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A Simple Method for Orthogonal Polynomial Approximation of Linear Time-Varying System Gramians

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Abstract: The paper considers the problem of reachability and observability finite interval gramians computation and approximation for linear time-varying systems. Both gramians are obtained from system trajectories. The reachability gramian is derived from the adjoint state impulse response and the observability gramian is obtained from the zero-input output response. The application of the adjoint system is discussed and the two basic impulse response characteristics, namely the regular and the adjoint state impulse responses are presented. The relation with the linear time-invariant case is also discussed and the role of the state transition matrix for computing the gramians is shown. An algorithm for derivation of the state transition matrix is proposed, which is based on the integration of the state equation by using the method of Runge-Kutta. The gramians are approximated in terms of Legendre orthogonal series representations of system trajectories.

Keywords: linear systems, regular state impulse response, adjoint state impulse response, zero-input output response, reachability and observability gramians.

1. INTRODUCTION

The reachability and observability gramians have important energy interpretation in linear system theory. The reachability gramian contains the energy of the state impulse response and accounts for the energy distribution at the system input. The observability gramian contains the energy of the zero-input output response and accounts for the energy distribution at the system output. Both gramians are related to some important system properties and their nonsingulatity is a criterion for system complete reachability and observability. The reachability gramian participates in the expression for minimum energy least squares control, which transforms the system from one state to another. The observability gramian is used to determine the initial state vector by observing the output signal. Both gramians are part of the balancing algorithms and can be used for solving the model order reduction problem. The most important part in deriving the similarity transformation matrices for system balancing is the computation of the gramians. For linear time-invariant systems, balanced model reduction is well developed area, where the computation of system gramians is mainly based on solving certain Lyapunov equations, see Antoulas (2005). This is not the case for linear time-varying systems. The system matrices for such systems are functions of time, and the algebraic Lyapunov equations for computing the gramians are replaced by differential Lyapunov equations. Solving such equations for large scale systems is computationally cumbersome task and requires serious computational resources. Moreover, there exist different types of balancing and gramians definitions. Three types of gramians for linear time-varying systems are defined in Verriest & Kailath (1983) : finite interval gramians, infinite interval gramians and sliding interval gramians. Sliding interval gramians are initially used in Verriest & Kailath (1983) and then in Verriest (2008) for system balancing. However, the procedure for determining the gramians requires computing high order matrix derivatives, which creates certain numerical problems for large-scale systems. The concept of sliding interval gramians is also used in Shokoohi et al. (1983) for defining uniformly balanced realizations. Computing the gramians by solving differential equations of Lyapunov is presented in Lang et al. (2016). The proposed method uses backward differentiation formulas and the procedure of Rosenbrock. Different approach for computing the gramians for linear time-varying systems is proposed in Sandberg & Rantzer (2004). The method is based on solving time-dependent linear matrix inequalities. Solving linear matrix inequalities for obtaining the gramians in the discrete domain is also suggested in Lall & Beck (2003). Error bounds on the error of approximation are presented in Sandberg & Rantzer (2004) and Lall & Beck (2003), where the time-dependence of the Hankel singular values leads to different expressions for the bounds. Another approach, which proposes an algorithm for computing the gramians in the discrete domain for linear time-varying periodic systems is presented in Ma et al. (2010). The periodic discrete-time case is also considered in Varga (2000). A different approach for computing the gramians is to use the trajectories of the system, and to obtain the solution by using data snapshots from system trajectories, see Sirovich (1987). The state trajectory is discretized in equally distributed state points called snapshots, which are further employed for low dimensional approximation of system

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states. The trajectories based approach avoids solving the usual Lyapunov equations, while rather uses the snapshots matrices, which one can obtain either from experiment or from simulation. The trajectories based approach relates closely to the approach of empirical gramians, see Lall et al. (1999) and Himpe & Ohlberger (2013). The trajectories based approach is also considered in Perev (2018a), where the main derivations and results are obtained for linear time-invariant systems. It is shown that the gramians can be computed from the system state impulse and zero-input output responses. The paper gives the general framework for orthogonal polynomial approximation of system gramians, and presents basic information about the errors of approximation. The obtained results are extended for the linear time-varying case by employing the empirical gramians approach.

