a \psi_0 - b \right) + c_{13} \left( y_0 - a \psi_0 + b \right) + c_{14} \left( y_0 - a \psi_0 - b \right) \right] \ddot{\psi} = \mathbf{M} \mathbf{\ddot{q}} - \mathbf{F}
\]

\[
f_{x\dot{x}} + \left[ f_{x\dot{x}} \left( y_0 + a \psi_0 + b \right) + f_{x\dot{y}} \left( y_0 - a \psi_0 - b \right) + f_{x\dot{z}} \left( y_0 - a \psi_0 + b \right) + f_{x\dot{z}} \left( y_0 - a \psi_0 - b \right) \right] \ddot{\psi} = \mathbf{M} \mathbf{\ddot{q}} - \mathbf{F}
\]

For the differentials equations describing the system from figure 2 are valid:

\[
[M] \ddot{q} + [C] \dot{q} = [F], \text{или}
\]

\[M\] is the matrix of inertia which is symmetrical to the main diagonal, with dimension 2x2 and has the following form:

\[
[M] = \begin{pmatrix} m_0 & 0 \\ 0 & J_{xy} \end{pmatrix}
\]

\[C\] is the matrix of elasticity which is also symmetrical to the main diagonal and has dimension 2x2:

\[
[C] = \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{12} & c_{11} & c_{14} & c_{13} \\ c_{13} & c_{14} & c_{11} & c_{12} \\ c_{14} & c_{13} & c_{12} & c_{11} \end{pmatrix}
\]

\[B\] is the matrix of dissipative forces, showing the influence of damper. Also symmetric with dimension 2x2, in our case \([B] = 0\).

The generalized coordinates and their derivatives are:

\[
[q] = \begin{pmatrix} y_0 \\ \psi_0 \\ y_0 \\ \psi_0 \end{pmatrix}
\]

To obtain natural frequencies of the system, the equations are represented in Cauchy’s normal form: \( y + Ly = 0 \), where \( L \) has the following form:

\[
L = \begin{pmatrix} M^{-1}B & M^{-1}C \\ 0 & I \end{pmatrix}
\]

The output parameters of the system – displacement, velocity and acceleration are obtained from the equations: \( y + Ly = Y \), where \( Y \) is:

\[
Y = \begin{pmatrix} M^{-1} \cdot F(t) \\ 0 \end{pmatrix}
\]

All solutions to the definite time interval are obtained after integration of the system using the method of Runge - Kutta.

**Plane model examine only the slip angle**

Figure 3 shows the schema of the model examining the car stability with reporting the wheel slip angle. The model is known as Rocards model.

**Fig. 3 Plane model describing only the slip angle**

The initial conditions were assumed for modeling:

- The point 1, 2, 3 and 4 to the fig.3 are the projection of the wheels;
- Considering that the wheels are undeformables and can ignore their lateral elasticity around to the axis \( O_y \);
- The car is shown at the angle \( \psi \) to the axis \( O_y \).

From fig. 4 the slip angles are \( \delta_1, \delta_2, \delta_3, \delta_4 \) can be determinate by coordinate \( y \), so we have:

**Fig. 4 Slip angle**

\[
tg(\psi) = \frac{dy}{dx} \rightarrow \psi = \frac{dy}{dx} \rightarrow dy = \psi dx
\]

\[
tg(\psi + \delta) = \frac{dy_1}{dx} \rightarrow \delta = \frac{dy_1}{dy} - \psi \rightarrow \delta = \frac{dy_1 - dy}{dy} = \frac{dy_1 - \psi dx}{v dt}
\]

If the vehicle moved with a pure rolling motion (the slip angle for the wheels were all zero) then the displacements of the point of contact of the wheels with the ground would be the same as the
displacement of the center of gravity C in any interval $\delta t$: 
x=dx\ dt, dx=dx,\ e=Vdt, where V is the speed of the vehicle in direction $O_x$.

The equations of motion are:

$$m\ddot{y} + k_1(\delta_1 + \delta_2) + k_2(\delta_3 + \delta_4) = F_y$$
$$J_{oz}\ddot{\psi} + k_1a_1(\delta_4 + \delta_1) - k_2a_2(\delta_1 + \delta_3) = M_\nu$$

Substituting for $\delta_1, \delta_2, \delta_3, \delta_4$ we obtain:

$$m\ddot{y} + 2\left(\begin{array}{c} k_1 + k_2 \\ k_1 - k_2 \\ k_1 - k_2 \\ k_1 - k_2 \end{array}\right) + 2\left(\begin{array}{c} k_1a_1 + k_2a_1 \\ k_2a_1 - k_1a_1 \\ k_1a_1 - k_2a_2 \\ k_2a_1 - k_1a_2 \end{array}\right) = 0$$

