Electromagnetic shock wave in nonlinear vacuum: exact solution

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An analytical approach to the theory of electromagnetic waves in nonlinear vacuum is developed. The evolution of the pulse is governed by a system of nonlinear wave vector equations. An exact solution with its own angular momentum in the form of a shock wave is obtained.

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Contemporary high-power laser facilities can generate optical pulses with intensities of the order of $10^{22}\ \mathrm{W/cm^2}$. At the same time, the critical power for observation self-action effects due to virtual electron–positron pairs is of order [1–3] $P_{\mathrm{cr}} = \lambda^2/8n_0n_2 = 2.5\text{–}4.4\times 10^{24}$ W, at a wavelength of 1 μm. Thus, for a laser pulse with waist $r_\perp = 1$ mm the corresponding intensity becomes $I_{\mathrm{cr}}^{\mathrm{vac}} = P_{\mathrm{cr}}/r_\perp^2 = 2.5\text{–}4.4\times 10^{26}\ \mathrm{W/cm^2}$, which is above the range of the new high-power lasers. The nonlinear addition to the refractive index in vacuum depends also on the magnetic field. That is why new different nonlinear effects can be expected. There are not only self-action effects, but also vacuum birefringence [4,5,7], different kinds of four-wave interaction [6,8,9], and higher-order harmonic generation [10]. In this Letter, we shall investigate the self-action effect only for intensities of the order of $I_{\mathrm{cr}}^{\mathrm{vac}}$.

Euler, Heisenberg, and Kockel [11,12] predicted intrinsic nonlinearity of the electromagnetic vacuum due to the electron–positron nonlinear polarization. The classical field-dependent nonlinear vacuum dielectric tensor can be written in the form

$$\epsilon_{ik} = \delta_{ik} + \frac{7e^4\hbar}{45\pi m^4c^7}[2(|\vec{E}|^2 - |\vec{B}|^2) + 7B_iB_k], \qquad (1)$$

where a complex form of presenting of the electrical E_i and magnetic B_i components is used. Note that the term containing B_iB_k vanishes, when a localized electromagnetic wave with only one magnetic component B_l is investigated. The dielectric response relevant to such optical pulse is thus

$$\epsilon_{ik} = \delta_{ik} + \frac{14e^4\hbar}{45\pi m^4c^7}(|\vec{E}|^2 - |\vec{B}|^2). \qquad (2)$$

In the case when the spectral width of a pulse Δk_z exceeds the values of the main wave vector, i.e., $\Delta k_z \simeq k_0$, the system of amplitude equations can be reduced to wave type [13] and in nonlinear vacuum becomes

$$\Delta\vec{E} - \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} + \gamma(|\vec{E}|^2 - |\vec{B}|^2)|\vec{E} = 0,$$
$$\Delta\vec{B} - \frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial t^2} + \gamma(|\vec{E}|^2 - |\vec{B}|^2)|\vec{B} = 0, \qquad (3)$$

where $\gamma = \frac{7k_0^2e^4\hbar}{90\pi m^4c^7}$ and $\vec{E}$, $\vec{B}$ are the amplitude functions. Initially, we can write the components of the electrical and magnetic fields as a vector sum of circular and linear components E_z; $E_c = iE_x - E_y$; $B_l = -B_z$. Thus Eq. (3) is transformed into the following scalar system of equations:

$$\Delta E_z - \frac{1}{c^2}\frac{\partial^2 E_z}{\partial t^2} + \gamma(|E_z|^2 + |E_c|^2 - |B_l|^2)E_z = 0,$$
$$\Delta E_c - \frac{1}{c^2}\frac{\partial^2 E_c}{\partial t^2} + \gamma(|E_z|^2 + |E_c|^2 - |B_l|^2)E_c = 0,$$
$$\Delta B_l - \frac{1}{c^2}\frac{\partial^2 B_l}{\partial t^2} + \gamma(|E_z|^2 + |E_c|^2 - |B_l|^2)B_l = 0. \qquad (4)$$

Let us now parameterize the three-dimensional (3D) + 1 space-time through pseudospherical coordinates (r,τ,θ,φ):$z = r\cosh(\tau)\cos(\theta)$, $y = r\cosh(\tau)\sin(\theta)\sin(\varphi)$, $x = r\cosh(\tau)\sin(\theta)\cos(\varphi)$, and $ct = r\sinh(\tau)$, where $r = \sqrt{x^2 + y^2 + z^2 - c^2t^2}$. After calculations, the corresponding d'Alambert operator in pseudospherical coordinates becomes [14]

$$\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} = \frac{3}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} - \frac{1}{r^2}\frac{\partial^2}{\partial \tau^2} - 2\frac{\tanh\tau}{r^2}\frac{\partial}{\partial\tau} + \frac{1}{r^2\cosh^2\tau}\Delta_{\theta,\varphi}, \qquad (5)$$

where with $\Delta_{\theta,\varphi}$ is denoted the angular part of the usual Laplace operator

$$\Delta_{\theta,\varphi} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}. \qquad (6)$$