The present paper considers the problem of finite interval gramians computation by using orthogonal polynomials approximations of the adjoint system state impulse response and the zero-input output response. A new feature of the proposed method is the application of the adjoint system state trajectories for computing the reachability gramian. The presented algorithm is restricted only for computing finite interval gramians. This algorithm can not be used for computing infinite interval gramians, since the adjoint system is unstable and on infinite time scale, the adjoint state impulse response will approach infinity. For difference with the empirical gramians approach, the presented approach for computing the reachability gramian is not experimental, because the adjoint state impulse response can only be obtained by simulation. Both gramians for linear time-varying systems are functions of two variables, namely the initial time moment and the final time moment. If the initial time moment is fixed, then the state impulse response of the adjoint time-varying system can be considered as its Green function. In order to obtain the system trajectories we can use the Runge-Kutta method for numerical integration, requiring stepwise only several evaluations of system functions. The proposed method avoids solving the matrix differential equations of Lyapunov, which in the simplest case, requires solving algebraic equations of Lyapunov at each step of the algorithm. We claim that, the proposed method is more efficient than the Lyapunov's approach, since it replaces integrating matrix differential equations with the simpler operation of integrating vector differential equations and thus, reducing the computational cost.

2. LINEAR TIME-VARYING SYSTEM GRAMIANS

Consider the stable linear time-varying system described by its state space model:

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t), \qquad t \geq 0 \\ y(t) &= C(t)x(t), \qquad x(0) = x_0, \end{aligned} \tag{1}$$

where $x(t) \in R^n$, $u(t) \in R^m$ and $y(t) \in R^p$. The reachability map of system (1) on the interval $[t_0, t_1]$ is defined by the expression $L_r : PC([t_0, t_1]) \to R^n$ as $L_r : u_{[t_0,t_1]} \to \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)u(\tau)d\tau$, where $\Phi(t_1, t)$ is the state transition matrix on the interval $[t, t_1]$. The finite interval reachability gramian of system (1) on the interval $[t_0, t_1]$ is defined by the expression:

$$W_r(t_0, t_1) = \int_{t_0}^{t_1} \Phi(t_1, \tau)B(\tau)B(\tau)^T \Phi(t_1, \tau)^T d\tau \tag{2}$$

From (2) is clear that the reachability gramian is a symmetric positive semi-definite matrix. If we fix t_0 and consider the function $X : t_1 \to W_r(t_0, t_1)$, the reachability gramian can be computed as a solution of the following differential Lyapunov equation, see Callier & Desoer (1991):

$$\dot{X}(t) = A(t)X(t) + X(t)A(t)^T + B(t)B(t)^T \tag{3}$$

with initial condition $X(t_0) = 0$. Similarly, the observability map of system (1) on the interval $[t_0, t_1]$ is defined by the expression: $L_0 : R^n \to PC([t_0, t_1])$ as $L_o : x_0 \to C(t)\Phi(t, t_0)x_0$ for every $t \in [t_0, t_1]$. The finite interval observability gramian of system (1) on the interval $[t_0, t_1]$ is defined by the expression:

$$W_o(t_0, t_1) = \int_{t_0}^{t_1} \Phi(\tau, t_0)^T C(\tau)^T C(\tau)\Phi(\tau, t_0)d\tau \tag{4}$$

From expression (4) is clear that the observability gramian is also a symmetric positive semi-definite matrix. If we fix t_1 and consider the function $Y : t_0 \to W_o(t_0, t_1)$, the observability gramian can be computed as a solution of the following differential Lyapunov equation, see Callier & Desoer (1991):