After matrix development is obtained:

$$p^2 + 2\left(\begin{array}{c} k_1 + k_2 \\ k_1 - k_2 \\ k_1 - k_2 \\ k_1 - k_2 \end{array}\right) + 2\left(\begin{array}{c} k_1a_1 + k_2a_1 \\ k_2a_1 - k_1a_1 \\ k_1a_1 - k_2a_2 \\ k_2a_1 - k_1a_2 \end{array}\right) = 0$$

This may be divides by $p^4$:

$$p^2 + 2\left(\begin{array}{c} k_1 + k_2 \\ k_1 - k_2 \\ k_1 - k_2 \\ k_1 - k_2 \end{array}\right) + 2\left(\begin{array}{c} k_1a_1 + k_2a_1 \\ k_2a_1 - k_1a_1 \\ k_1a_1 - k_2a_2 \\ k_2a_1 - k_1a_2 \end{array}\right) = 0$$

The analysis of stability is made with the criteria of Raus-Kurvits by creating a square matrix from the characteristic equation $A(s) = a_0s^n + a_1s^{n-1} + ... + a_{n-1}s + a_n = 0$.

For being systems stability is necessary $a_0 > 0$ and $\Delta_1, \Delta_2, ..., \Delta_n$ to be always positive. Where $\Delta_1, \Delta_2, ..., \Delta_n$ are:

$$\Delta_1 = \begin{bmatrix} a_1 \ v_1 \\
                         a_2 \ v_2 \\
                         a_3 \ v_3 \\
                         a_n \ v_n \end{bmatrix}$$

In our case $a_0 > 0$, so to have a cars stability motion is necessary the moment of intertia of front wheels to be smaller than that of rear wheels.

For the vehicles speed is obtained:

$$v^2 = \frac{4k_1a_1 + k_2a_2}{2k_1a_1 - k_2a_2}$$

Plane model examine lateral elasticity and slip angle  

The initial conditions for this case of modeling were the same as those from fig.3. But this time the lateral elasticity of the wheel is added (fig.5).

Fig 5 Plane model about lateral elasticity and slip angle  

Applying the models initial condition for the differentials equations describing the system are valid:

$$m\ddot{y} + k_1(\delta_1 + \delta_2) + k_2(\delta_3 + \delta_4) + 2(c_1y + c_2) \psi = F_y$$
$$J_{oz}\ddot{\psi} + k_1a_1(\delta_4 + \delta_1) - k_2a_2(\delta_1 + \delta_3) + 2(c_1a_1^2 + c_1b^2) \psi + 2(c_1a_1 + c_1b) y = M_\nu$$

Substituting for $\delta_1, \delta_2, \delta_3, \delta_4$ we obtain:

$$m\ddot{y} + 2\left(\begin{array}{c} k_1 + k_2 \\ k_1 - k_2 \\ k_1 - k_2 \\ k_1 - k_2 \end{array}\right) + 2\left(\begin{array}{c} k_1a_1 + k_2a_1 \\ k_2a_1 - k_1a_1 \\ k_1a_1 - k_2a_2 \\ k_2a_1 - k_1a_2 \end{array}\right) = 0$$

1. Computer Simulation

All vehicles models permit by variable of the input information to explore the influence of some construction parameters into the directional stability.

Vehicle characteristics and parameters’ numerical values were taken directly from the literature.

To solve the equations was made a program with MATLAB software. It simulates the behavior of car with different speed, with different position of center of gravity and variable characteristic of wheels. The investigations were made in two parts – without and with disturbing force. This force works only for a few seconds.

| m | Vehicle mass | 1 500 | kg |
| a | Distance from CG to front axel | 1 | m |
| b | Distance from CG to rear axel | 1.5 | m |
| I_o | Moment of inertia to axel O_z | 2 500 | Kg.m2 |
| c_o | Lateral stiffness of each tire to axel O_y | 57 000 | N/m |

2. Results

Simulation results of the first model - displacements around the axes O_x and the angular displacement around the axes O_z when the center of gravity changes the position and for different value of lateral elasticity are given in table 1.
Simulations results of the second model shown in fig. 6...9 demonstrate the displacements are around the axes $O_y$ and the angular displacement around the axes $O_z$ as function of coordinate of center of gravity, the steering force characteristics for the front $k_1$ and rear tires $k_2$ and vehicles speed.

