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Generalized Solutions of Protter Problem for (3+1)-D Keldysh Type Equations

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Abstract. This paper deals with Protter problems for Keldysh type equations in $\mathbb{R}^4$. Originally such type problems are formulated by M. Protter for equations of Tricomi type. Now it is well known that Protter problems for mixed type equations of the first kind are ill-posed and for smooth right-hand side functions they have singular generalized solutions. In the present paper Protter problem for equations of second kind (Keldysh type) is formulated and it is shown that in the frame of classical solvability this problem is not well posed. Further, a notion for a generalized solution in suitable functional space is given. Results for existence and uniqueness of generalized solution of the considered problem are obtained. Some a priori estimates are stated.

1. INTRODUCTION

In the present paper we consider some boundary value problems for the Keldysh type equation ($m \in \mathbb{R},\ 0 < m < 2$):

$$L_m[u] \equiv u_{x_1x_1} + u_{x_2x_2} + u_{x_3x_3} - (t^m u_t)_t = f(x_1, x_2, x_3, t), \tag{1}$$

expressed in Cartesian coordinates $(x, t) = (x_1, x_2, x_3, t) \in \mathbb{R}^4$ in a simply connected region

$$\Omega_m := \left\{(x,t) : \quad t > 0, \quad \frac{2}{2-m} t^{\frac{2-m}{2}} < \sqrt{x_1^2 + x_2^2 + x_3^2} < 1 - \frac{2}{2-m} t^{\frac{2-m}{2}}\right\},$$

bounded by the ball $\Sigma_0 := \left\{(x,t) :\ t = 0, \quad \sqrt{x_1^2 + x_2^2 + x_3^2} < 1\right\}$, centered at the origin $O = (0, 0, 0, 0)$ and two characteristic surfaces of equation (1)

$$\Sigma_1^m := \left\{(x,t) : t > 0,\ \sqrt{x_1^2 + x_2^2 + x_3^2} = 1 - \frac{2}{2-m} t^{\frac{2-m}{2}}\right\},\quad \Sigma_2^m := \left\{(x,t) : t > 0,\ \sqrt{x_1^2 + x_2^2 + x_3^2} = \frac{2}{2-m} t^{\frac{2-m}{2}}\right\}.$$

We are interested in finding sufficient conditions for the existence and uniqueness of a generalized solution of **Problem PK**. Find a solution to equation (1) in Ω_m that satisfies the boundary conditions

$$u|_{\Sigma_1^m} = 0; \quad t^m u_t \to 0, \text{as } t \to +0.$$

The adjoint problem to **PK** is as follows:

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Problem PK*. Find a solution to the self-adjoint equation (1) in Ω_m that satisfies the boundary conditions

$$u|_{\Sigma_2^m} = 0; \quad t^m u_t \to 0, \text{as } t \to +0.$$

The problems PK and PK^* can be considered as analogues of so called Protter problems for Tricomi type equations. About sixty years ago Murray Protter [37, 38] proposed a multidimensional analogue of classical 2-D Guderley -Morawetz problem for the Gellerstedt equation of hyperbolic-elliptic type. Actually the Guderley-Morawetz problem models flows around airfoils and it is well studied (see Lax and Phillips [20] and Morawetz [26, 27]). However, its multidimensional analogue - the Protter-Morawetz problem is rather different and even now, there is no general understanding of the situation. Even the question of well posedness is not completely resolved. Differences with the 2-D BVPs are illustrated by the related Protter problems in the hyperbolic part of the domain also formulated in [37] for degenerating hyperbolic equations of the first kind in $\mathbb{R}^4$ ($m \in \mathbb{R}$, $m > 0$):

$$t^m[u_{x_1x_1} + u_{x_2x_2} + u_{x_3x_3}] - u_{tt} = f(x_1, x_2, x_3, t). \tag{2}$$

M. Protter considered the equation (2) in the domain

$$\tilde{\Omega}_m := \left\{(x_1, x_2, x_3, t): \quad t > 0, \quad \frac{2}{m+2} t^{\frac{m+2}{2}} < \sqrt{x_1^2 + x_2^2 + x_3^2} < 1 - \frac{2}{m+2} t^{\frac{m+2}{2}} \right\},$$

bounded by the ball Σ_0 and two characteristics surfaces of (2):

$$\tilde{\Sigma}_1^m = \left\{t > 0, \ \sqrt{x_1^2 + x_2^2 + x_3^2} = 1 - \frac{2}{m+2} t^{\frac{m+2}{2}} \right\}, \quad \tilde{\Sigma}_2^m = \left\{t > 0, \ \sqrt{x_1^2 + x_2^2 + x_3^2} = \frac{2}{m+2} t^{\frac{m+2}{2}} \right\}.$$

The homogeneous boundary conditions are prescribed on a characteristic surface and on the non-characteristic part of the boundary:

Protter problems. Find a solution of the equation (2) in $\tilde{\Omega}_m$ with one of the following boundary conditions

$$P1: \quad u|_{\Sigma_0 \cup \tilde{\Sigma}_1^m} = 0, \quad P1^*: \quad u|_{\Sigma_0 \cup \tilde{\Sigma}_2^m} = 0$$
$$P2: \quad u|_{\tilde{\Sigma}_1^m} = 0, u_t|_{\Sigma_0} = 0, \quad P2^*: \quad u|_{\tilde{\Sigma}_2^m} = 0, u_t|_{\Sigma_0} = 0.$$

The boundary conditions $P1^*$ (respectively $P2^*$) are the adjoint boundary conditions to $P1$ (respectively $P2$) for (2). It turns out that in Tricomi case there are two boundary conditions in each of problems $P1$ and $P2$, while in Keldysh case they reduced to only one condition given on the characteristic Σ_1^m. The condition $t^m u_t \to 0$ as $t \to +0$ in problem PK means that the derivative u_t can have singularity on the parabolic boundary, while in problem $P2$ in Tricomi case this derivative is zero on Σ_0.