$$\dot{Y}(t) = -A(t)^T Y(t) - Y(t)A(t) - C(t)^T C(t), \tag{5}$$

with final condition $Y(t_1) = 0$. In the linear time-invariant case, i.e. when the system matrices are constant matrices, the gramians are defined as follows: $W_r(0, T) = \int_0^T e^{At}BB^T e^{A^T t}dt$ and $W_o(0, T) = \int_0^T e^{A^T t}C^T Ce^{At}dt$. In the time-invariant case, the gramians are functions of one variable only, namely the difference between the final and initial time moments. For such systems, the state impulse response when $u(t) = \delta(t)$ is obtained as $x(t) = e^{At}B$. If we apply at the input a shifted signal $u(t) = \delta(t - t_0)$, the obtained state impulse response is also a shifted function $x(t) = e^{A(t-t_0)}B$. Yet, it has the same shape and form and this is the reason to excite the linear time-invariant system at $t_0 = 0$. The reachability gramian can be computed by using the state impulse response $x(t) = e^{At}B$ as $W_r(0, T) = \int_0^T x(\tau)x(\tau)^T d\tau$. Similarly, the observability gramian can be obtained as $W_o(0, T) = \int_0^T y(\tau)^T y(\tau)d\tau$, where $y(t) = Ce^{At}$ is the zero-input output response of the linear system due to sequentially selected unity initial conditions. This is not the case for linear time-varying systems. The state impulse response is obtained as $w(t, t_0) = \Phi(t, t_0)B(t_0)$ by exciting the system with delta impulse $u(t) = \delta(t - t_0)$. It is clear that, the state impulse response depends on two variables: the initial time moment t_0 and the final time moment t. Moreover, for different values of t_0 the obtained characteristics will be different. The reason for this conjecture is that the system parameter values change with time, and if we change the initial time moment, the parameter values will also change and therefore, the time response will be different. For example, by applying a shifted delta impulse $u(t) = \delta(t - t_0^1)$, the obtained state impulse response is $w(t, t_0^1) = \Phi(t, t_0^1)B(t_0^1)$ and if the input signal is $u(t) = \delta(t - t_0^2)$, then the response will be $w(t, t_0^2) = \Phi(t, t_0^2)B(t_0^2)$. The impulse responses $w(t, t_0^1)$ and $w(t, t_0^2)$ are not only shifted in time, but they are with different shape and form. Therefore, the reachability gramian computed for different initial time moments and the same duration, will also be different. In this sense, the gramians $W_r(t_0^1, t_1^1)$ and $W_r(t_0^2, t_1^2)$ are different, although the length of the time intervals, where the gramians are defined are the same, i.e. $t_1^1 - t_0^1 = t_1^2 - t_0^2$. The reachability gramian can be computed as:

$$W_r(t_0, t_1) = \int_{t_0}^{t_1} w(t_1, \tau) w(t_1, \tau)^T d\tau \tag{6}$$

The state impulse response of the linear time-varying system is Green function and for $u(t) = \delta(t - t_0)$ can be computed as $w(t, t_0) = \Phi(t, t_0)B(t_0)$. The Green function for a given system is determined by the kernel of the integral operator and denotes the response of the system to a concentrated into a given point unit input signal. The unit input signal is concentrated at the time moment t_0 and is presented by a delta function $u(t) = \delta(t - t_0)$. In the empirical gramians approach, we derive the state impulse response $w(t, t_0)$, where the fixed time variable is t_0 and the time variable which changes is t. In the finite interval gramians case however, the fixed time variable for the integrand $w(t_1, \tau)$ is the final time moment t_1, while the changing one is the initial time moment τ, see expression (2). Therefore, the integration variables of the integral kernel have to be replaced. This can be achieved by switching the time variables in the state transition matrix. This switching of time variables in the state transition matrix can be obtained by using the adjoint system description, see Perev (2018b). The homogeneous adjoint of system (1) is defined by the equation:

$$\dot{p}(t) = -A(t)^T p(t), \tag{7}$$

where $p(t) \in R^n$ is the state vector of the adjoint system (7). We denote by $\Psi(t_1, t_0)$ the state transition matrix of the adjoint system (7) on the time interval $[t_0, t_1]$. The relation between the state transition matrices of the original system (1) and the adjoint (7) is given by the expression, see Callier & Desoer (1991):

$$\Psi(t, t_0) = \Phi(t_0, t)^T \tag{8}$$

It is clear that the ordering of the time variables for the adjoint system is reversed with respect to the ordering of these time variables in the original system. Using (8), the state transition matrix of the original system can be written in the form $\Phi(t, \tau) = \Phi(t, t_0)\Phi(t_0, \tau) = \Phi(t, t_0)\Psi(\tau, t_0)^T$. Therefore, the computation of the state transition matrix as a function of its first variable depends on the computation of the state transition matrix of the original system (1) by computing $\Phi(t, t_0)$ and the computation of the state transition matrix of the adjoint system (7) in terms of computing $\Psi(\tau, t_0)$. The reachability gramian of the original system (1) can be computed as:

$$W_r(t_0, t_1) = \Phi(t_1, t_0) \int_{t_0}^{t_1} \Psi(\tau, t_0)^T B(\tau) B(\tau)^T \cdot \Psi(\tau, t_0) d\tau \Phi(t_1, t_0)^T, \tag{9}$$

where expression (8) has been used. Therefore, in order to compute the reachability gramian, the state transition matrix of the adjoint system (7) has to be derived and appropriately used in the integral (9). However, there exists a major obstruction in computing $\Psi(t, t_0)$ due to instability of system (7). There exists a fundamental difference between the regular and adjoint state impulse responses. While the regular state impulse response for a stable system converges to zero, the adjoint state impulse response will diverge. Therefore, the adjoint state impulse response can not be computed on an infinite time interval. The state impulse response for an unstable system can be computed only on a finite interval of time. The procedure of computing the finite interval reachabilty gramian follows the idea behind equation (3), where we fix the initial time moment t_0. The application of the trajectory-based approach to the adjoint system depends also on the trajectories, obtained from the regular system. Since the adjoint system is unstable, the interval of integration for this system trajectories is determined entirely from the interval of integration of the original regular system trajectories. The algorithm for computing the system gramians is further presented:

Algorithm for computation of system gramians

The reachyability gramian is obtained as follows:

- Fix t_0. Apply the Runge-Kutta algorithm to system (1), compute its state impulse response and determine the state transition matrix $\Phi(t, t_0)$, $t \geq t_0$
- Determine the final time moment and therefore, the interval of integration $[t_0, t_1]$ from stability considerations
- Apply the Runge-Kutta algorithm to the adjoint system on this interval and compute the matrices $\Psi(\tau, t_0)^T$ as a function of the current time moment
- Use the expression $\Phi(t, \tau) = \Phi(t, t_0)\Phi(t_0, \tau) = \Phi(t, t_0)\Psi(\tau, t_0)^T$ to obtain the state transition matrix of system (1) as function of its first argument
- Determine the state trajectory $w(t, \tau) = \Phi(t, \tau)B(\tau)$ and compute $W_r(t_0, t_1) = \int_{t_0}^{t_1} w(t, \tau) w(t, \tau)^T d\tau$.

The observability gramian is obtained as follows:

- Apply the Runge-Kutta algorithm to system (1) and determine the state transition matrix $\Phi(t, t_0)$, $t \geq t_0$
- Obtain the transpose of the zero-input output response $h(t, t_0)$, where $h(t, t_0) = \Phi(t, t_0)^T C(t_0)^T$ is obtained by sequentially selecting the initial conditions as the columns of the identity matrix. In order to see this, consider the zero-input output response $y(t) = C(t)\Phi(t, t_0)x_0$. If we fix the initial time moment t_0 and apply the computation of the output with respect to different initial state vectors, we obtain the following expression:
$$Y(t) = [\, y_1(t) \; y_2(t) \; \cdots \; y_n(t) \,] = [\, C(t)\Phi(t, t_0)x_{0,1} \; \cdots \; C(t)\Phi(t, t_0)x_{0,n} \,], \tag{10}$$
where the initial state vectors are selected as columns of the identity matrix, i.e. $x_{0,j} = e_j$, $j = 1, 2, \cdots, n$. Therefore, $Y(t)^T = \Phi(t, t_0)^T C(t)^T = h(t, t_0)$ is a $[n \times 1]$ vector.
- Compute the observability gramian as:

$$W_o(t_0,t_1) = \int_{t_0}^{t_1} h(\tau,t_0)h(\tau,t_0)^T d\tau, \quad (11)$$

After computing both gramians for the initial time moment t_0, we can change the initial time moment as $t_0 \to t_0 + \delta$ and repeat the whole procedure again. We apply the trajectories based approach for computing the finite interval gamians by utilizing the fourth order Runge-Kutta algorithm. The order of accuracy for the presented method is $O(h^4)$, where h is the discretization step. The main computational task is to integrate numerically the adjoint state equation, which is a vector differential equation. Different approach for computing the gramians is by solving the matrix differential equations of Lyapunov. In its simplest form, when the system matrices are constant, most of the popular algorithms require solving algebraic Lyapunov equations at each step of the algorithm, see Lang et al. (2015), Behr et. al (2018). However, in the case when system matrices are functions of time, the numerical integration of the Lyapunov matrix differential equations is unavoidable, see Benner & Stykel (2017), Behr et. al (2018). The main advantage of the proposed method is that, it replaces integrating matrix differential equations with integrating vector differential equations. Additional feature of its efficiency is the possibility for parallelization of the computing processes, which further reduces the time of calculations.

3. LEGENDRE POLYNOMIAL APPROXIMATION OF SYSTEM GRAMIANS

The Legendre polynomials form a complete set of orthogonal functions in the Hilbert space $L_2[-1,1]$. The n-th order Legendre polynomial is defined as in Abramowitz & Stegun (1972):

$$P_n(t) = \frac{1}{2^n n!}\frac{d^n}{dt^n}(t^2-1)^n, \quad n = 0,1,2,\cdots \quad (12)$$

The Legendre polynomials can also be computed by using the following recurrence relation, see Abramowitz & Stegun (1972):

$$P_{n+1}(t) = \frac{(2n+1)tP_n(t) - nP_{n-1}(t)}{n+1}, \quad (13)$$
$$P_0(t) = 1, \quad P_1(t) = t, \quad n = 1,2,\cdots$$

The Legendre polynomials can be normalized by using the functions $\varphi_n(t) = \sqrt{\frac{2n+1}{2}}P_n(t)$ and they satisfy the orthonormality condition with an weighting function $w(t) = 1$. When the definition interval is different than $[-1,1]$, the Legendre functions are rescaled and the so called shifted Legendre functions are computed. For the Hilbert space $L_2[t_0,t_1]$, the shifted Legendre functions are obtained as $\varphi_n(t) = \sqrt{\frac{2n+1}{2}}P_n(\frac{2}{t_1-t_0}t - \frac{t_1+t_0}{t_1-t_0})$. Then every function $f(t) \in L_2[t_0,t_1]$ can be approximated on the interval $[t_0,t_1]$ by the Legendre polynomial series as $f(t) \approx \sum_{n=0}^{N} q_n \sqrt{\frac{2n+1}{2}}P_n(\frac{2}{t_1-t_0}t - \frac{t_1+t_0}{t_1-t_0})$, where the Fourier coefficients are computed as:

$$q_n = \frac{2}{t_1-t_0}\sqrt{\frac{2n+1}{2}}\int_{t_0}^{t_1} f(t)P_n(\frac{2}{t_1-t_0}t - \frac{t_1+t_0}{t_1-t_0})dt,$$

with N being the order of series truncation in the Legendre orthogonal series approximation. The mean square error of approximation is determined by the expression, see Schetzen (1989), as $\varepsilon_N^2 = \int_{t_0}^{t_1} f(t)^2 dt - \sum_{n=0}^{N} q_n^2$. In the vector case, the error of approximation is determined from the expression: $\varepsilon_N^2 = \int_{t_0}^{t_1} f(t)^T f(t)dt - \sum_{n=0}^{N} q_n^T q_n$

We consider the Legendre orthogonal series approximation of system gramians. Assume first the SISO case. As a first step we fix the initial time moment t_0. Based on energy considerations, we determine the final time moment t_1. We partition the time interval $[t_0,t_1]$ uniformly with a step $\delta = \frac{t_1-t_0}{n}$. The next step is to compute the adjoint state impulse response $w(t_1,\tau)$, $\tau \in [t_0,t_1]$ by using the algorithm presented in the previous section. Then, we determine its Legendre series approximation as $w(t_1,\tau) \approx \sum_{k=0}^{N} q_k \sqrt{\frac{2n+1}{2}}P_n(\frac{2}{t_1-t_0}\tau - \frac{t_1+t_0}{t_1-t_0})$, where q_k, $k = 0,1,2,\cdots N$ are the Fourier vector coefficients of the Legendre series expansion of the impulse response, which are determined as $q_k = \frac{2}{t_1-t_0}\sqrt{\frac{2k+1}{2}}\int_{t_0}^{t_1} w(t_1,\tau)P_k(\frac{2}{t_1-t_0}\tau - \frac{t_1+t_0}{t_1-t_0})d\tau$. The reachability gramian is determined as:

$$W_r(t_0,t_1) = \int_{t_0}^{t_1} w(t_1,\tau)w(t_1,\tau)^T d\tau \approx \frac{t_1-t_0}{2}\sum_{k=0}^{N} q_k q_k^T$$