<table>
<thead>
<tr>
<th>$c_r$, kN/m</th>
<th>Front/rear wheel</th>
<th>Distance from CG to front axle $a_1$, m</th>
</tr>
</thead>
<tbody>
<tr>
<td>50/50</td>
<td>Displacement around $O_y$, m</td>
<td>0.008714</td>
</tr>
<tr>
<td></td>
<td>Angular displacement around $O_z$, grad</td>
<td>0.3924</td>
</tr>
<tr>
<td>60/60</td>
<td>Displacement around $O_y$, m</td>
<td>0.00796</td>
</tr>
<tr>
<td></td>
<td>Angular displacement around $O_z$, grad</td>
<td>0.3586</td>
</tr>
<tr>
<td>70/70</td>
<td>Displacement around $O_y$, m</td>
<td>0.007391</td>
</tr>
<tr>
<td></td>
<td>Angular displacement around $O_z$, grad</td>
<td>0.3321</td>
</tr>
<tr>
<td>80/80</td>
<td>Displacement around $O_y$, m</td>
<td>0.006928</td>
</tr>
<tr>
<td></td>
<td>Angular displacement around $O_z$, grad</td>
<td>0.3105</td>
</tr>
<tr>
<td>90/90</td>
<td>Displacement around $O_y$, m</td>
<td>0.006534</td>
</tr>
<tr>
<td></td>
<td>Angular displacement around $O_z$, grad</td>
<td>0.2921</td>
</tr>
<tr>
<td>100/100</td>
<td>Displacement around $O_y$, m</td>
<td>0.006364</td>
</tr>
<tr>
<td></td>
<td>Angular displacement around $O_z$, grad</td>
<td>0.2859</td>
</tr>
<tr>
<td>110/110</td>
<td>Displacement around $O_y$, m</td>
<td>0.006287</td>
</tr>
<tr>
<td></td>
<td>Angular displacement around $O_z$, grad</td>
<td>0.3611</td>
</tr>
<tr>
<td>120/120</td>
<td>Displacement around $O_y$, m</td>
<td>0.006191</td>
</tr>
<tr>
<td></td>
<td>Angular displacement around $O_z$, grad</td>
<td>0.3365</td>
</tr>
<tr>
<td>130/130</td>
<td>Displacement around $O_y$, m</td>
<td>0.006078</td>
</tr>
<tr>
<td></td>
<td>Angular displacement around $O_z$, grad</td>
<td>0.3154</td>
</tr>
<tr>
<td>140/140</td>
<td>Displacement around $O_y$, m</td>
<td>0.005965</td>
</tr>
<tr>
<td></td>
<td>Angular displacement around $O_z$, grad</td>
<td>0.2976</td>
</tr>
<tr>
<td>150/150</td>
<td>Displacement around $O_y$, m</td>
<td>0.005832</td>
</tr>
<tr>
<td></td>
<td>Angular displacement around $O_z$, grad</td>
<td>0.2815</td>
</tr>
</tbody>
</table>

Simulations results of the third model shown in fig. 10...17 demonstrate the displacements are around the axes $O_y$ and the angular displacement around the axes $O_z$ as function of coordinate of center of gravity, the steering force characteristics for the front $k_1$ and rear tires $k_2$ and vehicles speed.

**fig. 6** Displacements are around the axes $O_y$ and the Angular displacement around the axes $O_z$, Distance from CG to front axle $a_1 = 1.2$, $V_y = 0$ km /h

**fig. 7** Displacements are around the axes $O_y$ and the Angular displacement around the axes $O_z$, Distance from CG to front axle $a_1 = 1.2$, $V_y = 0$ km /h

**fig. 8** Displacements are around the axes $O_y$ and the Angular displacement around the axes $O_z$, Distance from CG to front axle $a_1 = 1.3$, $V_y = 294$ km /h

**fig. 9** Displacements are around the axes $O_y$ and the Angular displacement around the axes $O_z$, Distance from CG to front axle $a_1 = 1.3$, $V_y = 294$ km /h
fig. 10 Displacements are around the axes $O_x$ and the Angular displacement around the axes $O_y$. Distance from CG to front axle $a_1 = 1.2, \tau_c = 0$ km/h

fig. 11 Displacements are around the axes $O_x$ and the Angular displacement around the axes $O_y$. Distance from CG to front axle $a_1 = 1.2, \tau_c = 0$ km/h

fig. 12 Displacements are around the axes $O_x$ and the Angular displacement around the axes $O_y$. Distance from CG to front axle $a_1 = 1.2, \tau_c = 0$ km/h

fig. 13 Displacements are around the axes $O_x$ and the Angular displacement around the axes $O_y$. Distance from CG to front axle $a_1 = 1.2, \tau_c = 0$ km/h

fig. 14 Displacements are around the axes $O_x$ and the Angular displacement around the axes $O_y$. Distance from CG to front axle $a_1 = 1.3, \tau_c = 294$ km/h

fig. 15 Displacements are around the axes $O_x$ and the Angular displacement around the axes $O_y$. Distance from CG to front axle $a_1 = 1.3, \tau_c = 294$ km/h

fig. 16 Displacements are around the axes $O_x$ and the Angular displacement around the axes $O_y$. Distance from CG to front axle $a_1 = 1.3, \tau_c = 294$ km/h

fig. 17 Displacements are around the axes $O_x$ and the Angular displacement around the axes $O_y$. Distance from CG to front axle $a_1 = 1.3, \tau_c = 294$ km/h

3. Conclusions

The analysis in this paper shows the influence of some construction parameters into the directional stability as coordinate of center of gravity, cars speed, the steering force characteristic.

4. Acknowledgement

This work is a part of the research project to support of PhD students № 122ПД0006-04 funded by Research Centre at Technical University – Sofia.

References: