

# AN AUTOMATED OPTIMIZATION USING GRAPH THEORY

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**Abstract:** In the study is developed automated approach for solving network problems with graph theory based on the theory of matrices. It is composed the program with Matlab. The model is experimented to optimize the energy efficiency of the movement of road train. This method of approach can be used for solving different problems presented by the graph theory.

**KEYWORDS:** GRAPH THEORY, MATRIX, ROAD TRAIN, OPTIMIZATION, MATLAB, FUEL CONSUMPTION, ENERGY EFFICIENCY, RELATIVE PREFORMANCE

## 1. Introduction

Graph theory is used for solving many optimization problems in transport. The logical sequence of the method and the possibility of visualization allow surveys problem to be provided in form of a network and to respond simultaneously to two questions - to what purpose should achieve and how to do this. In transport management the application of graph theory allows to solve many tasks, such as: transport - distribution of routes, energy efficient traffic management of the vehicle; technology - distribution of cargo and passenger flows, control of technological units, management - calendar-network planning and management, modeling of organizational structures; information; financial and others. Depending on the specific problem graph has a different structure.

The aim of the study is to develop a model of presentation the graph structure in a matrix and to automate the optimization process.

## 2. Matrix representation of the structure of the graph

### 2.1. Modeling by the graph theory

The formation of the structure of the graph depends on the problem that should be solved. The object of study is graph structure shown in Figure 1. Lines connecting two adjacent vertices determine arc of graph. All arcs and peaks define the network box. The graph contains the initial, final and intermediate peaks. The presented graph consists of N number of sections formed between the vertical lines where are the peaks. The last section is called fictitious because in this place all the arcs of the column ends in a peak. The flow is passed from the ended peak to the initial peak. All arcs determine network of graph. For as constructed structure for all arcs are assigned values which depend on the selected parameter of optimization. It's necessary to determine the passage of a unit flow through the network of the initial vertex 1 to the last vertex Z, to realize the maximum or minimum of an optimization function. Optimization model is:

$$(1) \quad Z = \sum_{i=1}^n \sum_{j=1}^n f w_{ij} \cdot f_{ij} \Rightarrow \min(\max)$$

where:  $f_{ij}$  is the flow through the arc of graph,  $f w_{ij}$  is the value of the techno-operational parameter for optimization;  $n$  is the number of vertices of the graph. Values of  $f_{ij}$  can be 0 or 1.

The objective function (1) indicates the minimum (maximum) value of the flow passing through the graph.

Restrictive conditions are:

• Limiting the flow of the minimum and maximum capacity. In this case, the minimum capacity is 0 and the maximum is 1:

$$(2) \quad f_{ij}^{\max} \geq f_{ij} \geq f_{ij}^{\min}, \text{ i.e. } 1 \geq f_{ij} \geq 0.$$

• In each vertex of the graph the total input flow must be equal to the total output flow:

$$(3) \quad \sum_{j=1}^n f_{ji} = \sum_{j=1}^n f_{ij}.$$

- The flow is positive,  $f_{ij} \geq 0$ .

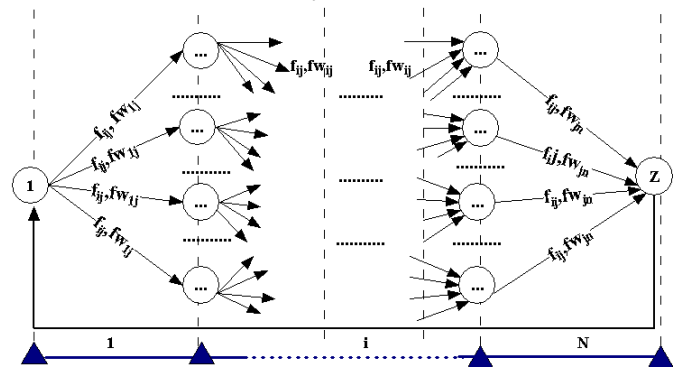


Fig.1. Type of graph structure

### 2.2. Transformation of the graph structure in a matrix

The graph composed of a number of sections (without considering the fictitious section, ie,  $k_1 = N-1$ ) and the number of variables for each section (Fig. 1) can be represented as a matrix  $\tilde{f}$  which is composed of 16 - set submatrix. The number of variables for each segment  $k_2$  is determined by the number of arcs coming out of peak i.

The structure of the matrix  $\tilde{f}$  represented by submatrix has the form:

	$[f_{11}]$	$[f_{12}]$	$[f_{13}]$	$[f_{14}]$
(5)	$[f_{21}]$	$[f_{22}]$	$[f_{23}]$	$[f_{24}]$
	$[f_{31}]$	$[f_{32}]$	$[f_{33}]$	$[f_{34}]$
	$[f_{41}]$	$[f_{42}]$	$[f_{43}]$	$[f_{44}]$

Each sub matrix  $[f_{ij}]$  of matrix  $\tilde{f}$  (5) has the same structure independent of the number of sections and the number of variables of surveys graph. To perform the optimization of the graph theory setting are only sub matrix  $[f_{11}]$ ,  $[f_{12}]$ ,  $[f_{23}]$ ,  $[f_{34}]$  and  $[f_{41}]$ . They differ by content and dimensionality, which depends on  $k_1$  and  $k_2$ . In (5) are marked with red the zero matrices, and yellow nonzero matrices defining the structure and dimensionality of the matrix  $\tilde{f}$ . Other matrices are zero.

The dimension of the matrix  $\tilde{f}$  and the individual submatrix that make it up depends on the number of individual sections, which is divided the graph and variables for each area. They are defined as follows:

- (6)  $[k_1, k_2 + 2, k_1, k_2 + 2]$  for the matrix  $\tilde{f}$ ;
- (7)  $[1, 1]$  for zero sub matrix  $[f_{11}]$ ;
- (8)  $[1, 1]$  for zero sub matrix  $[f_{41}]$ ;
- (9)  $[k_2, 1]$  for zero sub matrix  $[f_{34}]$ ;
- (10)  $[1, k_2]$  for submatrix of the values of the flow for first section  $[f_{12}]$ ;

(11)  $[(k_1-1).k_2, (k_1-1).k_2]$  for submatrix of the values of the flows to other areas  $[f_{23}]$ .

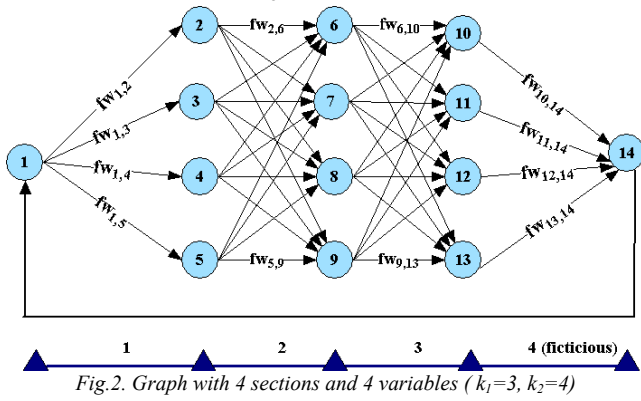


Fig.2. Graph with 4 sections and 4 variables ( $k_1=3, k_2=4$ )

**Example:** Figure 2 shows a graph with three real ( $k_1 = 3$ ) and a fictional section, and four ( $k_2 = 4$ ) peaks (variables) in each area.

The matrix  $\tilde{f}$  of input parameters has the form (12).

For the example nonzero submatrix  $[f_{12}]$  has dimensionality  $[1,4]$  and is shown in (13).

(13)

$fw_{1,2}$	$fw_{1,3}$	$fw_{1,4}$	$fw_{1,5}$
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For the example nonzero submatrix  $[f_{23}]$  has dimensionality  $[8,8]$  and is shown in (14).

(12)

	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	$fw_{1,1}$	$fw_{1,2}$	$fw_{1,3}$	$fw_{1,4}$	$fw_{1,5}$									
2						$fw_{2,6}$	$fw_{2,7}$	$fw_{2,8}$	$fw_{2,9}$					
3						$fw_{3,6}$	$fw_{3,7}$	$fw_{3,8}$	$fw_{3,9}$					
4						$fw_{4,6}$	$fw_{4,7}$	$fw_{4,8}$	$fw_{4,9}$					
5						$fw_{5,6}$	$fw_{5,7}$	$fw_{5,8}$	$fw_{5,9}$					
6										$fw_{6,10}$	$fw_{6,11}$	$fw_{6,12}$	$fw_{6,13}$	
7										$fw_{7,10}$	$fw_{7,11}$	$fw_{7,12}$	$fw_{7,13}$	
8										$fw_{8,10}$	$fw_{8,11}$	$fw_{8,12}$	$fw_{8,13}$	
9										$fw_{9,10}$	$fw_{9,11}$	$fw_{9,12}$	$fw_{9,13}$	
10														$fw_{10,14}$
11														$fw_{11,14}$
12														$fw_{12,14}$
13														$fw_{13,14}$
14														$fw_{14,1}$

(14)

$fw_{2,6}$	$fw_{2,7}$	$fw_{2,8}$	$fw_{2,9}$	0	0	0	0
$fw_{3,6}$	$fw_{3,7}$	$fw_{3,8}$	$fw_{3,9}$	0	0	0	0
$fw_{4,6}$	$fw_{4,7}$	$fw_{4,8}$	$fw_{4,9}$	0	0	0	0
$fw_{5,6}$	$fw_{5,7}$	$fw_{5,8}$	$fw_{5,9}$	0	0	0	0
0	0	0	0	$fw_{6,10}$	$fw_{6,11}$	$fw_{6,12}$	$fw_{6,13}$
0	0	0	0	$fw_{7,10}$	$fw_{7,11}$	$fw_{7,12}$	$fw_{7,13}$
0	0	0	0	$fw_{8,10}$	$fw_{8,11}$	$fw_{8,12}$	$fw_{8,13}$
0	0	0	0	$fw_{9,10}$	$fw_{9,11}$	$fw_{9,12}$	$fw_{9,13}$

For the example nonzero submatrix  $[f_{34}]$  has dimensionality  $[4,1]$  and is shown in (15).

(15)

$fw_{10,14}$
$fw_{11,14}$
$fw_{12,14}$
$fw_{13,14}$

The sub matrices  $[f_{11}]$  and  $[f_{41}]$  are zero and they have dimensionality  $[1,1]$ . The dimension and contents depend on the number of vertices and arcs. The elements of these sub matrices are:

(16)

$[f_{11}]$	$fw_{1,1}$	$[f_{41}]$	$fw_{14,14}$
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Indices  $i$  and  $j$  of the components  $fw_{ij}$  of the matrix  $\tilde{f}$  define the relationship between peaks of an arc. The first index  $i$  indicates the number at the vertex, where the flow starts  $fw_{ij}$ , the second index  $j$  indicates the number of vertex, where the flow ends.

For example for the graph of Figure 2:

- $fw_{1,2}$  – flow between a vertex 1 and vertex 2 of the first section;
- $fw_{1,3}$  – flow between a vertex 1 and vertex 3 of the first section;

- $fw_{2,6}$  – flow between a vertex 2 and vertex 6 of the second section;
- $fw_{5,9}$  – flow between a vertex 5 and vertex 9 of the second section;
- $fw_{6,10}$  – flow between a vertex 6 and vertex 10 of the third section;
- $fw_{9,13}$  – flow between a vertex 9 and vertex 13 of the third section;
- $fw_{10,14}$  – flow between a vertex 10 and vertex 14 of the fourth section.

In the graph with a larger number of sections the sub matrix  $[f_{23}]$  increases, according to the number of intermediate sections. Fig.3 shows an example of the general type of the matrix  $\tilde{f}$  with its constituent sub matrices of a case with a large number of intermediate sections. For sub matrix in gray are colored its nonzero elements corresponding to the number of intermediate sections.

$f_{11}$	$f_{12}$	$f_{13}$	$f_{14}$
$f_{21}$	$f_{22}$	$f_{23}$	$f_{24}$
$f_{31}$	$f_{32}$	$f_{33}$	$f_{34}$
$f_{41}$	$f_{42}$	$f_{43}$	$f_{44}$

Fig.3. Scheme of the components of the matrix  $\tilde{f}$  in a large number of intermediate stations

### 3. An automated optimization in MATLAB

#### 3.1. A linear optimization with MATLAB

The presentation of the graph structure in a matrix allows to be used Matlab and optimization function:

$$(17) \quad [x, fval] = \text{linprog}(fw1, [], [], Aeq, beq, lb, ub, \dots)$$

It is used for solving linear optimization problems of the type:

$$(18) \quad \min \left( f_{w1}^T x \right) \begin{cases} Ax \leq b \\ Aeq \cdot x = beq, \\ lb \leq 0 \leq ub \end{cases}$$

where:

- **fw1** is the vector row. It serves to input data necessary to define the objective function of research graph. It is determined by the transformation the matrix of the input parameters  $\tilde{f}$ .

- **Aeq** is the matrix of coefficients of the linear system of equations defining the specific boundary conditions for the estimated optimization problem (matrix coefficients of the unknown);

- **beq** is the solution of a system of linear equations defining the specific boundary conditions for the optimization problem (right side of the restrictive conditions);

- **lb** is the lower limit of the possible coefficients of a system of linear equations. For the task is 0;

- **ub** is the upper limit of the possible coefficients of a system of linear equations. For the task is 1;

- **x** is the flow  $f_{ij}$  on the arc of the graph.

**A** and **b** are not considered.

The objective function for the example of Figure 2 is:

$$(19) \quad \sum_{i=1}^n \sum_{j=1}^n fw_{i,j} f_{i,j} = fw_{1,2} f_{1,2} + fw_{1,3} f_{1,3} + fw_{1,4} f_{1,4} + \dots + fw_{i,j} f_{i,j} + \dots + fw_{12,14} f_{12,14} + fw_{13,14} f_{13,14} + fw_{14,1} f_{14,1} \rightarrow \min$$

където  $n = k_1 \cdot k_2 + 2$ .

It is used the function linprog.

#### 3.2. Peculiarities in the formation of the vector fw1.

The vector  $fw1$  is given by the matrix of the input parameters of the graph  $\tilde{f}$ . For this, use the built Matlab function  $\text{nonzeros}(\tilde{f}^T)$  to retrieve non-zero elements of matrices, the result is a vector of nonzero elements of the matrix. The built function search the nonzero elements of the matrix of the graph successively from top to bottom, while finding a non-zero element it consistently recorded in a vector. It is necessary the reading and extracting nonzero elements in the input matrix to be organized in rows and therefore the matrix  $\tilde{f}$  transpose. The vector  $fw1$  is obtained by functional recording:

$$(20) \quad fw1 = \text{nonzeros}(\tilde{f}^T).$$

As a result of the transformation the matrix  $\tilde{f}$  of dimension  $[n, n]$  transform in a vector  $fw1$  with dimension  $[(k_1 - 1) \cdot k_1 \cdot k_2 + 2 \cdot k_2 + 1, 1]$ . It is changed the indexing of the input data. For the example of Figure 2 the change is:

- 1)  $fw_{1,2} \rightarrow fw_{1,1}$
- 2)  $fw_{1,3} \rightarrow fw_{1,2}$
- 3)  $fw_{1,2} \rightarrow fw_{1,4}$
- 4)  $fw_{1,2} \rightarrow fw_{1,5}$
- ...
- 38)  $fw_{11,14} \rightarrow fw_{11,38}$
- 39)  $fw_{12,14} \rightarrow fw_{12,39}$
- 40)  $fw_{13,14} \rightarrow fw_{13,40}$

$$41) \quad fw_{14,1} \rightarrow fw_{14,41}$$

#### 3.3. Characteristics in the formation of a matrix Aeq.

The matrix Aeq is formed using the restrictive conditions. For example, the restrictive conditions of the graph presented in Figure 2 are eighteen ( $n + k_1 + 1 \Rightarrow 14 + 3 + 1 = 18$ ) and can be classified in eight groups. Below each of the equations is presented in two versions - one by indexing the matrix  $\tilde{f}$  and by indexing in a positive single vector  $fe$  with dimensionality  $[(k_1 - 1) \cdot k_1 \cdot k_2 + 2 \cdot k_2 + 1, 1]$ . In the equations the first index of a vector indicates the number of a vertex, which is given the condition and the second index - the number of the variable in the vector  $fw1$ . After each group of equations is presented the total record of the equations of the group.

Each equation determines the values of the elements of one row of the matrix Aeq, as its elements can have only values 0 or 1. The type of matrix is shown in Fig.4.

**Group I:** The entering flow at the initial peak should be equal to the exiting.

$$1) \quad -f_{1,2} - f_{1,3} - f_{1,4} - f_{1,5} + f_{14,1} = 0 \\ -fe_{1,1} - fe_{1,2} - fe_{1,3} - fe_{1,4} + fe_{14,41} = 0$$

The general form of the equation has the form:

$$(21) \quad - \sum_{j=1}^{k_2} fe_{1,j} + fe_{n,n1} = 0,$$

where:  $n1 = (k_1 - 1) \cdot k_1 \cdot k_2 + 2 \cdot k_2 + 1$ , a  $n = k_1 \cdot k_2 + 2$ ;

**Group II:** The entering flow at each vertex of the first section must be equal to the exiting.

$$2) \quad f_{1,2} - f_{2,6} - f_{2,7} - f_{2,8} - f_{2,9} = 0 \\ fe_{1,1} - fe_{2,5} - fe_{2,6} - fe_{2,7} - fe_{2,8} = 0$$

$$3) \quad f_{1,3} - f_{3,6} - f_{3,7} - f_{3,8} - f_{3,9} = 0 \\ fe_{1,2} - fe_{3,9} - fe_{3,10} - fe_{3,11} - fe_{3,12} = 0$$

$$4) \quad f_{1,4} - f_{4,6} - f_{4,7} - f_{4,8} - f_{4,9} = 0 \\ fe_{1,3} - fe_{4,13} - fe_{4,14} - fe_{4,15} - fe_{4,16} = 0$$

$$5) \quad f_{1,5} - f_{5,6} - f_{5,7} - f_{5,8} - f_{5,9} = 0 \\ fe_{1,4} - fe_{5,17} - fe_{5,18} - fe_{5,19} - fe_{5,20} = 0$$

The general form of the equations has the form:

$$(22) \quad fe_{1,i} - \sum_{j=1}^{k_2} fe_{i+1,i \cdot k_2 + j} = 0,$$

where:  $i$  is the integer and varies from 1 to  $k_2$  ( $i = 1:1:k_2$ ).

**Group III:** The entering flow at each vertex of the second section must be equal to the exiting.

$$6) \quad f_{2,6} + f_{3,6} + f_{4,6} + f_{5,6} - f_{6,10} - f_{6,11} - f_{6,12} - f_{6,13} = 0 \\ fe_{2,5} + fe_{3,9} + fe_{4,13} + fe_{5,17} - fe_{6,21} - fe_{6,22} - fe_{6,23} - fe_{6,24} = 0$$

$$7) \quad f_{2,7} + f_{3,7} + f_{4,7} + f_{5,7} - f_{7,10} - f_{7,11} - f_{7,12} - f_{7,13} = 0 \\ fe_{2,6} + fe_{3,10} + fe_{4,14} + fe_{5,18} - fe_{7,25} - fe_{7,26} - fe_{7,27} - fe_{7,28} = 0$$

$$8) \quad f_{2,8} + f_{3,8} + f_{4,8} + f_{5,8} - f_{8,10} - f_{8,11} - f_{8,12} - f_{8,13} = 0 \\ fe_{2,7} + fe_{3,11} + fe_{4,15} + fe_{5,19} - fe_{8,29} - fe_{8,30} - fe_{8,31} - fe_{8,32} = 0$$

$$9) f_{2,9} + f_{3,9} + f_{4,9} + f_{5,9} - f_{9,10} - f_{9,11} - f_{9,12} - f_{9,13} = 0$$

$$f_{e_{2,8}} + f_{e_{3,12}} + f_{e_{4,16}} + f_{e_{5,20}} - f_{e_{9,33}} - f_{e_{9,34}} - f_{e_{9,35}} - f_{e_{9,36}} = 0$$

The general form of the equations has the form:

$$(23) \quad \sum_{j=2}^{k_2+1} f_{e_{j,(k_2+i)+(j-2),k_2}} - \sum_{j=1}^{k_2} f_{e_{k_2+1+i,(k_2+1),k_2+j}} = 0,$$

where:  $i$  is the integer and varies from 1 to  $k_2$  ( $i = 1:1:k_2$ );  
 $i$  is the number of equations.

	$f_{1,1}$	$f_{1,2}$	$f_{1,3}$	$f_{1,4}$	$f_{1,5}$	$f_{2,6}$	$f_{2,7}$	$f_{2,8}$	$f_{2,9}$	$f_{2,10}$	$f_{2,11}$	$f_{2,12}$	$f_{2,13}$	$f_{2,14}$	$f_{2,15}$	$f_{2,16}$	$f_{2,17}$	$f_{2,18}$	$f_{2,19}$	$f_{2,20}$	$f_{2,21}$	$f_{2,22}$	$f_{2,23}$	$f_{2,24}$	$f_{2,25}$	$f_{2,26}$	$f_{2,27}$	$f_{2,28}$	$f_{2,29}$	$f_{3,30}$	$f_{3,31}$	$f_{3,32}$	$f_{3,33}$	$f_{3,34}$	$f_{3,35}$	$f_{3,36}$	$f_{10,37}$	$f_{10,38}$	$f_{10,39}$	$f_{10,40}$	$f_{11,41}$						
1	-1	-1	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1						
2	1	0	0	0	-1	-1	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0				
3	0	1	0	0	0	0	0	0	0	-1	-1	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0				
4	0	0	1	0	0	0	0	0	0	0	0	0	-1	-1	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0				
5	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	-1	-1	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0				
6	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	-1	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0				
7	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	-1	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0			
8	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	-1	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0			
9	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0			
10	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0		
11	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0		
12	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
13	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
14	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
15	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
16	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1		
17	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Fig.4. Matrix Aeq

**Group IV:** The entering flow at the each peak of fictitious (last) section should be equal to the exiting.

$$10) f_{6,10} + f_{7,10} + f_{8,10} + f_{9,10} - f_{10,14} = 0$$

$$f_{e_{6,21}} + f_{e_{7,25}} + f_{e_{8,29}} + f_{e_{9,33}} - f_{e_{10,37}} = 0$$

$$11) f_{6,11} + f_{7,11} + f_{8,11} + f_{9,11} - f_{11,14} = 0$$

$$f_{e_{6,22}} + f_{e_{7,26}} + f_{e_{8,30}} + f_{e_{9,34}} - f_{e_{11,38}} = 0$$

$$12) f_{6,12} + f_{7,12} + f_{8,12} + f_{9,12} - f_{12,14} = 0$$

$$f_{e_{6,23}} + f_{e_{7,27}} + f_{e_{8,31}} + f_{e_{9,35}} - f_{e_{12,39}} = 0$$

$$13) f_{6,13} + f_{7,13} + f_{8,13} + f_{9,13} - f_{13,14} = 0$$

$$f_{e_{6,24}} + f_{e_{7,28}} + f_{e_{8,32}} + f_{e_{9,36}} - f_{e_{13,40}} = 0$$

The general form of the equations has the form:

$$(24) \quad \sum_{j=2}^{k_2+1} f_{e_{j+k_2,k_2(j-1)+k_2,k_2+i}} - f_{e_{1+2.k_2+i,n_1-k_2-1+i}} = 0,$$

where:  $i$  is the integer and varies from 1 to  $k_2$   
( $i = 1:1:k_2$ );  $i$  is the number of equations.

**Group V:** The entering flow at the end peak should be equal to the exiting.

$$14) f_{10,14} + f_{11,14} + f_{12,14} + f_{13,14} - f_{14,1} = 0$$

$$f_{e_{10,37}} + f_{e_{11,38}} + f_{e_{12,39}} + f_{e_{13,40}} - f_{e_{14,1}} = 0$$

The general form of the equation has the form:

$$(25) \quad \sum_{j=2}^{k_2+1} f_{e_{j+2.k_2,n_1-k_2+j-2}} - f_{e_{n_1,n_1}} = 0.$$

**Group VI:** They show the flow in first section of the graph.

$$15) f_{1,2} + f_{1,3} + f_{1,4} + f_{1,5} = 1$$

$$f_{e_{1,1}} + f_{e_{1,2}} + f_{e_{1,3}} + f_{e_{1,4}} = 1$$

The general form of the equation has the form:

$$(26) \quad \sum_{j=1}^{k_2} f_{e_{1,j}} = 0.$$

**Group VII:** They show the flow in intermediate section of the graph.

$$16) f_{2,6} + f_{2,7} + f_{2,8} + f_{2,9} + f_{3,6} + f_{3,7} + f_{3,8} + f_{3,9} + \dots$$

$$+ f_{4,6} + f_{4,7} + f_{4,8} + f_{4,9} + f_{5,6} + f_{5,7} + f_{5,8} + f_{5,9} = 1$$

$$f_{e_{2,5}} + f_{e_{2,6}} + f_{e_{2,7}} + f_{e_{2,8}} + f_{e_{3,9}} + f_{e_{3,10}} + f_{e_{3,11}} + f_{e_{3,12}} + \dots$$

$$+ f_{e_{4,13}} + f_{e_{4,14}} + f_{e_{4,15}} + f_{e_{4,16}} + f_{e_{5,17}} + f_{e_{5,18}} + f_{e_{5,19}} + f_{e_{5,20}} = 1$$

$$17) f_{6,10} + f_{7,10} + f_{8,10} + f_{9,10} + f_{6,11} + f_{7,11} + f_{8,11} + f_{9,11} + \dots$$

$$+ f_{6,12} + f_{7,12} + f_{8,12} + f_{9,12} + f_{6,13} + f_{7,13} + f_{8,13} + f_{9,13} = 1$$

$$f_{e_{6,21}} + f_{e_{7,25}} + f_{e_{8,29}} + f_{e_{9,33}} + f_{e_{6,22}} + f_{e_{7,26}} + f_{e_{8,30}} + f_{e_{9,34}} + \dots$$

$$+ f_{e_{6,23}} + f_{e_{7,27}} + f_{e_{8,31}} + f_{e_{9,35}} + f_{e_{6,24}} + f_{e_{7,28}} + f_{e_{8,32}} + f_{e_{9,36}} = 1$$

The general form of the equations has the form:

$$(27) \quad \sum_{m=1}^{(k_2)(k_2+1)} \sum_{j=2} f_{e_{m+1+k_2.(i-1),k_2+j-1+(m-1).k_2+k_2.(i-1)}} = 1,$$

where:  $i$  is the integer and varies from 1 to  $k_1 - 1$

$i = 1:1:k_1$

**Group VIII:** They show the flow in last (fictitious) section of the graph.

$$18) f_{10,14} + f_{11,14} + f_{12,14} + f_{13,14} = 1$$

$$f_{e_{10,37}} + f_{e_{11,38}} + f_{e_{12,39}} + f_{e_{13,40}} = 1$$

The general form of the equation has the form:

$$(28) \quad \sum_{j=2}^{k_2+1} f_{e_{j+2.k_2,n_1-k_2+j-2}} = 1.$$

Using these restrictive conditions are made the matrix Aeq with coefficients of a system of linear equations, defining the specific boundary conditions for the optimization problem.

### 3.4. Characteristics in the formation of matrix *beq*.

The matrix *beq* is a vector column consisting of zero and non zero elements and has dimensionality  $[2.k_2 + (k_1 - 1)k_2 + 1, 1]$ . The last  $(k_1 + 1)$  elements are 1.

### 3.5. Characteristics in the formation of matrices *lb* and *ub*.

The matrix *lb* is zero vector column and has dimensionality  $[2.k_2 + (k_1 - 1)k_2 + 1, 1]$ .

The matrix  $ub$  is a single vector column and has dimensionality  $[2k_2 + (k_1 - 1)k_2 + 1, 1]$ .

### 3.6. The structural scheme for automated approach

In Figure 5 is shown the sequence of the automated approach – a conversion of the graph structure in a matrix, extraction of nonzero elements in vector row and application of linear optimization in Matlab.

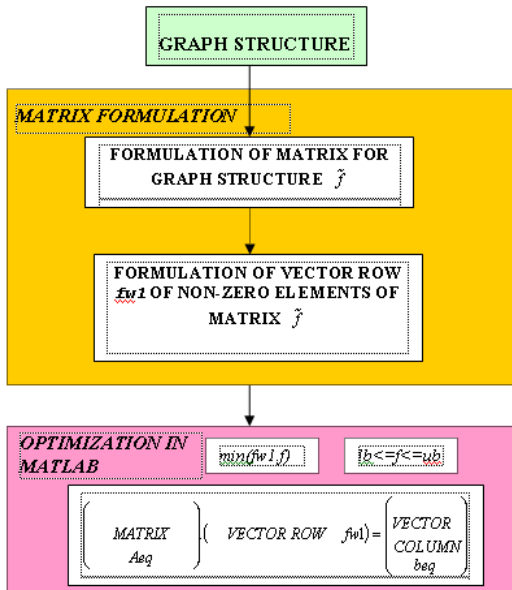


Fig.5. Structural scheme

## 4. Program in MATLAB

### 4.1. Block scheme of the program

On Fig.6 is shown a block scheme of the algorithm for a transformation of the graph in the matrix structure and use of linear optimization in MATLAB. It is elaborated a program by this algorithm that is shown in 4.2.

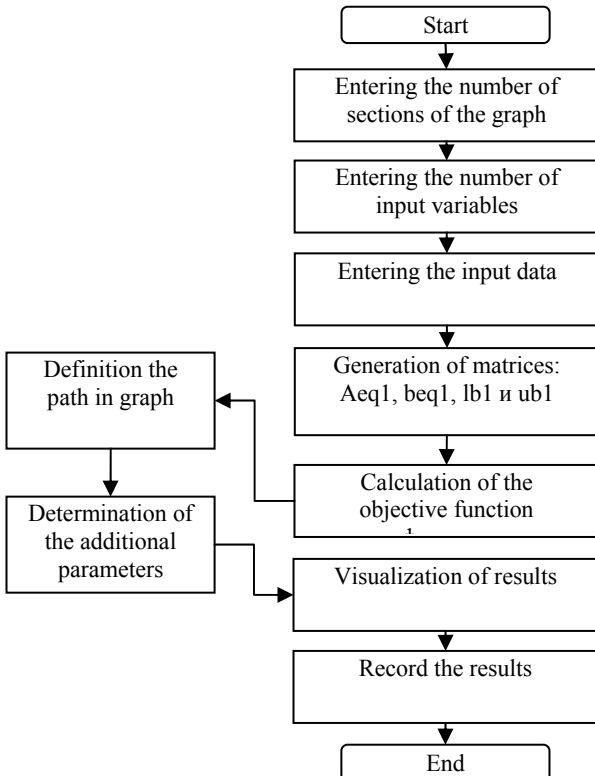


Fig.6. Block scheme of the program

### 4.2. A main program

```

% Name of the main program: Generation_Main.m
clear all
clc
% Generation of the symbols of input variables
Generation_var_Input
% Entering the input data for individual sections
Read_from_xls
% Generation of the matrices characterizing the links between
the elements of graph: Aeq1, beq1, lb1, ub1
Generation_var_Matrix
% Setting the maximum number of iterations to determine the
objective function
nIter = k2+2;
% Defining the objective function and the optimal way through
graph
[x, fval, exitflag, output, lambda] =
linprog(fw1, [], [], Aeq1, beq1, lb1, ub1, [], optimset('Display','iter',
MaxIter,nIter,'LargeScale','off','Simplex','on'));
% Visualization of the way through the graph and the values of
optimization parameter in the vertex
x_n = find(x);
Xs = [b3 x fw1'];
for i = 1:k2
    Xs_n(i) = conj(Xs(x_n(i)));
    xs_n(i) = x(x_n(i));
    fw1_n(i,1) = fw1(x_n(i));
end
xp = [Xs_n', xs_n', fw1_n']
xpc = [Xs_n', xs_n'];
xs = sum(x.*fw1')
cXp = cell(k1, 2);
% Determination and visualization of the values of parameter
in the vertex connected with optimization
name_kol{1,1} = sym('path'); name_kol{1,2} = sym('W');
name_kol{1,3} = sym('ik');
name_kol{1,4} = sym('v'); name_kol{1,5} = sym('q');...
name_kol{1,6} = sym('L'); name_kol{1,7} = sym('qL');
for i = 1:k2
    cXp{i,1} = (Xs(x_n(i)));
    cXp{i,2} = fw1(x_n(i)); % W
    cXp{i,3} = f_ik(x_n(i)); % ik predavka
    cXp{i,4} = f_v(x_n(i)); % v skorost
    cXp{i,5} = f_q(x_n(i)); % q, l/100 km
    cXp{i,6} = f_L(x_n(i)); % L, km
    cXp{i,7} = f_qL(x_n(i)); % qL, l/L km
    if i == 1
        sXp(i,:) = [name_kol{1,1}, name_kol{1,2},
name_kol{1,3}, name_kol{1,4}, ...
name_kol{1,5}, name_kol{1,6}, name_kol{1,7}];
    end
    sXp(i+1,:) = [cXp{i,1} num2str(cXp{i,2}) num2str(cXp{i,3})
num2str(cXp{i,4}) ...
num2str(cXp{i,5}) num2str(cXp{i,6})
num2str(cXp{i,7})];
end
sXp
% Save the result in XLS fail
Save_result_data
  
```

Before starting the main program "Generation\_Main.m" in Matlab is necessary to establish the input data file Input\_var\_all\_data\_H.xls, where X is the number of actual sections of the graph. File is placed in the main directory file. After starting the program in the window that appears is recorded on the first line the number of actual sections of the graph ( $k_1$ ). In the second line is placed the number of variants of parameters ( $k_2$ ). Press "OK" button, then determines and displays the objective function and the optimal path through the graph.



## 5. Approbation of automated optimization using graph theory to study the energy efficiency of movement of a road train

The automated approach and constructed program in Matlab has been applied for modeling the movement of road train. It is necessary to identify such traffic management of the road train for which the overall fuel costs for the individual sections of road to be minimal [1]. In each section of road fuel costs depend on the speed and technical resistance of movement (road profile). Developed in [1] a methodology is used for solving automated approach and program. In Fig.7 is shown a graph for the route Sofia - Karlovo - Burgas, consisting of 47 sections and four variations of speeds of the road train, according to the method in [1]. In this case  $k_1 = 3, k_2 = 4$ .

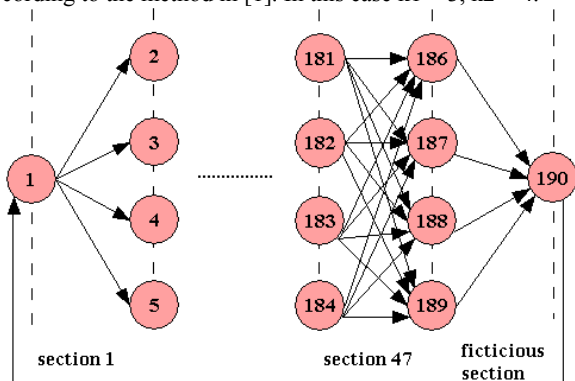


Fig.7. Graph for route Sofia - Karlovo - Burgas

The input data are configured in the file in Excel Input\_var\_all\_data\_47.xls. In Fig.8 is shown the type of input data. In column "A" is recorded the number of nodes from which flow out, and in column "B" the number of nodes in which it enters. In column "C" is entered the criterion of optimization. In this case it is fuel efficiency (the reciprocal of the relative performance). = 4. The other five columns (D, E, F, G and H in green) are used to enter additional parameters directly related to the optimization. In fig.8 column D is the number of gear which moves the road train (VC), column E is the speed that moves VC, column F is the average fuel consumption, l/100km; G column is the length of road section, km; column H is the amount of fuel needed for passage of L, km with a velocity v, km/h on certain road resistance. Of necessity shall be added additional parameters for comparison by introducing additional columns in the input file.

	A	B	C	D	E	F	G	H
1	from	to	flow(1/W)	gear	speed	q, l/100	l, km	ql, lkm
2	1	2	0,00070	12	70	29,4	3	0,882
3	1	3	0,00072	11	70	30,42	3	0,9126
4	1	4	0,00072	12	65	27,92	3	0,8376
5	1	5	0,00075	11	65	29,34	3	0,8802
6	2	6	0,0003712	11	50	25,7	1,3	0,3341
7	2	7	0,0004315	10	50	29,87	1,3	0,36831
8	2	8	0,0004672	10	45	29,11	1,3	0,37843
9	2	9	0,0005088	9	45	31,7	1,3	0,4121
10	3	6	0,0003712	11	50	25,7	1,3	0,3341
11	3	7	0,0004315	10	50	29,87	1,3	0,36831
12	3	8	0,0004672	10	45	29,11	1,3	0,37843
13	3	9	0,0005088	9	45	31,7	1,3	0,4121
14	4	6	0,0003712	11	50	25,7	1,3	0,3341
15	4	7	0,0004315	10	50	29,87	1,3	0,36831

Fig.8. Input data

In figure 9 are shown calculation procedure in Matlab. In a dialog box are set the number of consecutive sections ( $k_1$ ) and the number of parameters ( $k_2$ ).

The results are displayed in Matlab and are saved in a file in Excel. Figures 10 and 11 show the results. In column A is displayed optimal path for each section. For example, f12 means that in the first section the optimal path is between peak 1 and peak 2; f3034 means that in section 9 the optimal path is between peaks 30 and 34, etc. In the last section f188190 shows the optimal path between peak 188 and peak 190. In column B is displayed the value of the optimization parameters. The columns C, D, E, F and G is used to introduce additional parameters.

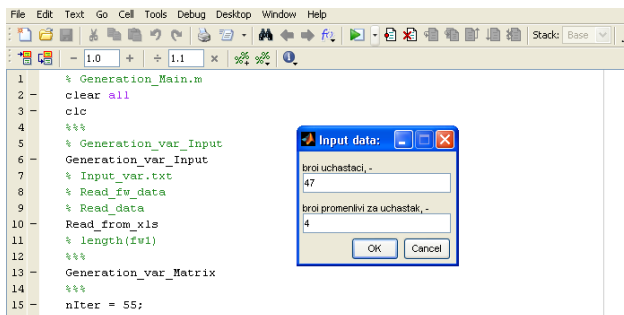


Fig.9. Calculation procedure in Matlab

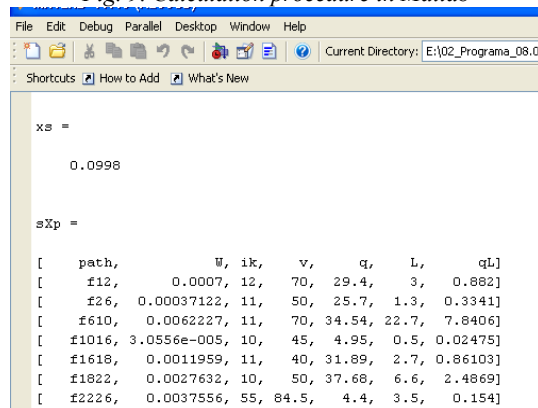


Fig.9. Results in Matlab

	A	B	C	D	E	F	G
1	path	1/W	ik	v, km/h	q, l/100km	L, km	qL, l/Lkm
2	f12	0,0007	12	70	29,4	3	0,882
3	f26	0,000371	11	50	25,7	1,3	0,3341
4	f610	0,006223	11	70	34,54	22,7	7,84058
5	f1016	3,0556e-005	10	45	4,95	0,5	0,02475
6	f1618	0,0011959	11	40	31,89	2,7	0,86103
7	f1822	0,0027632	10	50	37,68	6,6	2,48688
8	f2226	0,0037556	55	84,5	4,4	3,5	0,154
9	f2630	0,000401	12	70	5	10,1	0,505
10	f3034	0,002948	10	70	45,3	8,2	3,7146
11	f3439	0,00094	10	50	24,16	3,5	0,8456
12	f3942	0,000603	11	70	34,54	2,2	0,75988
13	f4246	0,00159	11	50	34,91	4,1	1,43131

Fig.11. Results in Excel

## 6. Conclusions

Investigation gives reason to make the following important conclusions:

- It is made a presentation of the graph in the matrix type.
- It is done an automated approach for optimization using graph theory.
- It is elaborated a program using Matlab.
- It is experimented a method for modeling the movement of auto train.
- The program can be applied to solve other optimization problems in graph theory but also in other graph structures.

## 7. Acknowledgments

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## 8. References

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