Gramians Computation for Parabolic Distributed Parameter Systems

Kamen Perev
Systems and Control Department
Technical University of Sofia
Sofia, Bulgaria
email:kperev@tu-sofia.bg

***Abstract*—The paper considers the problem of gramians computation for linear parabolic distributed parameters systems like the heat flow in a rod. The solution of the described partial differential equation is derived by applying the approach of time-space separation and by using the spectral method. By analogy to the finite dimensional case, it is shown that the derived solution consists of two parts. The first part is due to the initial conditions and participates in forming the observability gramian of the system. The second part is due to the system input and takes part in computing the reachability gramian. Different cases for problem definition in terms of different boundary conditions are given, showing how in such cases, the presented method can be used for gramians computation.**

Keywords — distributed parameters system, time-space separation, spectral method, Green's function, reachability and observability gramians

I. Introduction

Most of the explored linear dynamical systems are described in terms of ordinary differential equations, where the state vector at every time moment consists of finite number of state variables. The finite dimensionality is the specific feature of the so-called lumped parameter systems. The more complicated case is when the linear system is infinite dimensional. The state vector of such systems has infinitely many components, but these components are still functions of one independent variable and the derivatives are computed with respect to this variable. Such systems contain lumped as well as distributed elements [1]. These systems are called distributed systems and their time domain description contains a regular part and a singular atomic part, consisting of a train of delta impulses [2]. A more general definition for infinite dimensional linear systems is given in [3]. These systems are quartets of linear operators mapping between various infinite dimensional vector spaces. A specific feature of such systems is that they may lead to the existence of nonrational transfer functions [3]. For example, such systems may contain delays in their states, input and output signals, or their impulse response may not be an analytic function. The approximation of such systems with finite-dimensional ones is discussed in [3], where expressions for the approximation error are also developed. For this class of systems, balanced realizations are also obtained and various error bounds on the approximation errors are deduced. The gramians are obtained by utilizing C_0 semigroups with infinitesimal generator on the underlying Hilbert space.

The more general case for infinite dimensional systems is the case when the system state is a function of both time and spatial independent variables. A major approach for modeling such systems is by using time-space separation. In [4], the authors present a review for the basic methods, which apply time-space separation for building distributed parameter system models. The authors discuss partial differential equations models obtained from first principle knowledge, as well as the application of system identification methods for deriving distributed parameters descriptions. Most of the presented methods use Fourier series to approximate the time or spatial function representations. The application of the method of lines is used in [5] for balanced truncation of the semidiscretized Stokes equation. Based on time-space separation, the distributed parameter process is divided in two parts: time domain and spatial domain descriptions. The spatial function is discretized and thus increasing the order of the regular time domain representation, which is subject to model reduction by balanced truncation. The gramians are computed by solving projected generalized Lyapunov equations. A similar approach is undertaken in [6]. The authors use the concept of semistability to extend some results from H_2-norm problems to the infinite-dimensional setting. The gramians computation reduces to solving certain Lyapunov equations, where the condition for semistability plays important role and transforms to a condition for their computation at infinity. Another publication where the concept of semistability finds application is [7], where the model reduction problem for a linear directed network is considered. The pseudo controllability and observability gramians are proposed giving account for the interconnections among the network vertices. Bounds on the approximation error are determined by using the concept of vertex clusterability by generating appropriate graph clustering. The proposed model reduction procedure preserves the network structure and uses the pseudo gramians for computing the approximation error. The controllability problem for parabolic distributed systems is presented in [8], where the optimal actuator design problem is considered. The authors use the spectral approach for solving the partial differential equation by assuming the existence of orthonormal basis functions, which are built from the eigenfunctions of a densely defined operator, generating a strongly continuous semigroup on a Hilbert space. The gramian operator is defined as infinite-dimensional symmetric nonnegative matrix, whose elements are computed by separate integration in both time and spatial domains. The spectral method is also used in [9], where the problem of finding optimal projection spaces for the calculation of reduced order distributed systems models is considered. The authors propose a numerical construction of a databased spectral expansion for spatial-temporal measurements. The balanced truncation model reduction method of a flexible beam for the purpose of vibration control is considered in [10]. The authors use the special structure of the beam model to separate the system dynamics from the output observation equations. The special feature of such flexible structures is that the system output is contained in a finite dimensional subspace of an infinite dimensional one. The system gramians are computed by integration over the

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spatial domain. It is shown that the reduced order model based on balanced truncation of the infinite dimensional system guarantees an upper bound on the approximation error. In [11], the authors consider the problem of model order reduction of linear magneto-quasistatic systems obtained from Maxwell's equations, which find application in modeling of low-frequency electromagnetic devices. The method is based on finite element discretization of the partial differential equations, which leads to differential algebraic equations in 3D domain. The authors present a balanced truncation model reduction algorithm preserving stability and passivity, where the gramians are computed by solving generalized Lyapunov equations.

This paper considers the problem of gramians computation for a linear parabolic distributed parameter system. We examine the partial differential equation description of heat flow in a rod with one spatial variable. The weak solution of the parabolic partial differential equation is obtained by applying the principle of time-space separation. The spatial basis functions are determined by utilizing the spectral method for the obtained solution. By analogy to the finite dimensional case, the reachability and observability gramians of the distributed parameter system are obtained from the solution separation into zero-input and zero-state parts. We suggest gramians derivation based on scalar products by using the Laplace operator eigenvalues and eigenfunctions. Different cases for different types of boundary conditions are considered. It is shown how the presented method for gramians computation can be applied in such cases.

II. MATHEMATICAL PRELIMINARIES ON PARABOLIC DISTRIBUTED PARAMETER SYSTEMS

We consider the general form of a second order partial differential equation with two independent variables:

$$au_{tt} + bu_{tx} + cu_{xx} + du_t + eu_x + gu = f, \tag{1}$$

where $u = u(x,t)$ and $f = f(x,t)$.The partial differential equation reflects the distribution of a physical quantity and this is the reason to call the corresponding system distributed parameter system. The state space of a distributed parameter system is infinite dimensional and its transfer function is irrational function of the Laplace variable. One of the most obvious difference between rational and irrational transfer functions is that irrational transfer functions often have infinitely many poles and zeros, but there are many functions that have neither poles nor zeros [12]. The location of the poles and zeros depends critically on the choice of boundary conditions. Many definitions for rational transfer functions, like minimum phase, relative degree and limits at infinity, are not valid in the irrational case [12]. The equation (1) is homogeneous if the condition on the right hand side $f(x,t) \equiv 0$ is satisfied. If the discriminant $b^2 - 4ac = 0$, the equation (1) is of parabolic type. One of the most popular parabolic type partial differential equation is the heat equation.

We consider the problem of control of the heat flow in a rod of length l with constant thermal conductivity κ_0, mass density ρ and specific heat c_p. From the principle of conservation of energy for small volumes, we build the partial differential equation for the temperature distribution $u(x,t)$ at time t at position in the rod x in the following form:

$$\frac{\partial u(x,t)}{\partial t} + \alpha^2 \frac{\partial^2 u(x,t)}{\partial x^2} = f(x,t), \quad x \in (0,l), \ t \geq 0, \tag{2}$$

where $\alpha^2 = \frac{\kappa_0}{\rho c_p}$ is the conductivity coefficient and $f(x,t)$ is the heat energy, which source is generated inside the rod. In addition to modeling heat flow, equation (2) can model all types of diffusion processes. In more than one spatial dimensions, the homogeneous equation can be written in the form $u_t = k\Delta u$, where Δ is the Laplace operator. In order to determine the temperature field in the rod, one needs to specify the initial temperature profile:

$$u(x,0) = \varphi(x) \tag{3}$$

and the boundary conditions at each boundary point. The problem of obtaining the temperature profile based on equations (2) and (3) is called the Cauchy problem. In other words, the Cauchy problem is related to determining the temperature field based on the initial distribution of temperature. The problem of determining the temperature field based on the initial and boundary conditions is called the mixed problem. The boundary conditions can be presented in three basic forms. The first form is when the boundary condition is given as $u|_{\partial\Omega} = \psi_1(x,t)$. The second type of boundary conditions is presented in the form $\left.\frac{\partial u}{\partial x}\right|_{\partial\Omega} = \psi_2(x,t)$ and the third type is $\left.\left(\frac{\partial u}{\partial x} + u\right)\right|_{\partial\Omega} = \psi_3(x,t)$, where $\partial\Omega$ denotes the geometric boundary of the physical process under study and ψ_1, ψ_2 and ψ_3 are given continuous functions. In the problem under consideration, we accept as boundary points the positions $x = 0$ and $x = l$.

There exist different methods for solving the presented mixed problems. The main approach is by the unifying framework of time-space separation of the solution $u(x,t)$. The principle of time-space separation is based on the possibility of Fourier series approximation of every continuous function describing the physical reality. Based on this principle, every time-space function can be expanded in terms of a set of spatial basis functions $\{\phi_i(x)\}_{i=1}^{\infty}$ as follows [4]:

$$u(x,t) = \sum_{i=1}^{\infty} u_i(t)\phi_i(x) \tag{4}$$

We apply the Fourier method for obtaining the solution of equation (2). The Fourier method can be used to solve the mixed problem and is based on spectral properties of the differential operator representing the partial differential equation [4]. The goal is to obtain the weak solution of the mixed heat problem. First assume the case of homogeneous boundary conditions, i.e. $u(0,t) = u(l,t) = 0$. The initial condition is represented in the form (3). For fixed values of the spatial coordinate x, the functions $f(x,t) = f(t)$ and $u(x,t) = u(t)$ can be considered as elements of a Hilbert space $L_2(0,\infty)$ and $\varphi(x) = \varphi$. Thus the mixed problem can be reduced to a Cauchy problem. The function $u(t)$ is called a weak solution of the mixed problem, if it is continuous on $t \geq 0$ and continuously differentiable on $t > 0$, and also satisfies the equality $\left(\frac{du}{dt} + \Delta u - f, \eta\right) = 0$ for every $t > 0$ and every function $\eta(t)$ belonging to the Hilbert space $L_2(0,\infty)$, where Δu is the Laplace differential operator acting on the function u. We assume that, the weak solution of the problem [13]:

$$\left(\frac{du(t)}{dt}, \eta(t)\right) + \left(\Delta u(t), \eta(t)\right) = \left(f(t), \eta(t)\right) \tag{5}$$

$$u(0) = \varphi \tag{6}$$

$u(t) = u(x,t)$ exists. For every $t \geq 0$, this solution is an element of the Hilbert space $L_2(0,l)$ and can be expanded in a series for every complete system of orthonormal basis functions of the Hilbert space $L_2(0,l)$. In particular, this is the system of eigenfunctions of the Laplace operator Δ, where the Laplace operator in this case has the form $\Delta u = \alpha^2 \frac{\partial^2 u}{\partial x^2}$. We denote these elements as $\phi_i = \phi_i(x)$ and the corresponding eigenvalues as λ_i. We also denote the elements of the inner product $(u(t), \phi_i) = u_i(t)$. Then we have:

$$u(t) = \sum_{i=1}^{\infty} u_i(t)\phi_i \tag{7}$$

The problem now reduces to computation of the coefficients $u_i(t)$. We substitute in (5) $\eta(t) = \phi_i$ and since the basis function ϕ_i does not depend from t, we can write [13]:

$$\left(\frac{du(t)}{dt}, \phi_i\right) = \frac{d}{dt}(u(t), \phi_i) = \frac{du_i(t)}{dt} \tag{8}$$

Since $\phi_i(x)$, $x \in (0,l)$, are eigenfunctions of the Laplace operator Δ, we can write:

$$(\Delta u(t), \phi_i) = \lambda_i(u(t), \phi_i) = \lambda_i u_i(t), \tag{9}$$

where λ_i is the corresponding eigenvalue. If we denote by $(f(t), \phi_i) = f_i(t)$, from (5) we finally obtain the differential equation:

$$\frac{du_i(t)}{dt} + \lambda_i u_i(t) = f_i(t) \tag{10}$$

Equation (10) is an ordinary differential equation of first order with initial condition given as in (6). The solution of this equation can be represented in the following form:

$$u_i(t) = e^{-\lambda_i t} u_i(0) + \int_0^t e^{-\lambda_i(t-\tau)} f_i(\tau) d\tau, \tag{11}$$

where $u_i(0) = (u(0), \phi_i) = (\varphi, \phi_i)$. Thus, we obtain the solution of the temporal part of the temperature distribution. By using (4), we can write [13]:

$$u(x,t) = \sum_{i=1}^{\infty} (\varphi, \phi_i) e^{-\lambda_i t} \phi_i(x) + \sum_{i=1}^{\infty} \int_0^t e^{-\lambda_i(t-\tau)} f_i(\tau) d\tau \phi_i(x) \tag{12}$$

Expression (12) represents the weak solution of the mixed heat flow problem by using its Fourier series representation and the spectral characteristics of the Laplace operator Δ. The eigenvalues of the Laplace operator are $\lambda_n = \left(\frac{n\pi\alpha}{l}\right)^2$ and the corresponding eigenfunctions $\phi_n(x) = \sin\frac{n\pi}{l}x$, $n = 1,2,\cdots$. When the boundary conditions are nonhomogeneous, we have to transform them into homogeneous ones. This can be done as follows. Assume that the nonhomogeneous boundary conditions are constants: $u(0,t) = \mu_1$ and $u(l,t) = \mu_2$. Divide the weak solution into two parts: the stationary one and the transitional one [14]:

$$u(x,t) = \mu_1 + \frac{x}{l}(\mu_2 - \mu_1) + U(x,t), \tag{13}$$

where $U(x,t)$ is the weak solution of the heat problem with homogeneous boundary conditions. In this case, the initial conditions will be in the form:

$$U(x,0) = \varphi(x) - \left[\mu_1 + \frac{x}{l}(\mu_2 - \mu_1)\right] = \tilde{\varphi}(x) \tag{14}$$

The solution of the homogeneous problem is obtained in the form:

$$U(x,t) = \sum_{n=1}^{\infty} U_n e^{-(n\pi\alpha)^2 t} \sin\frac{n\pi}{l}x, \tag{15}$$

where

$$U_n = \frac{2}{l}\int_0^l \tilde{\varphi}(\zeta) \sin\frac{n\pi}{l}\zeta d\zeta, \qquad n = 1,2,\cdots \tag{16}$$

Consider now the case when the boundary conditions are in the form: $u(0,t) = \mu_1(t)$ and $u_x(l,t) + \gamma u(l,t) = \mu_2(t)$. Then the solution to the nonhomogeneous problem can be presented as follows [14]:

$$u(x,t) = A(t)\left[1 - \frac{x}{l}\right] + B(t)\frac{x}{l} + U(x,t) \tag{17}$$

where the functions $A(t)$ and $B(t)$ are such that the stationary part $S(x,t)$ of the solution (17) satisfies the boundary conditions of the problem:

$$S(x,t) = A(t)\left[1 - \frac{x}{l}\right] + B(t)\frac{x}{l} \tag{18}$$

The function $U(x,t)$ will satisfy the equation (2) with homogeneous boundary conditions and the function $S(x,t)$ will satisfy the equation with boundary conditions [14]:

$$S(0,t) = \mu_1(t) \text{ and } S_x(l,t) + \gamma S(l,t) = \mu_2(t) \tag{19}$$

From equations (19) we can determine the functions:

$$A(t) = \mu_1(t) \text{ and } B(t) = \frac{1}{1+\gamma l}[\mu_1(t) + l\mu_2(t)] \tag{20}$$

Finally, we obtain the solution of the heat problem with nonhomogeneous boundary conditions of the given type:

$$u(x,t) = \mu_1(t)\left[1 - \frac{x}{l}\right] + \frac{1}{1+\gamma l}[\mu_1(t) + l\mu_2(t)]\frac{x}{l} + U(x,t), \tag{21}$$

where $U(x,t)$ satisfies the nonhomogeneous problem with homogeneous boundary conditions [14]:

$U_t = \alpha^2 U_{xx} - S_t$, $U_x(l,t) + \gamma U(l,t) = 0$, $U(0,t) = 0$ with initial condition $U(x,0) = \varphi(x) - S(x,0)$ (22)

In general, the solution of the heat problem with nonhomogeneous boundary conditions can be presented in the form:

$$u(x,t) = U(x,t) + \beta(x,t) \tag{23}$$

where $\beta(x,t)$ is a certain twice-differentiable function satisfying the nonhomogeneous boundary conditions and $U(x,t)$ is a solution to a problem with homogeneous boundary conditions but with changed right hand side.

For the second type of nonhomogeneous boundary conditions, when the heat flow through the rod cross-section is not zero: $\left.\frac{\partial u}{\partial x}\right|_{x=0} = \mu_1(t)$ and $\left.\frac{\partial u}{\partial x}\right|_{x=l} = \mu_2(t)$ then, the eigenvalues of the Laplace operator include the zero value [13, 14], since they are the solutions of the equation $\tan\lambda = -\lambda$ and therefore, $\lambda_0 = 0$. The corresponding eigenfunctions are $\phi_n(x) = \cos\frac{n\pi}{l}x$, $n = 0,1,2,\cdots$.

Another method for obtaining the solution of the mixed heat problem is by using its Green's function, classified in [4] as kernel-based modeling. The Green's function for the heat problem $G(x,\zeta;t)$ is the distribution of the temperature in time t at position x obtained from the heat source with unit intensity when the excitation is at time $t = 0$ at a point $x = \zeta$. The Green's function is also known as the function of instantaneous excitation since it can be computed by the expression:

$$G(x,\zeta;t)=\int_0^l G(x,\sigma;t)\delta(\zeta-\sigma)\,d\sigma \qquad (24)$$

The method of Green's functions is when the initial and boundary conditions are replaced by the excitation of simple heat point sources and the problem is solved with respect to each of these sources. The final solution is obtained by superposition of the solutions obtained from the excitation of all elementary point sources.

In order to obtain the Green's function for the heat flow problem, we consider the weak solution (12), which can be written as follows:

$$u(x,t)=u_1(x,t)+u_2(x,t),$$

where $u_1(x,t)$ is this part of the solution, which is due to the initial condition $\varphi(x)$ and $u_2(x,t)$ is this part of the solution, which is due to the heat source of energy.

$$u_1(x,t)=\sum_{n=1}^{\infty} e^{-\lambda_n t}\,\phi_n(x)\frac{2}{l}\int_0^l \varphi(\zeta)\phi_n(\zeta)d\zeta = \int_0^l \varphi(\zeta)\,G(x,\zeta;t)d\zeta \qquad (25)$$

where $(\varphi,\phi_n)=\frac{2}{l}\int_0^l \varphi(\zeta)\,\phi_n(\zeta)d\zeta$ is the scalar product between the initial condition and the n-th basis function. Then, the Green's function is computed as:

$$G(x,\zeta;t)=\sum_{n=1}^{\infty} e^{-\lambda_n t}\,\frac{\phi_n(x)\phi_n(\zeta)}{\|\phi_n\|^2}, \qquad (26)$$

where $\lambda_n=\left(\frac{n\pi\alpha}{l}\right)^2$, $\phi_n(x)=\sin\frac{n\pi}{l}x$ and $\|\phi_n\|^2=\frac{l}{2}$.

$$u_2(x,t)=\sum_{i=1}^{\infty}\int_0^t e^{-\lambda_i(t-\tau)}\,f_i(\tau)d\tau\phi_i(x) = \sum_{n=1}^{\infty}\int_0^t e^{-\lambda_n(t-\tau)}\frac{2}{l}\int_0^l f(\zeta,\tau)\phi_n(\zeta)d\zeta d\tau\phi_n\,(x) = \int_0^t\int_0^l G(x,\zeta;t-\tau)f(\zeta,\tau)\,d\zeta d\tau \qquad (27)$$

where $G(x,\zeta;t-\tau)=\sum_{n=1}^{\infty} e^{-\lambda_n(t-\tau)}\,\frac{\phi_n(x)\phi_n(\zeta)}{\|\phi_n\|^2}$. Therefore, the weak solution of the heat flow problem can be presented in the form:

$$u(x,t)=\int_0^l G(x,\zeta;t)\varphi(\zeta)\,d\zeta+\int_0^t\int_0^l G(x,\zeta;t-\tau)\,f(\zeta,\tau)\,d\zeta d\tau \qquad (28)$$

In the second type of boundary conditions, which are given in terms of heat flows through the cross-section of the rod as $\frac{\partial u}{\partial x}(0,t)=\mu_1(t)$ and $\frac{\partial u}{\partial x}(l,t)=\mu_2(t)$, the Green's function is determined by the expression [15]:

$$G(x,\zeta;t)=\frac{2}{l}\left[\frac{1}{2}+\sum_{n=1}^{\infty} e^{-\left(\frac{\alpha\pi n}{l}\right)^2 t}\cos\frac{n\pi}{l}x\cos\frac{n\pi}{l}\zeta\right]. \qquad (29)$$

In the case when the boundary conditions are presented in the form: $\frac{\partial u}{\partial x}(0,t)=\mu_1(t)$ and $u(x,0)=\varphi(x)$, the Green's function is determined as [15]:

$$G(x,\zeta;t)=\frac{1}{2\alpha\sqrt{\pi t}}\sum_{n=-\infty}^{\infty}\left\{e^{\left[-\frac{(x-\zeta)^2}{4\alpha^2 t}\right]}+e^{\left[-\frac{(x+\zeta)^2}{4\alpha^2 t}\right]}\right\}. \qquad (30)$$

The Green's function approach is more useful when we consider an infinite length of the rod. We assume that the initial conditions are presented as $u(x,0)=\varphi(x)$ and additionally that, the solution is bounded for every x and every t. In this case, the solution for the homogeneous problem is given as [15]:

$$u(x,t)=\int_{-\infty}^{\infty} G(x,\zeta;t)\,\varphi(\zeta)d\zeta \qquad (31)$$

where

$$G(x,\zeta;t)=\frac{1}{2\alpha\sqrt{\pi t}}e^{-\frac{(x-\zeta)^2}{4\alpha^2 t}} \qquad (32)$$

For a half-infinite rod with initial condition $u(x,0)=\varphi(x)$ and insulated end, i.e. $u(0,t)=0$, the Green's function can be computed as [15]:

$$G(x,\zeta;t)=\frac{1}{2\alpha\sqrt{\pi t}}\left\{e^{-\frac{(x-\zeta)^2}{4\alpha^2 t}}-e^{-\frac{(x+\zeta)^2}{4\alpha^2 t}}\right\} \qquad (33)$$

The Green's function representation (28) is the one, which will be used for computing the distributed parameter system gramians.

III. DISTRIBUTED PARAMETER SYSTEM GRAMIANS

At the beginning, we present some preliminaries on gramians computation for linear, time-invariant, lumped parameter systems. Consider the system described by its state space model:

$$\dot{x}(t)=Ax(t)+Bu(t),\quad t\geq 0 \qquad (34.1)$$

$$y(t)=Cx(t),\quad x(0)=x_0, \qquad (34.2)$$

where $x(t)\in\mathbb{R}^n$

The state vector can be determined from the expression:

$$x(t)=e^{At}x_0+\int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau \qquad (35)$$

From (35) is clear that the state response consists of two components. The first component is due to the initial conditions and is called zero-input state response. The second component is due to the input signal and is called zero-state state response. The state impulse response of system (34) is determined from the expression $w(t)=e^{At}B$ and is equal to the state reaction when the input is a delta impulse and the initial conditions are zero. The reachability gramian of system (34) on the interval $[0,T]$ is determined from the expression:

$$W_r(0,T)=\int_0^T e^{At}\,BB^Te^{A^Tt}dt=\int_0^T w(t)\,w^T(t)dt \quad (36)$$

The observability gramian on the same interval is obtained as:

$$W_o(0,T)=\int_0^T e^{A^Tt}\,C^TCe^{At}dt \qquad (37)$$

It is clear that, the reachability gramian is related to the second component and the observability gramian is related to the first component of the state response of system (34).

We consider now the linear time-invariant distributed parameters system (2) with initial condition (3). Similarly to the finite dimensional case, the solution of the partial differential equation consists of two parts. The first part is due to the initial condition:

$$u_1(x,t)=\sum_{n=1}^{\infty}(\varphi,\phi_n)e^{-\lambda_n t}\,\phi_n(x)=\sum_{n=1}^{\infty} e^{-\lambda_n t}u_n(0)\phi_n(x)=\int_0^l G(x,\zeta;t)\varphi(\zeta)\,d\zeta$$

and the second part is due to the heat energy source:

$$u_2(x,t)=\sum_{n=1}^{\infty}\int_0^t e^{-\lambda_n(t-\tau)}\,f_n(\tau)d\tau\phi_n(x) = \int_0^t\int_0^l G(x,\zeta;t-\tau)f(\zeta,\tau)\,d\zeta d\tau$$

Similarly to the finite dimensional case, the reachability gramian is related to $u_2(x,t)$ and the observability gramian is related to $u_1(x,t)$.

For computing the reachability gramian, we consider $u_2(x,t)$. We build the following matrices: the infinite dimensional matrix $e^{-\Lambda t} = diag\{e^{-\lambda_n t}\}_{n=1}^{\infty}$, where $\lambda_n = \left(\frac{n\pi\alpha}{l}\right)^2$ are the distinct eigenvalues of the Laplace operator, the infinite dimensional vector column $f(\tau) = [f_1(\tau) \quad f_2(\tau) \quad \cdots \quad f_n(\tau) \quad \cdots]^T$, where $f_n(\tau) = \big(f(x,\tau), \phi_n(x)\big)$ are the coefficients in the Fourier series representation of the heat source energy inside the rod, i.e.:

$$f(x,\tau) = \sum_{n=1}^{\infty}\big(f(x,\tau), \phi_n(x)\big)\phi_n(x) = \sum_{n=1}^{\infty} f_n(\tau)\phi_n(x) \tag{38}$$

and the infinite dimensional vector column $\phi(x) = [\phi_1(x) \quad \phi_2(x) \quad \cdots \quad \phi_n(x) \quad \cdots]^T$, consisting of the Laplace operator eigenfunctions $\phi_n(x) = \sin\frac{n\pi}{l}x$ and representing an orthonormal basis in the Hilbert space $L_2(0,l)$. Then, the expression for $u_2(x,t)$ can be written as:

$$u_2(x,t) = \phi^T(x)\int_0^t e^{-\Lambda(t-\tau)} f(\tau)\, d\tau \tag{39}$$

By analogy to the reachability gramian for finite dimensional systems (36), we can derive the expression:

$$W_r(x;0,T) = \phi^T(x)\int_0^T e^{-\Lambda t}e^{-\Lambda^T t}\, dt\phi(x) = \phi^T(x)\int_0^T e^{-2\Lambda t}\, dt\phi(x), \tag{40}$$

which presentation is possible, since $e^{-\Lambda t}$ is a real, diagonal matrix. It is important here to mention that, the expression (40) is computed for a fixed value of the spatial coordinate x. More interesting is the case when the heat source energy function inside the rod can be separated in time and space coordinates and to obtain the separated spatial-temporal representation as: $f(x,t) = b(x)\tilde{f}(t)$. By developing $f(x,t)$ in Fourier series with respect the basis functions $\phi_n(x) = \sin\frac{n\pi}{l}x$, we obtain the series representation $f(x,t) = \sum_{n=1}^{\infty} b_n \tilde{f}(t)\phi_n(x)$, and compute $b(x) = \sum_{n=1}^{\infty} b_n\, \phi_n(x)$, where $b_n = \big(b(x), \phi_n(x)\big)$. Thus we obtain the solution for a separated source energy function $u_2(x,t) = \sum_{n=1}^{\infty}\int_0^t e^{-\lambda_n(t-\tau)}\, b_n\tilde{f}(\tau)d\tau\phi_n(x)$. If we build the infinite dimensional vector column $B = [b_1 \quad b_2 \quad \cdots \quad b_n \quad \cdots]^T$ and representing the infinite dimensional vector $f(\tau) = B\tilde{f}(\tau)$, we can write the expression (40) for a fixed value of x in the form:

$$W_r(x;0,T) = \phi^T(x)\int_0^T e^{-\Lambda t}BB^T e^{-\Lambda^T t}\, dt\phi(x) \tag{41}$$

Next, we consider the case when the spatial variable x is not fixed but is changing in the whole interval $[0,l]$. We partition the interval $[0,l]$ into N spatial points, where $x_k = \left(k - \frac{1}{2}\right)\delta$, $\delta = \frac{l}{N}$, $k = 1,2,\cdots N$. We build the matrix $\Phi = [\phi(x_1) \quad \phi(x_2) \quad \cdots \quad \phi(x_N)]$, each column of which is $\phi(x_k) = [\phi_1(x_k) \quad \phi_2(x_k) \quad \cdots \quad \phi_n(x_k) \quad \cdots]^T$, for spatial variable index $k = 1,2,\cdots,N$.

Finally, we suggest the following expression for the reachability gramian of a linear, time-invariant parabolic distributed parameters system:

$$W_r(0,T) = \Phi^T\int_0^T e^{-\Lambda t}BB^T e^{-\Lambda^T t}dt\Phi, \tag{42}$$

where $\lambda_n = \left(\frac{n\pi\alpha}{l}\right)^2$, $n = 1,2,\cdots$ and $\phi_{ik} = \sin\frac{i\pi}{l}x_k$, $k = 1,2,\cdots,N$, $i = 1,2,\cdots$. The reachability gramian contains an infinite dimensional middle part, which is determined from the infinite number of Laplace operator eigenvalues. The eigenvalues of the Laplace operator can be divided into fast and slow modes depending on their location on the real axis. The far remotely placed eigenvalues from zero, i.e. the fastest modes, contribute a little to the system dynamics and can be truncated. In this sense, the dimension of the approximated middle part will be equal to the number of the remaining eigenvalues. The dimension of the external matrices depends on the size of the partition set of the segment $[0,l]$. The more fine the partition set is, the larger will be the dimension of the external matrices.

The expression for the reachability gramian consists of two parts: one depending on spatial coordinates and the other one obtained from integration of certain infinite dimensional matrix on a finite interval of time. The second part is similar to the gramian for lumped parameter systems and similarly is positive definite or semidefinite matrix. If we fix the spatial coordinate, then expression (41) is quadratic form and therefore, is positive number or zero. Thus, for fixed spatial coordinates, the quantities $W_r(x;0,T)$ are quadratic forms, whose matrices are similar to the gamians for lumped parameter time-invariant systems. The difference with respect to the lumped parameter system gramians is that the integrand matrix is infinite dimensional. If we let changing the spatial coordinates, we obtain matrix $W_r(0,T)$ from (42) as a positive definite or semidefinite matrix. The expression for $u_2(x,t)$ in (27) determines the reachability map for parabolic distributed parameter systems defined as follows: $L_r: f(\xi,\tau) \to \int_0^t\int_0^l G(x,\zeta;t-\tau)f(\zeta,\tau)\, d\zeta d\tau$. In similarity to the lumped parameter system case, the reachability gramian for distributed parameter systems can be considered as the matrix representation of the operator $L_r L_r^*$ computed at different spatial points.

For computing the observability gramian, we consider the expression $u_1(x,t)$. Similarly to the case with the reachability gramian, we build the matrices: the infinite dimensional matrix $e^{-\Lambda t} = diag\{e^{-\lambda_n t}\}_{n=1}^{\infty}$, where $\lambda_n = \left(\frac{n\pi\alpha}{l}\right)^2$ are the distinct eigenvalues of the Laplace operator, the infinite dimensional vector column $u(0) = [u_1(0) \quad u_2(0) \quad \cdots \quad u_n(0) \quad \cdots]^T$, where the component $u_n(0) = (u(0), \phi_n) = (\varphi, \phi_n)$ from (6), $n = 1,2,\cdots$ and the infinite dimensional vector column $\phi(x) = [\phi_1(x) \quad \phi_2(x) \quad \cdots \quad \phi_n(x) \quad \cdots]^T$, consisting of the Laplace operator eigenfunctions $\phi_n(x) = \sin\frac{n\pi}{l}x$ and representing an orthonormal basis in the Hilbert space $L_2(0,l)$. In matrix-vector notation, the expression for $u_1(x,t)$ becomes:

$$u_1(x,t) = \phi^T(x)e^{-\Lambda t}u(0) \tag{43}$$

By analogy to the observability gramian for finite dimensional systems (37), for fixed value of x, we develop the expression:

$$W_o(x;0,T) = \phi^T(x)\int_0^T e^{-\Lambda^T t}\, e^{-\Lambda t}dt\phi(x) = \phi^T(x)\int_0^T e^{-2\Lambda t}\, dt\phi(x) \tag{44}$$

If we partition the segment $[0,l]$ into N spatial points, where $x_k = \left(k - \frac{1}{2}\right)\delta$, $\delta = \frac{l}{N}$, $k = 1,2,\cdots N$, we build the matrix

$\Phi = [\phi(x_1) \quad \phi(x_2) \quad \cdots \quad \phi(x_N)]$, each column of which is $\phi(x_k) = [\phi_1(x_k) \quad \phi_2(x_k) \quad \cdots \quad \phi_n(x_k) \quad \cdots]^T$. Finally, we obtain the observability gramian on the whole segment as:

$$W_o(0,T) = \Phi^T \int_0^T e^{-2\Lambda t}\, dt \Phi, \qquad (45)$$

where $\lambda_n = \left(\frac{n\pi\alpha}{l}\right)^2$, $n = 1,2,\cdots$ and $\phi_{ik} = \sin\frac{i\pi}{l}x_k$, $x_k = \left(k-\frac{1}{2}\right)\delta$, $\delta = \frac{l}{N}$, $k = 1,2,\cdots,N$, $i = 1,2,\cdots$. Similarly to the case of the reachability gramian, the middle part of (45) is infinite dimensional and depending on the separation of the eigenvalues on fast and slow modes, the fastest modes can be truncated. The number of the remaining eigenvalues determines the dimension of the truncated middle part of $W_o(0,T)$. The dimension of matrix Φ depends on the size of the partition set of the segment $[0,l]$. Similarly, for the case of lumped parameter systems, the observability gramian (45) can be positive definite or positive semidefinite matrix. The expression for $u_1(x,t)$ in (25) determines the observability map for parabolic distributed parameter systems $L_o\colon \varphi(\zeta) \to \int_0^l G(x,\zeta;t)\varphi(\zeta)\, d\zeta$. Then, the observability gramian for such systems given by (45), can be considered as the matrix representation of the operator $L_o^* L_o$ computed at different spatial points.

For the second type of boundary conditions including the heat flow through the cross-section of the rod at $x = 0$ and $x = l$, the set of eigenvalues includes the value $\lambda_0 = 0$ and the set of eigenfunctions of the form $\phi_n = \cos\frac{n\pi}{l}x$, $n = 0,1,2,\cdots$ includes the eigenfunction $\phi_0 = 1$. In this case, the integration in the middle term is only possible over a finite interval of time. This fact shows that the gramians for linear distributed parameter systems can only be computed on a finite interval of time. Integration over an infinite time interval will lead to elements of the middle term matrix with an infinite value and therefore, the computed gramians will also be infinitely large.

IV. CONCLUSION

This paper considers the problem of reachability and observability gramians computation for linear, time-invariant parabolic distributed parameters systems. The gramians are derived from the weak solution of the partial differential equation describing the heat flow problem with one spatial variable. The weak solution of the heat flow equation is obtained by applying the approach of time-space separation and by using the spectral method with spectral decomposition of the Laplace operator. By analogy to the finite dimensional case, it is shown that the temperature flow function can be divided in two parts. The first part is due to the initial conditions distribution and based on the eigenvalues and eigenfunctions of the Laplace operator, the observability gramian of the distributed parameter system is computed. The second part of the temperature flow function is due to the source energy inside the rod, which has the meaning of input function of the distributed parameters system. The reachability gramian is also derived from the eigenvalues and eigenfunctions of the Laplace operator by similar expression as in the case of a regular linear system with lumped parameters. The only difference with the finite dimensional case is presence of two matrices representing spatial distribution of the operator eigenfunctions. Different cases are considered depending on the boundary conditions describing the problem. It is shown that, the cases for different types of boundary conditions can be reduced to the basic case presented in the problem.

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