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Consecutive square-free values of the form $[\alpha \mathbf{p}], [\alpha \mathbf{p}] + \mathbf{1}$

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Abstract: In this short paper we shall prove that there exist infinitely many consecutive square-free numbers of the form $[\alpha p]$, $[\alpha p] + 1$, where p is prime and $\alpha > 0$ is irrational algebraic number. We also establish an asymptotic formula for the number of such square-free pairs when p does not exceed given sufficiently large positive integer.

Keywords: Consecutive square-free numbers, Asymptotic formula, Prime number.

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1 Notations

Let N be a sufficiently large positive integer. The letter p will always denote prime number. By ε we denote an arbitrary small positive number, not necessarily the same in different occurrences. We denote by $\mu(n)$ the Möbius function and by $\tau(n)$ the number of positive divisors of n. As usual $[t]$ and $\{t\}$ denote the integer part, respectively, the fractional part of t. Instead of $m \equiv n \pmod k$ we write for simplicity $m \equiv n\,(k)$. Moreover $e(t)$=exp($2\pi it$). Let $\alpha > 0$ be irrational algebraic number. As usual $\pi(N)$ is the prime-counting function.

Denote

$$\sigma = \prod_p \left(1 - \frac{2}{p^2}\right) . \tag{1}$$

2 Introduction and statement of the result

In 1932 Carlitz [3] proved that there exist infinitely many consecutive square-free numbers. More precisely he established the asymptotic formula

$$\sum_{n \le N} \mu^2(n)\mu^2(n+1) = \sigma N + \mathcal{O}\big(N^{\theta+\varepsilon}\big) , \tag{2}$$

where σ is denoted by (1) and $\theta = 2/3$.

Subsequently the reminder term of (2) was improved by Mirsky [9] and Heath-Brown [8]. The best result up to now belongs to Reuss [10] with $\theta = (26 + \sqrt{433})/81$.

In 2008 Güloğlu and Nevans [7] showed by asymptotic formula that the sequence

$$\{[\alpha n]\}_{n=1}^{\infty} \tag{3}$$

contains infinitely many square-free numbers, where $\alpha > 1$ is irrational number of finite type.

Recently Akbal [1] considered the sequence (3) with prime numbers and proved that when $k \geq 2$ and $\alpha > 0$ is of type $\tau \geq 1$, then there exist infinitely many k-free numbers of the form $[\alpha p]$. Akbal also established an asymptotic formula for the number of such k-free numbers when p does not exceed given sufficiently large real number x.

As a consequence of his result Akbal established that whenever $\alpha > 0$ is an algebraic irrational number, then

$$\sum_{p \leq N} \mu^2([\alpha p]) = \frac{6}{\pi^2}\pi(N) + \mathcal{O}\left(N^{\frac{9}{10}+\varepsilon}\right) ,$$

(see [1] , Corollary 1).

In 2018 the author [4] showed that for any fixed $1 < c < 22/13$ there exist infinitely many consecutive square-free numbers of the form $[n^c]$, $[n^c] + 1$. More precisely he proved that

$$\sum_{n \leq X} \mu^2([n^c])\mu^2([n^c] + 1) = \sigma X + \mathcal{O}\left(X^{1-\varepsilon^2/2}\right) ,$$

$$\sum_{p \leq X} \mu^2([p^c])\mu^2([p^c] + 1) = \sigma \int\limits_2^X \frac{dt}{\ln t} + \mathcal{O}\left(Xe^{-c_0\sqrt{\log X}}\right) ,$$

$$\sum_{\substack{n \leq X \\ \mu^2(n)=1}} \mu^2([n^c])\mu^2([n^c] + 1) = \frac{6}{\pi^2}\sigma X + \mathcal{O}\left(X^{1-\varepsilon^2/2}\right) ,$$

where σ is denoted by (1) and $c_0 > 0$ is an absolute constant.

Further in 2019 the author [5] showed that there exist infinitely many consecutive square-free numbers of the form $[\alpha n]$, $[\alpha n] + 1$, where n is natural and $\alpha > 1$ is irrational number with bounded partial quotient or irrational algebraic number. More precisely he showed that

$$\sum_{n \leq N} \mu^2([\alpha n])\mu^2([\alpha n] + 1) = \sigma N + \mathcal{O}\left(N^{\frac{5}{6}+\varepsilon}\right) ,$$

where σ is denoted by (1).

Recently the author [6] proved that there exist infinitely many consecutive square-free numbers of the form $x^2 + y^2 + 1$, $x^2 + y^2 + 2$. More precisely he proved that

$$\sum_{1 \leq x,y \leq H} \mu(x^2 + y^2 + 1)\, \mu(x^2 + y^2 + 2) = \mathfrak{S}H^2 + \mathcal{O}\left(H^{\frac{8}{5}+\varepsilon}\right) ,$$

where

$$\mathfrak{S} = \prod_p \left(1 - \frac{\lambda(p^2, 1) + \lambda(1, p^2)}{p^4}\right)$$

and

$$\lambda(q_1, q_2) = \sum_{\substack{1\leq x,y\leq q_1q_2 \\ x^2+y^2+1\equiv 0\,(q_1) \\ x^2+y^2+2\equiv 0\,(q_2)}} 1\,.$$

Define

$$\Sigma(N, \alpha) = \sum_{p\leq N} \mu^2([\alpha p])\mu^2([\alpha p]+1)\,. \tag{4}$$

Motivated by these results and following the method of Akbal [1] we shall prove the following theorem.

Theorem 1. *Let $\alpha > 0$ be irrational algebraic number. Then for the sum $\Sigma(N, \alpha)$ defined by* (4) *the asymptotic formula*

$$\Sigma(N, \alpha) = \sigma\pi(N) + \mathcal{O}\left(N^{\frac{9}{10}+\varepsilon}\right) \tag{5}$$

holds. Here σ is defined by (1).

From Theorem 1 it follows that there exist infinitely many consecutive square-free numbers of the form $[\alpha p]$, $[\alpha p]+1$, where p is prime and $\alpha > 0$ is irrational algebraic number.

3 Preliminary lemmas

Lemma 1. ***(Erdös-Turán inequality)*** *Let $\{t_k\}_{k=1}^K$ be a sequence of real numbers. Suppose that $\mathcal{I} \subset [0, 1)$ is an interval. Then*

$$\Big|\#\{k \leq K\,:\,\{t_k\} \in \mathcal{I}\} - K|\mathcal{I}|\Big| \ll \frac{K}{H} + \sum_{h\leq H} \frac{1}{h}\left|\sum_{k\leq K} e(ht_k)\right|$$

for any $H \gg 1$. The constant in the $\mathcal{O}$-term is absolute.

Proof. See ([2], Theorem 2.1). □

Lemma 2. *Suppose that $H, D, T, N \geq 1$. Let $\alpha > 0$ be irrational algebraic number. Then*

$$\sum_{H<h\leq 2H}\sum_{D<d\leq 2D}\sum_{T<t\leq 2T}\left|\sum_{p\leq N} e\left(\frac{\alpha hp}{d^2t^2}\right)\right| \ll (HDTN)^{\varepsilon}\Big(H^{1/2}D^2T^2N^{1/2} + H^{3/5}DTN^{4/5} + HDTN^{3/4} + H^{3/4}D^{3/2}T^{3/2}N^{3/4}\Big)\,. \tag{6}$$

Proof. This lemma is very similar to result of Akbal [1]. Inspecting the arguments presented in ([1], Lemma 3), the reader will easily see that the proof of Lemma 2 can be obtained by the same manner. □

4 Proof of the Theorem

Assume

$$2 \le z \le (\alpha N)^{2/3} . \tag{7}$$

We use (4) and the well-known identity $\mu^2(n) = \sum_{d^2|n} \mu(d)$ to write

$$\begin{aligned}\Sigma(N,\alpha) &= \sum_{p\le N} \mu^2([\alpha p])\mu^2([\alpha p]+1) = \sum_{p\le N} \sum_{d^2|[\alpha p]} \mu(d) \sum_{t^2|[\alpha p]+1} \mu(t) \\ &= \sum_{\substack{d,t \\ (d,t)=1}} \mu(d)\mu(t) \sum_{\substack{p\le N \\ [\alpha p]\equiv 0\,(d^2) \\ [\alpha p]+1\equiv 0\,(t^2)}} 1 = \Sigma_1(N) + \Sigma_2(N) ,\end{aligned} \tag{8}$$

where

$$\Sigma_1(N) = \sum_{\substack{dt\le z \\ (d,t)=1}} \mu(d)\mu(t) \sum_{\substack{p\le N \\ [\alpha p]\equiv 0\,(d^2) \\ [\alpha p]+1\equiv 0\,(t^2)}} 1 , \tag{9}$$

$$\Sigma_2(N) = \sum_{\substack{dt> z \\ (d,t)=1}} \mu(d)\mu(t) \sum_{\substack{p\le N \\ [\alpha p]\equiv 0\,(d^2) \\ [\alpha p]+1\equiv 0\,(t^2)}} 1 . \tag{10}$$

Estimation of $\mathbf{\Sigma_1(N)}$

From (9) and Chinese remainder theorem we obtain

$$\Sigma_1(N) = \sum_{\substack{dt\le z \\ (d,t)=1}} \mu(d)\mu(t) \sum_{\substack{p\le N \\ [\alpha p]\equiv q\,(d^2t^2)}} 1 , \tag{11}$$

where $1 \le q \le d^2t^2 - 1$.
It is easy to see that the congruence $[\alpha p] \equiv q\,(d^2t^2)$ is tantamount to

$$\frac{q}{d^2t^2} < \left\{\frac{\alpha p}{d^2t^2}\right\} < \frac{q+1}{d^2t^2} . \tag{12}$$

Bearing in mind (11), (12) and Lemma 1 we get

$$\begin{aligned}\Sigma_1(N) &= \pi(N) \sum_{\substack{dt\le z \\ (d,t)=1}} \frac{\mu(d)\mu(t)}{d^2t^2} + \mathcal{O}\left(\frac{N}{H}\sum_{dt\le z} 1\right) + \mathcal{O}\left(\sum_{\substack{dt\le z \\ (d,t)=1}} \sum_{h\le H} \frac{1}{h}\left|\sum_{p\le N} e\left(\frac{\alpha hp}{d^2t^2}\right)\right|\right) \\ &= \pi(N)\left(\sum_{\substack{d,t=1 \\ (d,t)=1}} \frac{\mu(d)\mu(t)}{d^2t^2} - \sum_{\substack{dt>z \\ (d,t)=1}} \frac{\mu(d)\mu(t)}{d^2t^2}\right) + \mathcal{O}\left(\frac{N}{H}\sum_{dt\le z} 1\right) \\ &\quad + \mathcal{O}\left(\sum_{\substack{dt\le z \\ (d,t)=1}} \sum_{h\le H} \frac{1}{h}\left|\sum_{p\le N} e\left(\frac{\alpha hp}{d^2t^2}\right)\right|\right) .\end{aligned} \tag{13}$$

It is well-known that

$$\sum_{\substack{d,t=1\\(d,t)=1}} \frac{\mu(d)\mu(t)}{d^2t^2} = \prod_p \left(1 - \frac{2}{p^2}\right). \tag{14}$$

On the other hand

$$\sum_{\substack{dt>z\\(d,t)=1}} \frac{\mu(d)\mu(t)}{d^2t^2} \ll \sum_{dt>z} \frac{1}{d^2t^2} = \sum_{n>z} \frac{\tau(n)}{n^2} \ll \sum_{n>z} \frac{1}{n^{2-\varepsilon}} \ll z^{\varepsilon-1}. \tag{15}$$

By the same way

$$\sum_{dt\le z} 1 = \sum_{n\le z} \tau(n) \ll z^{1+\varepsilon}. \tag{16}$$

From (13) – (16) it follows

$$\Sigma_1(N) = \sigma\pi(N) + \mathcal{O}\Big(\pi(N)z^{\varepsilon-1}\Big) + \mathcal{O}\left(\frac{N}{H}z^{1+\varepsilon}\right) + \mathcal{O}\left(\sum_{dt\le z}\sum_{h\le H}\frac{1}{h}\left|\sum_{p\le N} e\left(\frac{\alpha hp}{d^2t^2}\right)\right|\right), \tag{17}$$

where σ is denoted by (1).

Splitting the range of h, d and t of the exponential sum in (17) into dyadic subintervals of the form $H < h \le 2H$, $D < d \le 2D$, $T < t \le 2T$, where $DT < z$ and applying Lemma 2 we find

$$\begin{aligned}\sum_{dt\le z}\sum_{h\le H}\frac{1}{h}\left|\sum_{p\le N} e\left(\frac{\alpha hp}{d^2t^2}\right)\right| &\ll (HDTN)^{\varepsilon}\Big(D^2T^2N^{1/2} + DTN^{4/5} + D^{3/2}T^{3/2}N^{3/4}\Big)\\ &\ll (HzN)^{\varepsilon}\Big(z^2N^{1/2} + zN^{4/5} + z^{3/2}N^{3/4}\Big).\end{aligned} \tag{18}$$

Taking into account (7), (17), (18) and choosing $H = N^{1/5}$ we obtain

$$\Sigma_1(N) = \sigma\pi(N) + \mathcal{O}\Big(N^{\varepsilon}\Big(z^2N^{1/2} + zN^{4/5} + z^{3/2}N^{3/4} + Nz^{-1}\Big)\Big). \tag{19}$$

Estimation of $\mathbf{\Sigma_2(N)}$

By (7), (10), (15) and Chinese remainder theorem we get

$$\begin{aligned}\Sigma_2(N) &\ll \sum_{dt>z} \sum_{\substack{n\le N\\ [\alpha n]\equiv 0\,(d^2)\\ [\alpha n]+1\equiv 0\,(t^2)}} 1 = \sum_{dt>z} \sum_{\substack{n\le N\\ [\alpha n]\equiv l\,(d^2t^2)}} 1 \ll \sum_{dt>z} \sum_{\substack{m\le[\alpha N]\\ m\equiv l\,(d^2t^2)}} 1\\ &\ll N\sum_{dt>z}\frac{1}{d^2t^2} \ll N^{1+\varepsilon}z^{-1}.\end{aligned} \tag{20}$$

The end of the proof

Bearing in mind (8), (19), (20) and choosing $z = N^{1/10}$ we establish the asymptotic formula (5).

The theorem is proved.

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