The system of equations (4) in pseudospherical coordinates becomes

$$\frac{3}{r}\frac{\partial E_z}{\partial r}+\frac{\partial^2 E_z}{\partial r^2}-\frac{1}{r^2}\frac{\partial^2 E_z}{\partial \tau^2}-2\frac{\tanh\tau}{r^2}\frac{\partial E_z}{\partial \tau}+\frac{1}{r^2\cosh^2\tau}\Delta_{\theta,\varphi}E_z$$
$$+\gamma(|E_z|^2+|E_c|^2-|B_l|^2)E_z=0,$$
$$\frac{3}{r}\frac{\partial E_c}{\partial r}+\frac{\partial^2 E_c}{\partial r^2}-\frac{1}{r^2}\frac{\partial^2 E_c}{\partial \tau^2}-2\frac{\tanh\tau}{r^2}\frac{\partial E_c}{\partial \tau}$$
$$+\frac{1}{r^2\cosh^2\tau}\Delta_{\theta,\varphi}E_c+\gamma(|E_z|^2+|E_c|^2-|B_l|^2)E_c=0,$$
$$\frac{3}{r}\frac{\partial B_l}{\partial r}+\frac{\partial^2 B_l}{\partial r^2}-\frac{1}{r^2}\frac{\partial^2 B_l}{\partial \tau^2}-2\frac{\tanh\tau}{r^2}\frac{\partial B_l}{\partial \tau}+\frac{1}{r^2\cosh^2\tau}\Delta_{\theta,\varphi}B_l$$
$$+\gamma(|E_z|^2+|E_c|^2-|B_l|^2)B_l=0. \tag{7}$$

Equations (7) are solved using the method of separation of the variables.

$$E_i(r,\tau,\theta,\varphi)=R(r)T_i(\tau)Y_i(\theta,\varphi),$$
$$B_l(r,\tau,\theta,\varphi)=R(r)T_l(\tau)Y_l(\theta,\varphi), \tag{8}$$

where $i=z,\ c$. We use an additional constraint on the angular and "spherical" time parts

$$|T_z|^2|Y_z|^2+|T_c|^2|Y_c|^2-|T_l|^2|Y_l|^2=\text{const}. \tag{9}$$

The condition (9) separates the variables. The nonlinear terms appear in the radial part only. Thus the radial parts obey the equation

$$\frac{3}{r}\frac{\partial R}{\partial r}+\frac{\partial^2 R}{\partial r^2}-\frac{A_i}{r^2}R+\gamma|R|^2R=0, \tag{10}$$

where A_i, $i=z,c,l$ are separation constants. We look for solutions that possess more clearly expressed localization than the scalar soliton solution obtained in [13]

$$R=\frac{\operatorname{sech}(\ln(r^\alpha))}{r}, \tag{11}$$

where α, γ and the separation constants A_i, $i=z,\ c,\ l$ satisfy the relations $\alpha^2-1=A_i$; $2\alpha^2=\gamma$. The corresponding τ-dependent part of the equations (7) is linear:

$$\cosh^2\tau\frac{d^2T_i}{d\tau^2}+2\sinh\tau\cosh\tau\frac{dT_i}{d\tau}$$
$$+(C_i-A_i\cosh^2\tau)T_i,=0, \tag{12}$$

where $i=z,\ c,\ l$ and C_i are other separation constants connected with the angular part of the Laplace operator $Y_i(\theta,\varphi)$. Only the following solutions of Eq. (12) exist, which satisfy the condition (9): $T_z=\cosh\tau$, $T_c=\cosh\tau$, $T_l=\sinh\tau$, with separation constants—for the electrical part $A_z=A_c=3$, $C_z=C_c=2$ and for the magnetic part $A_l=3$, $C_l=0$. Thus the magnetic part of the system of equations (7) does not depend on the angular components, i.e., $Y_l(\theta,\varphi)=0$, and as for the electrical part $Y_z(\theta,\varphi)$, $Y_c(\theta,\varphi)$, we have the following linear system of equations:

$$\frac{\Delta_{\theta,\varphi}Y_i}{Y_i}=-2, \tag{13}$$

where now $i=z,\ c$. There are only two solutions of Eq. (13) that satisfy the condition (9): $Y_z=\cos\theta$, $Y_c=\sin\theta\exp(i\varphi)$, Using the relation between the separation constants A_i and the real number α, we obtain the following values for α and γ: $\alpha^2=4$, $\alpha=\pm2$; $\gamma=8$.