Actually the Protter problems are multidimensional analogues of the plane Darboux or Cauchy-Goursat problems. It is interesting that in contrast to 2-D case the multidimensional problems are not well posed. P. Garabedian [9] proved the uniqueness of classical solution to the problem $P1$ for the wave equation (i.e. the equation (2) with $m = 0$). A. Aziz and M. Schneider [2] prove uniqueness of generalized solution in more complicated case of Frankl-Moravetz problem for mixed-type equations of first kind in $\mathbb{R}^3$. N. Popivanov and M. Schneider [36] showed that the 3-D homogeneous problems $P1^*$ and $P2^*$ have an infinite number of linearly independent nontrivial classical solutions. This means that there are infinitely many orthogonality conditions on the right-hand side function for classical solvability of problems $P1$ and $P2$. That is the reason for introducing of generalized solutions of these problems. In [14, 36] uniqueness of generalized solutions of three-dimensional problems $P1$ and $P2$ is proved and existence of singular solutions of these problems even for smooth right-hand side functions is obtained. The behavior of singular solutions to 3-D Protter problem $P1$ is studied in [35]. The existence of bounded or unbounded solutions for equations of Tricomi type is considered in [1, 8, 18, 30, 31]. Tricomi type problems for the Lavrent'ev- Bitsadze equation are investigated in [17, 25, 40].

On the other hand different models in plasma physics, transonic flows and optics are described by various boundary value problems for equations of Keldysh type [5, 6, 33]. So it would be interesting to study Protter and Protter-Morawetz problems for Keldysh type equations and try to find new effects that can appear and their applications in real processes. Various statements of problems for mixed type equations of the first and the second kind can be found in O. Oleĭnik, E. Radkevič [32], A. Nakhushev [28] and T. Otway [33]. For different statements of multidimensional Darboux type problems or some connected with them Protter-Morawetz problems for mixed type equations see [3, 7, 19, 21, 22, 23, 24, 27, 34, 39]. Existence and uniqueness of generalized solutions to problem PK in $\mathbb{R}^3$ are discussed in [11, 12, 15] and some singular generalized solutions are announced in [13].

In this paper we firstly show that (3+1)-D problem PK is not correctly set. Following [15] we give definitions of classical solutions of problems PK and PK^*:

Definition 1 *We call a function $u \in C^2(\Omega_m) \cap C(\bar{\Omega}_m)$ a classical solution to Problem PK if $u(x,t)$ satisfies the equation $L_m[u] = f$ in Ω_m, the boundary condition $u|_{\Sigma_1^m} = 0$, and $t^m u_t \to 0$, as $t \to +0$.*

Definition 2 *We call a function $v \in C^2(\Omega_m) \cap C(\bar{\Omega}_m)$ a classical solution to Problem PK^* if $v(x,t)$ satisfies the equation $L_m[v] = g$ in Ω_m, the boundary condition $v|_{\Sigma_2^m} = 0$, and $t^m v_t \to 0$, as $t \to +0$.*

We find some nontrivial classical solutions of homogeneous problem PK^* that are connected with the following functions:

$$E^n_{k,m}(|x|,t) := \sum_{i=0}^{k} A_i^k |x|^{-n+2i-1} \left(|x|^2 - \frac{4}{(2-m)^2} t^{2-m}\right)^{n-k-i-\frac{m}{2(2-m)}},$$

where $k, n \in \mathbb{N} \cup \{0\}$, $|x| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ and the coefficients are:

$$A_i^k := (-1)^i \frac{(k-i+1)_i (n-k-i+(4-3m)/(4-2m))_i}{i!(n+1/2-i)_i}.$$

Here and farther we use the notations $(a)_i = \Gamma(a+i)/\Gamma(a)$, where Γ is the Gamma function of Euler. For $i \in \mathbf{N}$ $(a)_i = a(a+1)\dots(a+i-1)$, $(a)_0 = 1$.

To construct classical solutions of Problem PK^* we use the three-dimensional spherical functions $Y_n^s(x)$ with $n = 0, 1, 2, \dots; s = 1, 2, \dots 2n+1$. $Y_n^s(x)$ are defined usually on the unit sphere $S^2 := \{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 = 1\}$. Recall that Y_n^s form a complete orthonormal system in $L_2(S^2)$ (see [16]). For convenience of discussions that follow, we extend the spherical functions out of S^2 radially, keeping the same notation for the extended functions $Y_n^s(x) := Y_n^s(x/|x|)$ for $x \in \mathbb{R}^3 \setminus \{0\}$.

Lemma 1 *For all $m \in \mathbf{R}$, $0 < m < 2$, $k \in \mathbf{N} \cup \{0\}$, $n \geq N(m,k)$ an integer and $s = 1, 2, \dots, 2n+1$, the functions*

$$v^{n,s}_{k,m}(x,t) := E^n_{k,m}(|x|,t) Y_n^s(x)$$

are classical smooth solutions of the homogeneous problem PK^* for the equation (1).

Lemma 1 shows that a necessary condition for the existence of classical solution for the problem PK is the orthogonality of the right-hand side function $f(x,t)$ to all functions $v^{n,s}_{k,m}(x,t)$.

We mention here that in the case $0 < m < 1$ problem PK for the Keldysh type equation (1) can be formally reduced to the problem $P2$ for the Tricomi type equation (2) with right-hand side function, which vanishes on Σ_0. That implies many differences between investigation of the obtained problem and usual Protter problem $P2$. However, in this paper we study problem (3+1)-D Protter problem PK in the more general case when $\mathbf{0 < m < 4/3}$. To avoid an infinite number of necessary conditions in the frame of classical solvability, we give a notion of a generalized solution to problem PK which can have some singularity at the point O. In order to deal successfully with the encountered difficulties we introduce the region

$$\Omega_{m,\varepsilon} := \Omega_m \cap \left\{ \sqrt{x_1^2 + x_2^2 + x_3^2} > \varepsilon + \frac{2}{2-m} t^{\frac{2-m}{2}} \right\}, \ \varepsilon \in [0,1),$$

bounded by Σ_0, Σ_1^m and

$$\Sigma^m_{2,\varepsilon} := \left\{ (x,t) : 0 < t < t_\varepsilon,\ \sqrt{x_1^2 + x_2^2 + x_3^2} = \varepsilon + \frac{2}{2-m} t^{\frac{2-m}{2}} \right\},$$

where $t_\varepsilon := \left[\frac{2-m}{4}(1-\varepsilon)\right]^{\frac{2}{2-m}}$.

Investigation of the problem PK when $0 < m < 4/3$ leads to the following definition of generalized solution.

Definition 3 *We call a function $u(x,t)$ a generalized solution of problem PK in $\Omega_m,\ 0<m<\frac{4}{3}$, for equation (1) if:*

1. $u, u_{x_j} \in C(\bar{\Omega}_m \setminus O),\ j=1,2,3,\ u_t \in C(\bar{\Omega}_m \setminus \bar{\Sigma}_0)$;
2. *For each $\varepsilon \in (0,1)$ there exists a constant $C(\varepsilon)>0$, such that in $\Omega_{m,\varepsilon}$*

$$|u(x,t)| \leq C(\varepsilon)\left(1-\sqrt{x_1^2+x_2^2+x_3^2}-\tfrac{2}{2-m}t^{\frac{2-m}{2}}\right),\ |u_t(x,t)| \leq C(\varepsilon)t^{-\frac{3m}{4}}; \tag{3}$$

3. *The identity*

$$\int_{\Omega_m}\{t^m u_t v_t - u_{x_1}v_{x_1} - u_{x_2}v_{x_2} - u_{x_3}v_{x_3} - fv\}dx_1dx_2dx_3dt = 0 \tag{4}$$

holds for all v from

$$V_m := \left\{v(x,t) : v,\ v_{x_j} \in C(\bar{\Omega}_m),\ j=1,2,3,\ v_t \in C(\bar{\Omega}_m \setminus \bar{\Sigma}_0),\ |v_t| \leq c\,t^{-\frac{3m}{4}},\ v \equiv 0\ \textit{ in a neighbourhood of }\ \Sigma_2^m\right\},$$

$c = const > 0$ *dependent by $v(x,t)$.*

Remark 1 *It is interesting that in Tricomi case all the first derivatives of generalized solutions can have singularity on the boundary of the domain (see [14], [30], [36]). While in Keldysh case, according to Definition 3 the derivative u_t can be unbounded on Σ_0, but the derivatives u_{x_j}, $j=1,2,3$ are bounded in each $\bar{\Omega}_{m,\varepsilon}$, $\varepsilon>0$.*

The present paper is organized in Introduction and three more sections. When the right-hand side function $f(x,t)$ of equation (1) is fixed as a harmonic polynomial of order l with $l \in \mathbf{N} \cup \{0\}$ and having the representation:

$$f(x,t) = \sum_{n=0}^{l}\sum_{s=1}^{2n+1} f_n^s(|x|,t)Y_n^s(x). \tag{5}$$

with some coefficients $f_n^s(|x|,t)$, we look for a solution of the Protter problem PK of the form

$$u(x,t) = \sum_{n=0}^{l}\sum_{s=1}^{2n+1} u_n^s(|x|,t)Y_n^s(x). \tag{6}$$

Remark 2 *In the case when the right-hand side function $f(x,t)$ has the form (5) one can take test functions $v \in V_m$ in the identity (8) to have the form $v = w(|x|,t)Y_n^s(x)$, $n \in \mathbf{N}\cup\{0\}$, $s=1,2,\dots 2n+1$ and*

$$w \in W_m := \left\{w(r,t) : w, w_r \in C(\bar{G}_m),\ w_t \in C(\bar{G}_m \setminus \bar{S}_0), |w_t| \leq c\,t^{-\frac{3m}{4}}\right\},\ c = const > 0.$$

Here and farther $r=|x|$, $S_0 = \{(r,t) :\ 0<r<1,\ t=0\}$ and

$$G_m = \left\{(r,t) : 0<t<t_0,\ \frac{2}{2-m}t^{\frac{2-m}{2}} < r < 1-\frac{2}{2-m}t^{\frac{2-m}{2}}\right\}.$$

In Section 2 we formulate the 2-D boundary value problems PK_1 and PK_2, corresponding to the (3+1)-D problem PK. The Riemann-Hadamard function associated to the Goursat-Darboux problem PK_2 is constructed and an integral representation for generalized solution to this problem is found. Further, we obtain existence result for generalized solution of problem PK_2. This allows us to obtain the existence and uniqueness theorems for Problem PK_1. Using the results of the previous section, in Section 3 we prove main results in this paper for existence and uniqueness of a generalized solution of (3+1)-D Problem PK. More precisely, we formulate and prove the following theorems:

Theorem 2 *If $m \in (0,\frac{4}{3})$, then there exists at most one generalized solution of Problem PK in Ω_m.*

If in addition the right-hand side function $f(x,t)$ is a harmonic polynomial we give an existence result as well.

Theorem 3 *Let $m \in (0,\frac{4}{3})$, the right-hand side function $f(x,t)$ has the form (5) and $f, f_{x_j} \in C(\bar{\Omega}_m)$, $j=1,2,3$. Then there exists one and only one generalized solution of Problem PK in Ω_m, which has the form (6).*

Remark 3 *Actually, under the conditions for the right-hand side function in Theorem 3 using Riemann-Hadamard function we find explicit representation of generalized solution to the problem PK, which involves appropriated sum of hypergeometrical functions.*

Further, in case when the right-hand side function has the form (5) we give an a priori estimate for the generalized solution of the Problem PK in Ω_m.

Theorem 4 *Let the conditions in Theorem 3 are fulfilled. Then the unique generalized solution of the Problem PK in Ω_m has the form (6) and satisfies the a priori estimate*

$$|u(x,t)| \leq c \left(\max_{\bar{\Omega}_m} |f| \right) |x|^{-l-1}, \tag{7}$$

with a constant $c > 0$ independent on f.

The estimate (7) shows the maximal order of possible singularity at point O, when the right-hand side function $f(x,t)$ is a harmonic polynomial of fixed order l. We will point out that a similar estimate for generalized solutions to 3-D Protter problem $P1$ in Tricomi case is obtained in [35].

In the Appendix - Section 4 the Riemann-Hadamard function associated to two-dimensional Goursat-Darboux problem PK_2 is given.

2. Two-dimensional problem corresponding to Problem PK

Generalized solution of the problem PK in $\Omega_{m,\varepsilon}$, $0 < \varepsilon < 1$ in the case when the right-hand side function $f(x,t)$ has the form (5) we define in the following way:

Definition 4 *We call a function $u(x,t)$ a generalized solution of problem PK in $\Omega_{m,\varepsilon}$ $(0 < m < \frac{4}{3}, 0 < \varepsilon < 1)$, for equation (1) if:*

1. $u, u_{x_j} \in C(\bar{\Omega}_{m,\varepsilon})$, $j = 1,2,3$, $u_t \in C(\bar{\Omega}_{m,\varepsilon} \setminus \bar{\Sigma}_0)$;
2. *There exists a constant $C(\varepsilon) > 0$, such that the estimates (3) hold in $\Omega_{m,\varepsilon}$;*
3. *The identity*

$$\int_{\Omega_{m,\varepsilon}} \{t^m u_t v_t - u_{x_1} v_{x_1} - u_{x_2} v_{x_2} - u_{x_3} v_{x_3} - f v\} dx_1 dx_2 dx_3 dt = 0$$

holds for all $v = w(|x|,t) Y_n^s(x)$, $n \in \mathbf{N} \cup \{0\}$, $s = 1,2,\dots 2n+1$, such that $v \equiv 0$ in $\Omega_m \setminus \Omega_{m,\varepsilon}$ and $w \in W_m$.

Remark 4 *It is evident that a generalized solution of problem PK in Ω_m is a generalized solution in $\Omega_{m,\varepsilon}$, $0 < \varepsilon < 1$, for the function f restricted to $\Omega_{m,\varepsilon}$.*

In this paper we treat Problem PK in the spherical coordinates $(r,\theta,\varphi,t) \in \mathbb{R}^4$: $x_1 = r\sin\theta\cos\varphi$, $x_2 = r\sin\theta\sin\varphi$, $x_3 = r\cos\theta$, i.e. we consider the equation

$$L_m u = \frac{1}{r^2}(r^2 u_r)_r + \frac{1}{r^2\sin\theta}(\sin\theta\, u_\theta)_\theta + \frac{1}{r^2\sin^2\theta} u_{\varphi\varphi} - (t^m u_t)_t = f \tag{8}$$

in the region

$$\Omega_{m,\varepsilon} = \left\{ (r,\theta,\varphi,t) : t > 0, 0 \leq \theta < \pi, 0 \leq \varphi < 2\pi,\ \varepsilon + \frac{2}{2-m} t^{\frac{2-m}{2}} < r < 1 - \frac{2}{2-m} t^{\frac{2-m}{2}} \right\},$$

bounded by the surfaces:

$$\Sigma_0 = \{(r,\theta,\varphi,t) : t = 0, 0 \leq \theta < \pi, 0 \leq \varphi < 2\pi, r < 1\},$$
$$\Sigma_1^m = \{(r,\theta,\varphi,t) : t > 0, 0 \leq \theta < \pi, 0 \leq \varphi < 2\pi, r = 1 - \frac{2}{2-m} t^{\frac{2-m}{2}}\},$$
$$\Sigma_{2,\varepsilon}^m = \{(r,\theta,\varphi,t) : t > 0, 0 \leq \theta < \pi, 0 \leq \varphi < 2\pi, r = \varepsilon + \frac{2}{2-m} t^{\frac{2-m}{2}}\}.$$

The problem PK becomes to the following one: find solution to the equation (8) with the boundary conditions

$$u|_{\Sigma_1 \cap \partial\Omega_{m,\varepsilon}} = 0;\ t^m u_t \to 0, \quad as \ \ t \to +0.$$

In the special case when the right-hand side of the equation (8) has the form

$$f(r,\theta,\varphi,t) = f_n^s(r,t) Y_n^s(\theta,\varphi),$$

we may look for a solution of the form

$$u(r,\theta,\varphi,t) = u_n^s(r,t) Y_n^s(\theta,\varphi),$$

with unknown coefficients $u_n^s(r,t)$.

Recall that Y_n^s satisfy the differential equation (see [16])

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial}{\partial\theta} Y_n^s\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} Y_n^s + n(n+1) Y_n^s = 0.$$

For the coefficient $u_n^s(r,t)$ which correspond to the right-hand side $f_n^s(r,t)$ we obtain the 2-D equation

$$u_{rr} + \frac{2}{r} u_r - (t^m u_t)_t - \frac{n(n+1)}{r^2} u = f(r,t)$$

in the domain

$$G_{m,\varepsilon} = \left\{(r,t) : 0 < t < t_\varepsilon,\ \varepsilon + \frac{2}{2-m} t^{\frac{2-m}{2}} < r < 1 - \frac{2}{2-m} t^{\frac{2-m}{2}}\right\},$$

which is bounded by the segment S_0 and the characteristics

$$S_1^m := \left\{(r,t) : 0 < t < t_\varepsilon,\ r = 1 - \frac{2}{2-m} t^{\frac{2-m}{2}}\right\},\quad S_{2,\varepsilon}^m := \left\{(r,t) : 0 < t < t_\varepsilon,\ r = \frac{2}{2-m} t^{\frac{2-m}{2}} + \varepsilon\right\}.$$

In this case, for $u(r,t)$, the 2-D problem corresponding to PK is the problem

$$PK_1 : \begin{cases} u_{rr} + \frac{2}{r} u_r - (t^m u_t)_t - \frac{n(n+1)}{r^2} u = f \quad in \quad G_{m,\varepsilon}, \\ \\ u|_{S_1^m \cap \partial G_{m,\varepsilon}} = 0;\ t^m u_t \to 0,\ as\ t \to +0. \end{cases}$$

The generalized solution of the Problem PK_1 is defined by

Definition 5 *We call a function $u(r,t)$ a generalized solution of problem PK_1 in $G_{m,\varepsilon}$ $(0 < m < \frac{4}{3},\ 0 < \varepsilon < 1)$, if:*

1. $u,\ u_r \in C(\bar{G}_{m,\varepsilon}),\ u_t \in C(\bar{G}_{m,\varepsilon} \setminus \bar{S}_0)$,
2. *There exists a constant $C(\varepsilon) > 0$, such that in $G_{m,\varepsilon}$*

$$|u(r,t)| \le C(\varepsilon)\left(1 - r - \frac{2}{2-m} t^{\frac{2-m}{2}}\right),\ |u_t(r,t)| \le C(\varepsilon) t^{-\frac{3m}{4}};$$

3. *The identity*

$$\int_{G_{m,\varepsilon}} \left\{u_r v_r - t^m u_t v_t + \frac{n(n+1)}{r^2} uv + fv\right\} r^2 dr dt = 0 \tag{9}$$

holds for all

$$v \in V_{m,\varepsilon} := \left\{v(r,t) : v,\ v_r \in C(\bar{G}_m),\ v_t \in C(\bar{G}_m \setminus \bar{S}_0),\ |v_t| \le c\, t^{-\frac{3m}{4}},\ v \equiv 0 \text{ in } G_m \setminus G_{m,\varepsilon}\right\},$$

$c = const > 0$.

Substituting the new characteristic coordinates

$$\xi = 1 - r - \frac{2}{2-m} t^{\frac{2-m}{2}}, \quad \eta = 1 - r + \frac{2}{2-m} t^{\frac{2-m}{2}} \tag{10}$$

and the new functions

$$U(\xi,\eta) = r(\xi,\eta)u(r(\xi,\eta),t(\xi,\eta)), \quad V(\xi,\eta) = r(\xi,\eta)v(r(\xi,\eta),t(\xi,\eta)),$$
$$F(\xi,\eta) = \frac{1}{8}(2-\xi-\eta)f(r(\xi,\eta),t(\xi,\eta)),$$

we derive from (9) the identity

$$\int_{D_\varepsilon} (\eta-\xi)^{2\beta} \left\{ 2U_\xi V_\eta + 2U_\eta V_\xi + \frac{4n(n+1)}{(2-\xi-\eta)^2} UV + 4FV \right\} d\xi\, d\eta = 0, \tag{11}$$

where for $0 < m < 4/3$, $\beta := \dfrac{m}{2(2-m)} \in (0,1)$ and

$$D_\varepsilon = \{(\xi,\eta) : 0 < \xi < \eta < 1-\varepsilon\} \subset \mathbb{R}^2,\ \varepsilon \in [0,1).$$

If U is sufficiently smooth, from (11) we get the 2-D Goursat-Darboux problem:

Problem PK_2. Find a function $U(\xi,\eta)$ that satisfies the equation

$$U_{\xi\eta} + \frac{\beta}{\eta-\xi}(U_\xi - U_\eta) - \frac{n(n+1)}{(2-\xi-\eta)^2} U = F(\xi,\eta) \quad \text{in} \quad D_\varepsilon \tag{12}$$

and the boundary conditions

$$U(0,\eta) = 0, \quad \lim_{\eta-\xi\to+0} (\eta-\xi)^{2\beta}\Big(U_\xi - U_\eta\Big) = 0. \tag{13}$$

Remark 5 *For Protter problem P2 in the Tricomi case we get a similar two-dimensional Goursat-Darboux problem with an equation similar to the equation (12), but in it the right-hand side function $F(\xi,\eta)$ has singularity like $(\eta-\xi)^{-4\beta}$ on the line $\{\eta = \xi\}$. In that case $\beta = \frac{m}{2(m+2)} \in (0, \frac{1}{2})$, because $m > 0$. While in Keldysh case we see that $F(\xi,\eta)$ is bounded in $\bar{D}_\varepsilon$ if right-hand side function $f(x,t)$ in equation (1) is continuous in $\bar{\Omega}_m$.*

To investigate the smoothness and possible singularities of a solution to the original (3+1)-D problem PK on Σ_2^m, we are seeking for a generalized solution of the corresponding two-dimensional Goursat-Darboux problem PK_2 not only in the domain D_ε, but also in the domain

$$D_\varepsilon^{(1)} := \{(\xi,\eta) : 0 < \xi < \eta < 1, 0 < \xi < 1-\varepsilon\}, \quad \varepsilon > 0.$$

Clearly, $D_\varepsilon \subset D_\varepsilon^{(1)}$ and we give the following definition of a generalized solution to the problem PK_2 in $D_\varepsilon^{(1)}$:

Definition 6 *We call a function $U(\xi,\eta)$ a generalized solution of problem PK_2 in $D_\varepsilon^{(1)}$ $(0 < \beta < 1,\ 0 < \varepsilon < 1)$, if:*

1. *U, $U_\xi + U_\eta \in C(\bar{D}_\varepsilon^{(1)})$, $U_\xi - U_\eta \in C(\bar{D}_\varepsilon^{(1)} \setminus \{\eta = \xi\})$, $U_{\xi\eta} \in C(D_\varepsilon^{(1)})$;*
2. *$U(\xi,\eta)$ satisfies the equation (12) in $D_\varepsilon^{(1)}$ and the following estimates hold*

$$|U(\xi,\eta)| \le c\xi, \ \text{in } \bar{D}_\varepsilon^{(1)}, \tag{14}$$
$$|(U_\xi - U_\eta)(\xi,\eta)| \le c(\eta-\xi)^{-\beta} \text{ in } \bar{D}_\varepsilon^{(1)} \setminus \{\eta = \xi\}, \tag{15}$$

where $c > 0$ is a constant.

Now we use Riemann-Hadamard function associated to problem PK_2 to find integral representation for a generalized solution of this problem in $D_\varepsilon^{(1)}$. According to S. Gellerstedt [10] and A. Nakhushev [29] this function has the form

$$\Phi(\xi,\eta;\xi_0,\eta_0) = \begin{cases} \Phi^+(\xi,\eta;\xi_0,\eta_0), & \eta > \xi_0 \\ \Phi^-(\xi,\eta;\xi_0,\eta_0), & \eta < \xi_0, \end{cases} \tag{16}$$

for $(\xi_0,\eta_0)\in D_0$ and $(\xi,\eta)\in T\cup\Pi$, where

$$T:=\{(\xi,\eta):0<\xi<\eta<\xi_0\},\quad \Pi:=\{(\xi,\eta):0<\xi<\xi_0,\ \xi_0<\eta<\eta_0\}.$$

The Riemann-Hadamard function $\Phi(\xi,\eta;\xi_0,\eta_0)$ should have the following main properties (see [10], [29]):

(**i**) The function Φ as a function of (ξ_0,η_0) satisfies

$$E[\Phi]:=\frac{\partial^2\Phi}{\partial\xi_0\partial\eta_0}+\frac{\beta}{\eta_0-\xi_0}\left(\frac{\partial\Phi}{\partial\xi_0}-\frac{\partial\Phi}{\partial\eta_0}\right)-\frac{n(n+1)}{(2-\xi_0-\eta_0)^2}\Phi=0\ \text{ in }\ D_\varepsilon^{(1)},\ \eta\neq\xi_0. \tag{17}$$

and with respect to the first pair of variables (ξ,η)

$$E^*[\Phi]:=\frac{\partial^2\Phi}{\partial\xi\partial\eta}-\frac{\partial}{\partial\xi}\left(\frac{\beta\Phi}{\eta-\xi}\right)+\frac{\partial}{\partial\eta}\left(\frac{\beta\Phi}{\eta-\xi}\right)-\frac{n(n+1)}{(2-\xi-\eta)^2}\Phi=0\ \text{ in }\ D_\varepsilon^{(1)},\ \eta\neq\xi_0 \tag{18}$$

(**ii**) $\Phi^+(\xi_0,\eta_0;\xi_0,\eta_0)=1$;

(**iii**) $\Phi^+(\xi,\eta_0;\xi_0,\eta_0)=\left(\dfrac{\eta_0-\xi}{\eta_0-\xi_0}\right)^\beta$;

(**iv**) $\Phi^+(\xi_0,\eta;\xi_0,\eta_0)=\left(\dfrac{\eta-\xi_0}{\eta_0-\xi_0}\right)^\beta$;

(**v**) The jump of function Φ on the line $\{\eta=\xi_0\}$ is

$$\begin{aligned}[[\Phi]] &:= \lim_{\delta\to+0}\{\Phi^-(\xi,\xi_0-\delta;\xi_0,\eta_0)-\Phi^+(\xi,\xi_0+\delta;\xi_0,\eta_0)\}\\ &= \cos(\pi\beta)\lim_{\delta\to+0}\{\Phi^+(\xi,\xi_0+\delta;\xi_0,\xi_0+\delta)\Phi^+(\xi_0,\xi_0+\delta;\xi_0,\eta_0)\}\\ &= \cos(\pi\beta)\left(\frac{\xi_0-\xi}{\eta_0-\xi_0}\right)^\beta;\end{aligned}$$

(**vi**) Φ^- vanishes on the line $\{\eta=\xi\}$ of power 2β.

Actually, the function Φ^+ is the Riemann function for equation (12). Existence of function $\Phi(\xi,\eta;\xi_0,\eta_0)$ with properties $(i)\div(vi)$ is shown in Section 4.

In the case $\mathbf{0}<\beta<\mathbf{1/2}$ for generalized solution to the Problem PK_2 with right-hand side functions of the form $F(\xi,\eta)=(\eta-\xi)^{-4\beta}f(\xi,\eta)$, where $f\in C(\bar{D}_\varepsilon^{(1)})$, we have the following explicit integral representation for $(\xi_0,\eta_0)\in D_\varepsilon^{(1)}$(see S. Gellerstedt [10] and A. Nakhushev [29]):

$$U(\xi_0,\eta_0)=\int_0^{\xi_0}\int_\xi^{\eta_0}F(\xi,\eta)\Phi(\xi,\eta;\xi_0,\eta_0)\,d\eta\,d\xi. \tag{19}$$

In the case $\mathbf{0}<\beta<\mathbf{1}$, using (18) and (17) we obtain the same formula (19) for generalized solution of the problem PK_2 with right-hand side functions $F\in C(\bar{D}_\varepsilon^{(1)})$, by integrating the identity (differentiations are with respect to ξ and η)

$$\Phi(\xi,\eta;\xi_0,\eta_0)E[U(\xi,\eta)]-U(\xi,\eta)E^*[\Phi(\xi,\eta;\xi_0,\eta_0)]=F(\xi,\eta)\Phi(\xi,\eta;\xi_0,\eta_0)$$

over a triangle T_{δ_1} bounded by the characteristics $\xi=0$, $\eta=\xi_0-\delta_1$ and $\eta=\xi+\delta_1$ and then over the rectangle Π_{δ_1} bounded by the characteristics $\xi=0$, $\xi=\xi_0-2\delta_1$, $\eta=\xi_0+\delta_1$, $\eta=\eta_0$ and using the properties of the Riemann-Hadamard function $\Phi(\xi,\eta;\xi_0,\eta_0)$ and finally letting $\delta_1\to 0$.

In this way we showed that if $U(\xi,\eta)$ is a generalized solution to the problem PK_2 it should have the form (19). Further, we prove that if F, $F_\xi+F_\eta\in C(\bar{D}_0)$ and $U(\xi,\eta)$ is a function defined by (19) than it is a generalized solution to the problem PK_2. We introduce the notation

$$M_F:=\max\left\{\max_{\bar{D}_0}|F|,\max_{\bar{D}_0}|F_\xi+F_\eta|\right\}.$$

Using the representation (26) (see the Appendix below) of the function Φ and properties of Gauss hypergeometrical function we prove the following:

Theorem 5 *Let $0<\beta<1$; F, $F_\xi+F_\eta \in C(\bar D_0)$. Then the function $U(\xi,\eta)$, defined by (19) is a generalized solution of Problem PK in $D_\varepsilon^{(1)}$ and the following estimates hold*

$$\begin{aligned}
&|U(\xi,\eta)| \le K\left(\max_{\bar D_0}|F|\right)\xi(2-\xi-\eta)^{-n} \quad in \quad \bar D_\varepsilon^{(1)},\\
&|(U_\xi+U_\eta)(\xi,\eta)| \le KM_F(2-\xi-\eta)^{-n-1} \quad in \quad \bar D_\varepsilon^{(1)},\\
&|U_\eta(\xi,\eta)| \le K\left(\max_{\bar D_0}|F|\right)\xi(\eta-\xi)^{-\beta}(2-\xi-\eta)^{-n-1} \quad in \quad \bar D_\varepsilon^{(1)}\setminus\{\eta=\xi\},
\end{aligned} \tag{20}$$

where the constant $K>0$ does not depend on F.

Actually, the Theorem 5 is the essential result in this section and has the most difficult proof. Now, following [36], we are able to prove existence and uniqueness result for two-dimensional problem PK_1 :

Lemma 6 *Let $0<m<\frac{4}{3}$; $f, f_r \in C(\bar G_m)$. Then for each fixed $\varepsilon\in(0,1)$ there exists a generalized solution $u(r,t)$ of problem PK_1 in $G_{m,\varepsilon}$.*

Lemma 7 *Let $0<m<\frac{4}{3}$. Then for each fixed $\varepsilon\in(0,1)$ there exists at most one generalized solution of Problem PK_1 in $G_{m,\varepsilon}$.*

3. Proof of the main results.

In this section give a sketch of the proofs of Theorem 2, Theorem 3 and Theorem 4, formulated in Section 1.

Proof of Theorem 2.

(**i**) Let $0<\varepsilon<1$ and u_1 and u_2 are two generalized solutions of Problem PK in $\Omega_{m,\varepsilon}$. Then the function $u := u_1 - u_2$ solves the homogeneous Problem PK. Now let us define the Fourier coefficients

$$u_n^s(r,t) := \int_0^\pi\int_0^{2\pi} u(r,\theta,\varphi,t)Y_n^s(\theta,\varphi)\sin\theta\, d\varphi\, d\theta.$$

We will show that $u_n^s(r,t)\equiv 0$ for $n\in\mathbf{N}\cup\{0\}$, $s=1,2,\dots,2n+1$, i.e. $u\equiv 0$ in $\Omega_{m,\varepsilon}$.

For u we know that

$$\int_{\Omega_{m,\varepsilon}}\{t^m u_t v_t - u_{x_1}v_{x_1} - u_{x_2}v_{x_2} - u_{x_3}v_{x_3}\}dx_1dx_2dx_3dt = 0 \tag{21}$$

holds for all test functions $v = wY_n^s$ described in Definition 4. Therefore from (21) we derive

$$\int_{G_{m,\varepsilon}}\left\{u_{n,r}^s w_r - t^m u_{n,t}^s w_t + \frac{n(n+1)}{r^2}u_n^s w\right\} r^2 dr dt = 0$$

for all $w(r,t)\in V_{m,\varepsilon}$ (see Definition 5). From Definition 5 it follows that the functions $u_n^s(r,t)$, are generalized solutions of the 2-D homogeneous problem PK_1. Lemma 7 gives $u_n^s(r,t)\equiv 0$ in $G_{m,\varepsilon}$, $\varepsilon>0$, and thus $u=u_1-u_2\equiv 0$.

(**ii**) Let $\varepsilon=0$ and $u(x,t)$ is a generalized solution of the homogeneous Problem PK in Ω_m. Then it is easy to see that $u(x,t)$ is a generalized solution of the same homogeneous problem in $\Omega_{m,\varepsilon}$ for $0<\varepsilon<1$. From (*i*) follows that $u\equiv 0$ in Ω_{m,ε_0} for each $\varepsilon_0\in(0,1)$, so $u=u_1-u_2\equiv 0$ in Ω_m.

Proof of Theorem 3. From Theorem 2 it follows that there exists at most one generalized solution of Problem PK in Ω_m. Since $f(x,t)$ has the form (5) we look for a generalized solution of the form (6), i. e.

$$u(x,t) = \sum_{n=0}^{l}\sum_{s=1}^{2n+1} u_n^s(|x|,t)Y_n^s(x).$$

To find such solution means to find functions $u_n^s(r,t)$ that satisfy the equations

$$\int_{G_{m,\varepsilon}}\left[u_{n,r}^s v_r - t^m u_{n,t}^s v_t + \frac{n(n+1)}{r^2}u_n^s v + f_n^s v\right] r^2\, dr\, dt = 0$$

for all $v \in V_{m,\varepsilon}$, $\varepsilon \in (0,1)$ and satisfy the corresponding conditions (1.), (2.) and (3.) in the Definition 5. Lemma 6 gives existence of such functions u_n^s which are generalized solutions of Problem PK_1 in $G_{m,\varepsilon}$, $\varepsilon \in (0,1)$. This shows that the function $u(x,t)$, given by (6) is a generalized solution of Problem PK in $\Omega_{m,\varepsilon}$, $\varepsilon \in (0,1)$. In this way, we prove existence of generalized solution in $\Omega_{m,\varepsilon}$ for each $\varepsilon \in (0,1)$. Let mention also, that for each two fixed $\varepsilon_1, \varepsilon_2 : 0 < \varepsilon_1 < \varepsilon_2 < 1$, the corresponding generalized solution u_{ε_2} is a restriction of u_{ε_1} in the domain Ω_{m,ε_2}. So we have a function $u, u_{x_j} \in C(\bar{\Omega}_m \setminus O)$, $j = 1,2,3$, $u_t \in C(\bar{\Omega}_m \setminus \bar{\Sigma}_0)$. Therefore, there exists generalized solution of Problem PK in Ω_m in sense of Definition 3.

Proof of Theorem 4. Theorem 2 and Theorem 3 clime existence and uniqueness of generalized solutions $u(x,t)$ of Problem PK in Ω_m, which has the form (6). Using (10) for functions $U_n^s(\xi,\eta) = r(\xi,\eta)u_n^s(r(\xi,\eta),t(\xi,\eta))$ and $F_n^s(\xi,\eta) = \frac{1}{4}r(\xi,\eta)f_n^s(r(\xi,\eta),t(\xi,\eta))$ we obtain the 2-D problem PK_1. According to Theorem 5 the estimates (20) hold and we have

$$|U_n^s(\xi,\eta)| \le K\left(\max_{\bar{G}_m}|f_n^s|\right)(2-\xi-\eta)^{-n}$$

with a constant $K > 0$ independent of f_n^s. That implies

$$|u_n^s(r,t)| \le 2^{-n}K\left(\max_{\bar{G}_m}|f_n^s|\right)r^{-n-1}.$$

Therefore in view of (6) summing up over n and s we get the desired estimate (7).

4. Appendix. Riemann-Hadamard function

In the case $\mathbf{n} = \mathbf{0}$ the Riemann-Hadamard function associated to problem PK_2 is well known (see S. Gellerstedt [10], A. Nakhushev [29] and M. Smirnov [41]):

$$H(\xi,\eta;\xi_0,\eta_0) = \begin{cases} H^+(\xi,\eta;\xi_0,\eta_0), & \eta > \xi_0 \\ H^-(\xi,\eta;\xi_0,\eta_0), & \eta < \xi_0, \end{cases}$$

where $(\xi_0,\eta_0) \in D_0$, $(\xi,\eta) \in T \cup \Pi$ and

$$H^+(\xi,\eta;\xi_0,\eta_0) = \left(\frac{\eta-\xi}{\eta_0-\xi_0}\right)^{\beta} F(\beta,1-\beta,1;X),$$
$$H^-(\xi,\eta;\xi_0,\eta_0) = k\left(\frac{\eta-\xi}{\eta_0-\xi_0}\right)^{\beta} X^{-\beta} F\left(\beta,\beta,2\beta;\frac{1}{X}\right),$$

$$k = \frac{\Gamma(\beta)}{\Gamma(1-\beta)\Gamma(2\beta)}, \quad X = X(\xi,\eta,\xi_0,\eta_0) = \frac{(\xi_0-\xi)(\eta_0-\eta)}{(\eta-\xi)(\eta_0-\xi_0)}.$$

Here $F(a,b,c;\zeta)$ is the standard hypergeometric function of Gauss

$$F(a,b,c;\zeta) := \sum_{i=0}^{\infty} \frac{(a)_i(b)_i}{i!(c)_i}\zeta^i. \tag{22}$$

In the case $\mathbf{n} \ge \mathbf{0}$ we construct the following Riemann-Hadamard function for problem PK_2 of the form (16), where for $(\xi_0,\eta_0) \in D_0$ and $(\xi,\eta) \in T \cup \Pi$

$$\begin{aligned} \Phi^+ &= \left(\frac{\eta-\xi}{\eta_0-\xi_0}\right)^{\beta} F_3(\beta,n+1,1-\beta,-n,1;X,Y), \\ \Phi^- &= k\left(\frac{\eta-\xi}{\eta_0-\xi_0}\right)^{\beta} X^{-\beta} H_2\left(\beta,\beta,-n,n+1,2\beta;\frac{1}{X},-Y\right), \end{aligned} \tag{23}$$

$$Y = Y(\xi,\eta,\xi_0,\eta_0) := -\frac{(\xi_0-\xi)(\eta_0-\eta)}{(2-\xi-\eta)(2-\xi_0-\eta_0)}.$$

Here $F_3(a_1, a_2, b_1, b_2, c; x, y)$ is the Appell series

$$F_3(a_1, a_2, b_1, b_2, c; x, y) := \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(a_1)_j (a_2)_i (b_1)_j (b_2)_i}{(c)_{i+j} i! j!} x^j y^i \tag{24}$$

which converges absolutely for $\{|x| < 1, |y| < 1\}$(see [4], p. 220 - 223) and $H_2(a_1, a_2, b_1, b_2, c; x, y)$ is the Horn series

$$H_2(a_1, a_2, b_1, b_2, c; x, y) := \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(a_1)_{j-i} (a_2)_j (b_1)_i (b_2)_i}{(c)_j i! j!} x^j y^i \tag{25}$$

which converges absolutely for $\{|x| < 1, |y|(1 + |x|) < 1\}$ (see [4], p. 220 - 223).

We mention that for $(\xi_0, \eta_0) \in D_0$ we have $|X| < 1$ in Π and $1/|X| < 1$ in T, while $|Y| < 1$ in Π but $|Y|$ could be greater than 1 in T. However function Φ is well defined, because $n \in \mathbb{N}$ and we have finite sum with respect to i in function H_2(see (25)), which appears in (23).

Using the relation $(\beta)_{j-i}(1-\beta)_i = (-1)^i(\beta - i)_j$ it is easy to see that

$$\Phi(\xi, \eta; \xi_0, \eta_0) = H(\xi, \eta; \xi_0, \eta_0) + G(\xi, \eta; \xi_0, \eta_0) \tag{26}$$

with

$$G(\xi, \eta; \xi_0, \eta_0) = \begin{cases} G^+(\xi, \eta; \xi_0, \eta_0), & \eta > \xi_0 \\ G^-(\xi, \eta; \xi_0, \eta_0), & \eta < \xi_0, \end{cases}$$

where

$$G^+(\xi, \eta; \xi_0, \eta_0) := \left(\frac{\eta - \xi}{\eta_0 - \xi_0}\right)^{\beta} \sum_{i=1}^{n} c_i Y^i F(\beta, 1-\beta, i+1; X),$$

$$G^-(\xi, \eta; \xi_0, \eta_0) := k \left(\frac{\eta - \xi}{\eta_0 - \xi_0}\right)^{\beta} X^{-\beta} \sum_{i=1}^{n} d_i Y^i F\left(\beta - i, \beta, 2\beta; \frac{1}{X}\right)$$

and

$$c_i := \frac{(n+1)_i (-n)_i}{i!\, i!}, \;\; d_i := \frac{(n+1)_i (-n)_i}{(1-\beta)_i\, i!}.$$

Using properties of Gauss hypergeometrical function (22), the Appell series (24) and the Horn series (25) it is not difficult to see that the function Φ has the properties $(i) \div (vi)$ described in Section 2. To check the property (v) we establish that the function G has no jump on the line $\{\eta = \xi_0\}$, and $[[\Phi]] = [[H]]$.

We mention here that function (23) is closely connected to the Riemann-Hadamard function announced in [42](p. 25, example 7), which is associated to a Goursat-Darboux problem for an equation connected with (12) with some appropriate substitutions.

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