In the MIMO case, the reachability gramian approximation is computed by using the *Dyadic Expansion Lemma* of two matrices product, see Callier & Desoer (1991), as $W_r(t_0,t_1) \approx \frac{t_1-t_0}{2}\sum_{k=0}^{N}\sum_{i=1}^{m} q_k^i (q_k^i)^T$, where q_k^i, $k = 0,1,2,\cdots,N$ is the Fourier vector coefficient corresponding to the i^{th} input signal component $i = 1,2,\cdots,m$, when the other components are zero, see Perev (2018a). Similarly to the reachability gramian case, we approximate the observability gramian as $W_o(t_0,t_1) = \int_{t_0}^{t_1} h(\tau,t_0)h(\tau,t_0)^T d\tau \approx \frac{t_1-t_0}{2}\sum_{k=0}^{N} f_k f_k^T$, where f_k, $k = 0,1,2,\cdots,N$ are the Fourier vector coefficients of the Legendre series expansion of the zero-input output response of system (1) and are computed as $f_k = \frac{2}{t_1-t_0}\sqrt{\frac{2k+1}{2}}\int_{t_0}^{t_1} h(\tau,t_0)P_k(\frac{2}{t_1-t_0}\tau - \frac{t_1+t_0}{t_1-t_0})d\tau$, where $h(t,t_0) = \Phi(t,t_0)^T C(t_0)^T$. In the MIMO case the observability gramian is approximated by the expression $W_o(t_0,t_1) \approx \frac{t_1-t_0}{2}\sum_{k=0}^{N}\sum_{j=1}^{p} f_k^j (f_k^j)^T$, where f_k^j, $k = 0,1,2,\cdots,N$ is the Fourier vector coefficient in the Legendre series expansion for the j^{th} output response $j = 1,2,\cdots p$, see Perev (2018a).

4. NUMERICAL EXAMPLE

Consider the linear time-varying stable system (1), where the system matrices are determined as follows:

$$A(t) = \begin{bmatrix} -1+\cos^2 t & 1 - ae^{-0.2t}\sin t\cos t \\ -1 - ae^{-0.4t}\sin t\cos t & -1 + a\sin^2 t \end{bmatrix}$$

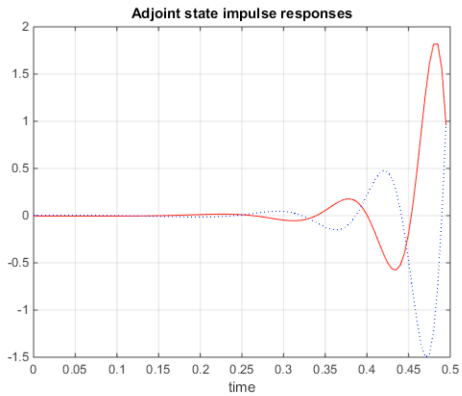

Fig. 1. Adjoint state impulse responses

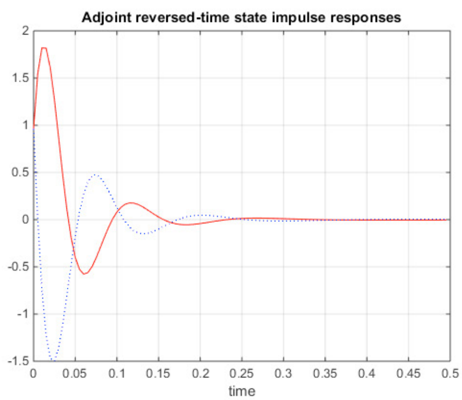

Fig. 2. Adjoint reversed-time state impulse responses

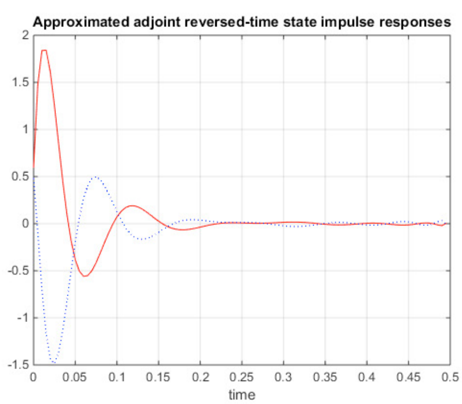

Fig. 3. Approximated adjoint reversed-time state impulse responses

$$B(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad C(t) = [\, 1.5 \;\; 0.5 \,],$$