Finally, we can write the exact solution of the system of nonlinear equations (4), which describes the propagation of a electromagnetic wave in nonlinear vacuum

$$E_z(r,\tau,\theta)=\frac{\operatorname{sech}(\ln(r^{\pm2}))}{r}\cosh\tau\cos\theta,$$
$$E_c(r,\tau,\theta,\varphi)=\frac{\operatorname{sech}(ln(r^{\pm2}))}{r}\cosh\tau\sin\theta\exp(i\varphi),$$
$$B_l(r,\tau)=\frac{\operatorname{sech}(ln(r^{\pm2}))}{r}\sinh\tau. \tag{14}$$

If we rewrite the solution in Cartesian coordinates, it is not difficult to show that the solution (14) of the system (4) admits finite energy and the electrical part possesses angular momentum $l=1$:

$$E_z=\frac{2z}{r^4+1},\qquad E_c=\frac{2(x+iy)}{r^4+1},\qquad B_l=\frac{2ct}{r^4+1}, \tag{15}$$

where $r=\sqrt{x^2+y^2+z^2-c^2t^2}$. The intensity profile of the solution now becomes

$$I(x,y,z,t)=\frac{4(x^2+y^2+z^2+c^2t^2)}{[(x^2+y^2+z^2-c^2t^2)^2+1]^2}. \tag{16}$$

For a comparison, in Fig. 1, we show the time evolution of the intensity profile I of a spherically symmetric analytical solution

$$E(x,y,z,t)=1\Big/\left[\frac{r^2}{r_0^2}+\left(1+\frac{ict}{r_0}\right)^2\right] \tag{17}$$

of the linear scalar wave equation

$$\Delta E=\frac{1}{c^2}\frac{\partial^2 E}{\partial t^2}, \tag{18}$$

obtained recently by us applying the Fourier method. We have used normalized scales $r_0=1$, $c=1$, and times of evolution $t=0$ and $t=10$. The initially localized

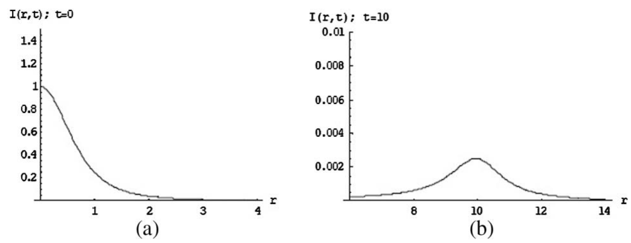

Fig. 1. Time evolution of the intensity profile I of the spherically symmetric analytical solution (17) of the linear wave equation (18) ($r_0=1$ and $c=1$). The initially ($t=0$) localized amplitude function [Fig. 1(a)] decreases with the generation of outside and inside fronts [Fig. 1(b)], while the energy density distributes over the whole space for a finite time ($t=10$).

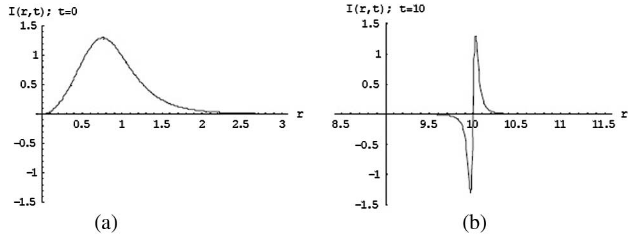

Fig. 2. Time evolution of the intensity (16) of the solution (15) of the nonlinear system of equations (4) in Euler's vacuum ($c = 1$) for $t = 0$ and $t = 10$ correspondingly. The nonlinear wave demonstrates entirely different evolution than the linear spherical one: the shock wave preserves its amplitude maximum and self compresses in r direction.

amplitude function in the linear case decreases with the generation of outside and inside fronts, while the energy density distributes over the whole space in a finite time. The evolution of the intensity profile (16) is presented in Figs. 2(a) and 2(b) for $t = 0$ and $t = 10$ correspondingly. It is clearly seen from Fig. 2 that solution (15) describes a nonlinear shock wave in vacuum. The wave admits entirely different evolution than the linear spherical ones: as the linear wave front enlarges spherically, the shock wave preserves its amplitude maximum and self compresses itself in r direction.

In this Letter, the nonlinear vector wave equations in nonlinear vacuum (2) are solved through the method of separation of the variables in a pseudospherical coordinate system. The obtained analytical solution (15) represents a spherical shock wave with its own angular momentum $l = 1$ for the electrical field. Such a high-intensity wave can be generated not only from the laser sources, but also in a nuclear reaction, where a nonlinear polarization of virtual electron–positron pairs appears at the beginning. If we compare the nonlinear vacuum shock wave with a spherically symmetric solution of the linear wave equation, the difference becomes obvious. While the spherically symmetric solution of the linear wave equation forms inside and outside wave fronts and the amplitude significantly decreases, the nonlinear shock wave preserves the amplitude maximum and self compress in r direction.

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