AIP Conference Proceedings

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Cite as: AIP Conference Proceedings **2172**, 030016 (2019); https://doi.org/10.1063/1.5133505
Published Online: 13 November 2019

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Explicit Solutions of Protter's Problem for a 4-D Hyperbolic Equation Involving Lower Order Terms with Constant Coefficients

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Abstract. The Protter's problems are multidimensional variants of the 2-D Darboux problems for hyperbolic and weakly hyperbolic equations and they are not well-posed in the frame of classical solvability, since their adjoint homogeneous problems have infinitely many nontrivial classical solutions. The generalized solutions of the Protter's problem may have strong singularities even for very smooth right-hand side functions of the equation. These singularities are isolated at one boundary point and do not propagate along the bicharacteristics which is unusually for the hyperbolic equations.

Here we treat a generalization of the well studied Protter's problem for the 4-D wave equation, considering a case of more general equation involving lower order terms with constant coefficients. First, we announce explicit formulas for the nontrivial classical solutions of the corresponding adjoint homogeneous problem. Further, we give an exact integral representation of the generalized solutions of the considered problem as well as an asymptotic expansion of their singularities.

INTRODUCTION

Denote the points in R^4 as $(x,t) := (x_1, x_2, x_3, t)$ and, respectively, $|x| := \sqrt{x_1^2 + x_2^2 + x_3^2}$.

For $a_1, a_2, a_3, b, c \in \mathrm{R}$ consider the following boundary value problem:

$$\sum_{i=1}^{3} v_{x_i x_i} - v_{tt} + \sum_{i=1}^{3} a_i v_{x_i} + b v_t + c v = g(x,t) \quad \text{in} \quad \Omega, \tag{1}$$

$$v|_{\Sigma_0 \cup \Sigma_1} = 0, \tag{2}$$

where the region

$$\Omega := \{(x,t): \ 0 < t < 1/2, \ t < |x| < 1 - t\}$$

is bounded by the ball

$$\Sigma_0 := \{(x,t): \ t = 0, \ |x| < 1\}$$

and by two characteristic surfaces of equation (1)

$$\Sigma_1 := \{(x,t): \ 0 < t < 1/2, \ |x| = 1 - t\}, \qquad \Sigma_2 := \{(x,t): \ 0 < t < 1/2, \ |x| = t\}.$$

For the sake of convenience instead of problem (1)–(2) we will treat another one, which is immediately derived from (1)–(2) applying the substitution

$$u = v \, \exp\left(\frac{a_1}{2}x_1 + \frac{a_2}{2}x_2 + \frac{a_3}{2}x_3 - \frac{b}{2}t\right).$$

Proceedings of the 45th International Conference on Application of Mathematics in Engineering and Economics (AMEE'19)
AIP Conf. Proc. 2172, 030016-1–030016-7; https://doi.org/10.1063/1.5133505
Published by AIP Publishing. 978-0-7354-1919-3/$30.00

Problem $\mathbf{P}_\gamma$. *For* $\gamma \in \mathrm{R}$ *find a solution* $u(x,t)$ *of the equation*

$$\sum_{i=1}^{3} u_{x_i x_i} - u_{tt} - \gamma u = f(x,t) \quad \text{in} \quad \Omega, \tag{3}$$

which satisfies the boundary condition

$$u|_{\Sigma_0 \cup \Sigma_1} = 0.$$

The constant γ is related to the coefficients in (1) by

$$\gamma = \frac{1}{4}\left(a_1^2 + a_2^2 + a_3^2 - b^2 - 4c\right)$$

and also we have

$$f(x,t) = g(x,t)\ \exp\left(\frac{a_1}{2}x_1 + \frac{a_2}{2}x_2 + \frac{a_3}{2}x_3 - \frac{b}{2}t\right).$$

The adjoint problem to P_γ is as follows:

Problem $\mathbf{P}^*_\gamma$. *Find a solution to the self-adjoint equation* (3) *in* Ω *which satisfies the boundary condition*

$$u|_{\Sigma_0 \cup \Sigma_2} = 0.$$

In 1954 M. H. Protter ([21, 22]) proposed some multidimensional boundary value problems for hyperbolic and weakly hyperbolic equations. These problems are multidimensional analogues of the planar Darboux problems, since the boundary data is imposed on one of the characteristic surfaces of the equation and on a non-characteristic surface. Actually, Protter arrived to his problems while studied multidimensional variants of the famous Guderley-Morawetz problem for mixed-type equations and restricted his investigation in the hyperbolic part of the considered domain.

However, while the two-dimensional Darboux problem is well posed, this is not true for the Protter's problems, since they have infinite-dimensional cokernels ([11, 19, 24]). This means that for the existence of classical solutions it is necessary infinitely many orthogonality conditions on the right-hand side of the equation to be fulfilled. For this reason (following [19]) it is suitable to study the Protter's problems in the frame of generalized solutions with possible big singularities. Today it is well-known that even for very smooth right-hand sides of the equation such singularities really exist. It is interesting that they are isolated at one boundary point and do not propagate along the bicharacteristics, which is not traditionally assumed for the hyperbolic equations.

Different Protter's problems for 3-D and 4-D equations were studied for example in [3, 7, 14, 15, 17, 18, 19, 20]. Generalizations of the Protter's problems (for mixed-type equations, for nonlinear equations etc) are considered in [1, 4, 10, 12, 16, 23].

In the papers [17, 20] it was studied at length a particular case of Problem P_γ, more precisely the case $\gamma = 0$, when equation (3) is the 4-D wave equation. Here we generalize this research, adding to the wave operator lower order terms with constant coefficients.

Some 3-D Protter's problems for hyperbolic and weakly hyperbolic equations involving lower terms (in a more general case, when the coefficients are not restricted to be constant) are studied in [5, 6, 8, 9, 13]. Different results on the existence and uniqueness of generalized solutions were proven. However, in the case of nontrivial lower order terms, any explicit formulas for the solutions of these problems or their adjoint problems were not given.

ILL-POSEDNESS AND GENERALIZED SOLVABILITY OF THE PROBLEM

For $k, n \in \mathrm{N} \cup \{0\}$ define the functions:

$$\mathcal{H}_k^n(r,t) := \frac{t\left(r^2 - t^2\right)^{n-2k-1}}{r^{n-2k+1}}\ {}_2F_1\left(n-k+\frac{1}{2}, -k, \frac{3}{2}; \frac{t^2}{r^2}\right)\ {}_0F_1\left(n-2k; \frac{\gamma}{4}\left(r^2 - t^2\right)\right)$$

and let $Y_n^s(x)$, $n \in \mathrm{N} \cup \{0\}$, $s = 1, 2, \ldots, 2n+1$ be the three-dimensional spherical functions, defined on the unit sphere $S^2 := \{x \in \mathrm{R}^3 : |x| = 1\}$.

Then we state the following lemma:

Lemma 1. *For $k = 0, \dots, [(n-1)/2] - 2$ and $s = 1, 2, \dots, 2n+1$ the functions*

$$v^n_{k,s}(x,t) := \begin{cases} \mathcal{H}^n_k\big(|x|, t\big)\, Y^s_n\big(x/|x|\big), & (x,t) \neq O, \\ 0, & (x,t) = O \end{cases}$$

with $O := (0,0,0,0)$, are linearly independent classical solutions from $C^2(\bar{\Omega})$ of the homogeneous Problem P^*_γ.

Then Problem P_γ is not well-posed in the frame of classical solvability. A necessary condition for the existence of a classical solution of Problem P_γ is the orthogonality of the right-hand side function $f(x,t)$ to all these functions $v^n_{k,s}(x,t)$. This means that an infinite number of orthogonality conditions $\mu^n_{k,s} = 0$ with

$$\mu^n_{k,s} := \int_\Omega v^n_{k,s}(x,t) f(x,t)\, dxdt \tag{4}$$

must be fulfilled. In this situation (which is typical for the Protter's problems at all) it is suitable to seek for solutions in a generalized sense. Define the generalized solutions of Problem P_γ similarly to [5, 17]:

Definition 1. *A function $u = u(x,t)$ is called a generalized solution of Problem P_γ in Ω if*:
(1) $u \in C^1(\bar{\Omega}\backslash O),\ u|_{\Sigma_1} = 0,\ u|_{\Sigma_0\backslash O} = 0;$
(2) *the identity*

$$\int_\Omega (u_t w_t - u_{x_1} w_{x_1} - u_{x_2} w_{x_2} - u_{x_3} w_{x_3} - \gamma u w - f w)\, dxdt = 0$$

holds for all w from

$$W_0 := \{w \in C^1(\bar{\Omega}) :\ w|_{\Sigma_0} = 0,\ w \equiv 0 \text{ in a neighborhood of } \Sigma_2\}.$$

This definition allows the generalized solutions to have strong singularities at the point O. The results we give below show that in the general case such singularities really exist.

We have the following result on the generalized solvability of Problem P_γ:

Theorem 1. *Problem P_γ has at most one generalized solution in Ω. Further, let the right-hand side of equation* (3) *be of the form*

$$f(x,t) = \sum_{n=0}^{l} \sum_{s=1}^{2n+1} f^s_n\big(|x|, t\big)\, Y^s_n\big(x/|x|\big), \qquad l \in \mathrm{N} \cup \{0\} \tag{5}$$

and $f \in C(\bar{\Omega})$. Then there exists a generalized solution $u(x,t)$ of Problem P_γ.

Actually, it is well known that the spherical functions form a complete orthonormal system in $L_2(S^2)$, i.e. the function in the right-hand side of (5) is a partial sum of a Fourier expansion.

2-D PROBLEM RELATED TO $\mathbf{P}_\gamma$

Next, consider the following 2-D problem:

Problem $\mathbf{P}_{\gamma 2}$. *Find a function $U(\xi,\eta)$ such that*:
(1) *$U(\xi,\eta)$ solves the differential equation*

$$U_{\xi\eta} - \left(\frac{n(n+1)}{(2-\xi-\eta)^2} + \frac{\gamma}{4}\right) U = F(\xi,\eta) \tag{6}$$

in the domain $D := \Big\{(\xi,\eta) :\ 0 < \xi < \eta < 1\Big\}$;
(2) *$U(\xi,\eta)$ satisfies the boundary conditions*:

$$U(0,\eta) = 0, \qquad U(\xi,\xi) = 0, \quad 0 \leq \xi < 1;$$

(3) $U \in C^2(D) \cap C(\bar{D} \setminus (1,1))$.

This problem is related to Problem P_γ in the following way:

Lemma 2. *Let the right-hand side of equation* (3) *be of the form* (5) *and* $f \in C(\bar\Omega)$. *Then the unique generalized solution* $u(x,t)$ *of Problem* P_γ*, stated in Theorem 1, can be represented as*

$$u(x,t) = \frac{1}{|x|} \sum_{n=0}^{l} \sum_{s=1}^{2n+1} U_n^s\Big(1 - |x| - t, 1 - |x| + t\Big) Y_n^s\Big(x/|x|\Big),$$

where the functions $U_n^s(\xi,\eta)$ *are solutions of Problem* $P_{\gamma 2}$ *with corresponding right-hand side functions in* (6)

$$F(\xi,\eta) = \frac{1}{8}(2 - \xi - \eta) f_n^s\left(\frac{2-\xi-\eta}{2}, \frac{\eta-\xi}{2}\right).$$

Actually, Problem $P_{\gamma 2}$ is obtained from Problem P_γ via the method of separation of variables. According to the ill-posedness of Problem P_γ and the possible singularity of the function $u(x,t)$ at the point O, the functions $U_n^s(\xi,\eta)$ are allowed to have singularity at the point $(1,1)$ (note that equation (6) has a coefficient with singularity at this point and the statement of Problem $P_{\gamma 2}$ allows its solutions to be discontinuous there).

We solve Problem $P_{\gamma 2}$ explicitly via the Riemann-Hadamard method. To write the solution, define the function:

$$R(\xi,\eta;\xi_0,\eta_0) := \Xi_2\left(n+1, -n, 1;\ \frac{-(\xi_0-\xi)(\eta_0-\eta)}{(2-\xi-\eta)(2-\xi_0-\eta_0)}, \frac{\gamma}{4}(\xi_0-\xi)(\eta_0-\eta)\right),$$

where

$$\Xi_2(a,b,c;x,y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(a)_i (b)_i}{(c)_{i+j}\, i!\, j!} x^i y^j$$

is a confluent hypergeometric series of two variables (this series is given for example in [2]). Note that in $\Xi_2(n+1,-n,1;x,y)$ we have a finite sum in respect to the index i and the series in respect to j has an infinite radius of convergence.

An integral representation of $U(\xi,\eta)$ is given in the next theorem.

Theorem 2. *Let* $F \in C(\bar D)$. *Then Problem* $P_{\gamma 2}$ *is uniquely solvable and its solution has the following integral representation at a point* $(\xi_0,\eta_0) \in D$:

$$U(\xi_0,\eta_0) = \int_0^{\xi_0} \int_{\xi}^{\eta_0} \Phi(\xi,\eta;\xi_0,\eta_0) F(\xi,\eta)\, d\eta d\xi, \tag{7}$$

where the Riemann-Hadamard function $\Phi(\xi,\eta;\xi_0,\eta_0)$ *is defined as*

$$\Phi(\xi,\eta;\xi_0,\eta_0) := \begin{cases} R(\xi,\eta;\xi_0,\eta_0), & \eta > \xi_0, \\ R(\xi,\eta;\xi_0,\eta_0) - R(\eta,\xi;\xi_0,\eta_0), & \eta < \xi_0. \end{cases}$$

From (7) one could see that in the general case, if no special conditions on the function $F(\xi,\eta)$ are imposed, the solution $U(\xi,\eta)$ should have a power-type singularity at the point $(1,1)$.

ASYMPTOTIC BEHAVIOR OF THE SINGULAR SOLUTIONS

A specific feature of the Protter's problems studied so far (including our problem with $\gamma = 0$) is that the asymptotic behavior of their singular solutions is determined by orthogonality conditions of the right-hand side of the equation in respect to the nontrivial solutions of the adjoint homogeneous problem. We find that this is true for Problem P_γ with $\gamma \neq 0$ as well.

First, we give the asymptotic expansion of the singular solutions of Problem $P_{\gamma 2}$.

From Lemma 1 it follows that for $k = 0, \dots, [n/2] - 2$ the functions

$$H_k^n(\xi, \eta) := (\eta - \xi)\, \frac{(1-\xi)^{n-2k-1}(1-\eta)^{n-2k-1}}{(2-\xi-\eta)^{n-2k}}\, {}_2F_1\left(n-k+\frac{1}{2}, -k, \frac{3}{2}; \frac{(\eta-\xi)^2}{(2-\xi-\eta)^2}\right) {}_0F_1\left(n-2k; \frac{\gamma}{4}\,(1-\xi)(1-\eta)\right),$$

continued at the point $(1,1)$ as $H_k^n(1,1) := 0$, are linearly independent classical solutions of the adjoint to $P_{\gamma 2}$ homogeneous problem, which means:

(1) $H_k^n \in C^2(\bar{D})$;

(2) $H_k^n(\xi, \eta)$ solve equation (6) with $F(\xi, \eta) \equiv 0$;

(3) $H_k^n(\xi, 1) = 0,\ H_k^n(\xi, \xi) = 0$.

Further, setting $\eta_0 = 1$, we find an expansion of the function $\Phi(\xi, \eta; \xi_0, 1)$ in powers of $1 - \xi_0$, namely:

$$R(\xi, \eta; \xi_0, 1) = \sum_{p=0}^{n} (1-\xi_0)^{-p}\, \frac{(n+1)_p(-n)_p}{p!\, p!}\, W_n^p(\xi, \eta) + \sum_{p=1}^{\infty} (1-\xi_0)^p\, \frac{\gamma^p}{4^p\, p!\, p!}\, W_n^p(\xi, \eta)$$

with

$$W_n^p(\xi, \eta) := \frac{(1-\xi)^p(1-\eta)^p}{(-1)^p(2-\xi-\eta)^p}\, {}_2F_1\left(p+n+1, p-n, p+1; \frac{1-\eta}{2-\xi-\eta}\right) {}_0F_1\left(p+1; \frac{\gamma}{4}(1-\xi)(1-\eta)\right)$$

and

$$R(\xi, \eta; \xi_0, 1) - R(\eta, \xi; \xi_0, 1) = \sum_{k=0}^{[(n-1)/2]} a_k^n (1-\xi_0)^{2k+1-n} H_k^n(\xi, \eta) + \gamma \sum_{k=-\infty}^{[(n-2)/2]} b_k^n (1-\xi_0)^{n-2k-1} H_k^n(\xi, \eta),$$

where a_k^n and b_k^n are nonzero constants.

Applying this expansion into (7) we obtain that

$$U(\xi, 1) = \sum_{k=0}^{[(n-1)/2]} c_k^n\, \mu_k^n\, (1-\xi_0)^{2k+1-n} + g(\xi)(1-\xi), \tag{8}$$

where

$$\mu_k^n := \int_D H_k^n(\xi, \eta) F(\xi, \eta)\, d\xi d\eta,$$

$c_k^n = \text{const} \neq 0$ and the function $g(\xi) \in C([0,1))$ is bounded on the segment $[0,1]$. Note that the order of singularity of $U(\xi, 1)$ is controlled by the coefficients μ_k^n, i.e. by orthogonality conditions of the right-hand side $F(\xi, \eta)$ in respect to the corresponding functions $H_k^n(\xi, \eta)$. The coefficients μ_k^n are related to the coefficients (4) as $\mu_k^n = \beta_k^n \mu_{k,s}^n$, where β_k^n are nonzero constants independent of $f(x, t)$.

Now, having a boundary data given by (8) and the boundary condition $U(0, \eta) = 0$, we solve Problem $P_{\gamma 2}$ by the Riemann method to obtain the following result:

Theorem 3. *Let $F \in C(\bar{D})$. Then the unique solution of Problem $P_{\gamma 2}$ has the following asymptotic representation at the singular point* $(1,1)$:

$$U(\xi, \eta) = \sum_{k=0}^{[(n-1)/2]} \mu_k^n\, G_k^n(\xi, \eta)\, (1-\xi_0)^{2k+1-n} + G(\xi, \eta),$$

where:

(1) *the functions $G_k^n(\xi, \eta)$ are bounded in $\bar{D}$ and independent of $F(\xi, \eta)$*;

(2) *the function $G(\xi, \eta)$ is bounded in $\bar{D}$*;

(3) $G_k^n(\xi, 1) = c_k^n$ *and* $G(\xi, 1) = g(\xi)(1-\xi)$.

Finally, using Theorem 3, we may describe the asymptotic behavior of the singular solutions of the 4-D Problem P_γ by the next theorem:

Theorem 4. *Suppose that the right-hand side function* $f \in C(\bar{\Omega})$ *has the form* (5). *Then the unique generalized solution* $u(x,t)$ *of Problem* P_γ *belongs to* $C^2(\bar{\Omega}\backslash O)$ *and has the following asymptotic expansion at the point* O:

$$u(x,t) = \frac{1}{|x|}\left(\sum_{p=1}^{l} |x|^{1-p} Q_p(x,t) + Q(x,t)\right),$$

where:

(1) *the function* $Q(x,t)$ *is bounded in* $\bar{\Omega}$ *and in the case* $\gamma = 0$ *it satisfies the a priori estimate*

$$|Q(x,t)| \le C\, t \max_{\bar{\Omega}} |f(x,t)|, \quad (x,t) \in \Omega$$

with a constant C *independent of* f;

(2) *the functions* Q_p, $p = 1,\dots,l$ *satisfy the equalities*

$$Q_p(x,t) = \sum_{k=0}^{[(l-p)/2]} \sum_{s=1}^{2p+4k+1} \mu_{k,s}^{p+2k} Q_{k,s}^{p+2k}(x,t) \tag{9}$$

with functions $Q_{k,s}^n(x,t)$ *bounded in* $\bar{\Omega}$ *and independent of* f, *which in the case* $\gamma = 0$ *have the following exact expression*:

$$Q_{k,s}^n(x,t) = h_k^n \frac{t}{|x|}\, {}_2F_1\left(n-k+\frac{1}{2}, -k, \frac{3}{2}; \frac{t^2}{|x|^2}\right) Y_n^s\big(x/|x|\big), \quad h_k^n = \text{const} \neq 0;$$

(3) *if at least one of the constants* $\mu_{k,s}^{p+2k}$ *in* (9) *is different from zero, then for the corresponding function* $Q_p(x,t)$ *there exists an unit vector* $\alpha \in \mathrm{R}^3$, *such that*[1]

$$\lim_{t\to+0} Q_p(\alpha t, t) = \text{const} \neq 0.$$

This means that in this case the order of singularity of $u(x,t)$ *will be no smaller than* p.

According to this theorem, the order of singularity of $u(x,t)$ can be strictly fixed by the coefficients (4), i.e. by choosing the right-hand side $f(x,t)$ to be orthogonal to the appropriate functions $v_{k,s}^n(x,t)$.

Remark 1. We mention that for the particular case when $\gamma = 0$ there is a similar theorem in [17], but here we have the following two improvements:

(1) the right-hand side $f(x,t)$, instead of $C^1(\bar{\Omega})$, is allowed to be only continuous;

(2) we give the exact form of the functions $Q_{k,s}^n(x,t)$.

Acknowledgments

This work was partially supported by the Bulgarian Ministry of Education and Science under the National Program for Research "Young Scientists and Postdoctoral Students" (approved with RMS No. 577/ 17.08.2018) and by Sofia University Grant 80-10-109/2019.

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