where the parameter a assumes the value $a = 1.5$. The state impulse response characteristics are obtained on the interval $[0, 0.5]$ with discretization step $\delta = 0.005$.

It is clear that the adjoint state impulse response is unstable and is bounded only on a finite interval of time as can be seen in fig.1. If we reverse the time variable by using $\tau = t_1 - t$, we obtain the adjoint reversed-time state impulse responses in fig.2.

The next step is to apply the Legendre orthogonal series approximation of these characteristics and to compute the system gramians based on the approximated curves. We use the same discretization step for the series expansion as $\delta = 0.005$ and the order of series truncation is $N = 25$. The approximated adjoint reversed-time state impulse responses are shown in fig.3.

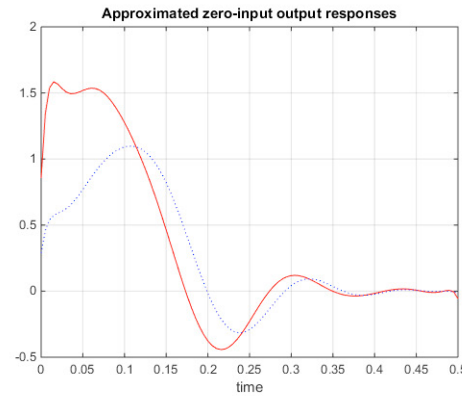

Fig. 4. Approximated zero-input output responses

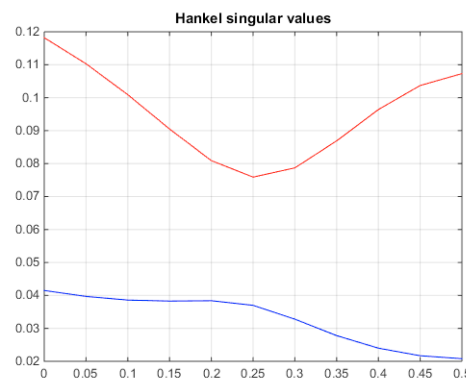

Fig. 5. Hankel singular values as functions of t_0

The zero-input output approximate responses obtained by using 25^{th} order Legendre series approximation of the output characteristics are shown in fig.4. Fig.5 contains the Hankel singular values of system (1) for the time interval $t \in [0, 0.5]$, when the time variable is the initial time moment $t = t_0$. The corresponding Hankel singular values on the same time interval, obtained by orthogonal approximation of the system (1) adjoint state impulse and the zero-input output responses are shown in fig.6, where both Hankel singular values are again functions of the initial time moment t_0.

5. CONCLUSION

The paper considers the problem for Legendre orthogonal polynomials approximation of system finite interval gramians. A simple method for computing the gramians instead of solving differential Lyapunov equations is presented. The proposed method is based on approximation of the adjoint system state impulse response and the regular system zero-input output response. The numerical implementation of the method uses the Runge-Kutta algorithm for performing linear time-varying system simulations. Both gramians are approximated by using Legendre orthogonal

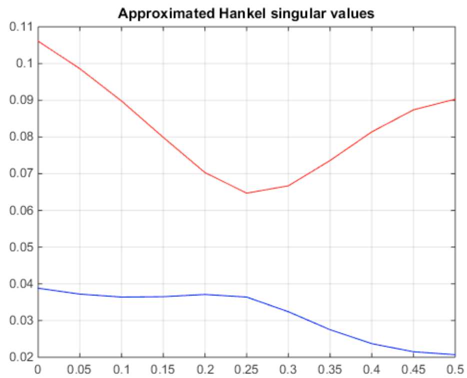

Fig. 6. Approximated HSV as functions of t_0

polynomial representations of the system adjoint state impulse response and the zero-input output response.

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