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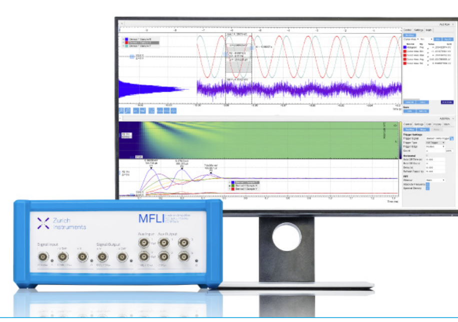

A Short-term Interest Rate Extended Merton's Model Influenced by A Risk Market Factor

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Abstract. In the context of the interest rate derivatives, a short-rate model is a mathematical model that can predict the random movement of the interest rates. In the present paper we introduce a short-term interest rate Extended Merton's model for which the movement of the interest rate is given by a stochastic differential equation. For this model we consider the zero-coupon bond's price which is determined by using the apparatus of the stochastic differential equations and the partial differential equations. We use the diffusion equation to calculate the bond's price for this model in the case of a risk market factor and without a risk market factor. Numerical experiments and graphics are presented to determine the zero-coupon bond's price. Results obtained by a Monte Carlo method for evaluation the zero-coupon bond's price in case with the risk market factor is a constant, demonstrate that this stochastic method could be applied in more complicated cases when the risk market factor depends on the time. The results also show that the zero coupon-bond's price depends on the risk market factor.

INTRODUCTION

The short-term models in finance are mathematical models which can describe the future evolution of the short interest rate $r(t)$. These models depend on many factors and this fact determines a source of uncertainty on the market. For example the one-factor model includes only one source of the market on which the interest rate depends on and analogously the two factor models includes two sources of uncertainty and etc. In the finance market two types of short-rate models can be found: (i) one-factor short-rate model and (ii) multi-factor short-rate model. These models have been of interest to many authors during the last century. This leads to different types of rate interest models which are derived. Examples for this are the works of [1], [2], [3], [4] and many others. In finance a bond is an instrument of indebtedness of the bond issuer to the holders. It is a debt security under which the issuer owes the holder a debt and the issuer is obligated to pay them a coupon interest or to repay the principal at the maturity date. Usually the maturity date is a fixed time (semiannual, annual or sometimes monthly). Every bond has specific characteristics such as issuers, priority, coupon rate (interest rate) and redemption. When there is no a coupon payment then the bond is called a zero-coupon bond.

In this article we consider a one-factor Extended Merton's model (a Ho-Lee model) for which the short interest rate is given by a stochastic differential equation. Such model's construction can be also found in the classical Vasicek model, see [5]. For some one-factor short-rate models can be found a finite number of free parameters. This fact gives the difficulty to specify them in a way to get closer to the market' prices. This problem is solved in the Extended Merton's model [6] when the parameters vary deterministically in time. Such model can be calibrated to the market data meaning that it can exactly return the bond's price. The assumption that we use in this model is that the bond market is non-arbitrary market where a risk-free portfolio with a quantity W exists. The main purpose for the Extended Merton's model is to determine the zero-coupon bond's price at time t with maturity time T. In this paper we consider two financial models. The first one is constructed without a risk factor on the market and the second one is influenced

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by the market risk factor. For the both models we use the connection between the stochastic differential equations and the partial differential equations, see [7]. The suggested numerical comparative analysis is made essentially on the partial differential equations and the specific assumptions for the initial and the boundary conditions. In section 2 we introduce the classical Extended Merton's model where we pay a great attention on the market price of risk and the value of the zero-coupon bond. Some comparative characteristics and a numerical analysis are made in Section 3. The concluding remarks are given in Section 4.

EXTENDED MERTON'S MODEL CONSTRUCTION

The construction of the short interest rate is the important step in creating a realistic and reliable financial model. It depends on the time t and its changing over the time is given by a stochastic differential equation. A simple assumption of the interest rate movements is that it follows a random walk with a drift - $a(t)$. In the terms of a stochastic process this statement can be written in the following way:

$$dr(t) = \underbrace{a(t)}_{drift}\, dt + \sigma dW_t, \tag{1}$$

with initial condition $r(0) = r_0 > 0$, where

- r is a deterministic interest rate function defined for $t \in [0, T]$; t is a time variable and T is the maturity date;
- $a(t)$ is a time-dependent drift term. The drift is the positive risk premium associated with the long-term horizons;
- σ is the instantaneous standard deviation of the interest rate r which characterizes the amplitude of the instantaneous incidental flow;
- W_t is the Standard Wiener process with normal distribution $N(0, 1)$.

The changes in r are a random walk with a non-zero drift. The common equation of (1) is given by the formula $r(t) = r_0 + b(t) + \sigma W_t$, where $b'(t) = a(t)$ and $b(0) = 0$. The assumption for the model is that the changes in the short-term rate of the interest are normally distributed. In the case when the drift function is a constant $(a(t) = a)$ the equation (1) can be rewritten in the following way: $r(t) = r_0 + at + \sigma W_t$, and the interest rate should be with a normal distribution $r_t \sim N(r_0 + at, \sigma^2 t)$. The simulation of the stochastic process when the drift is a constant is presented

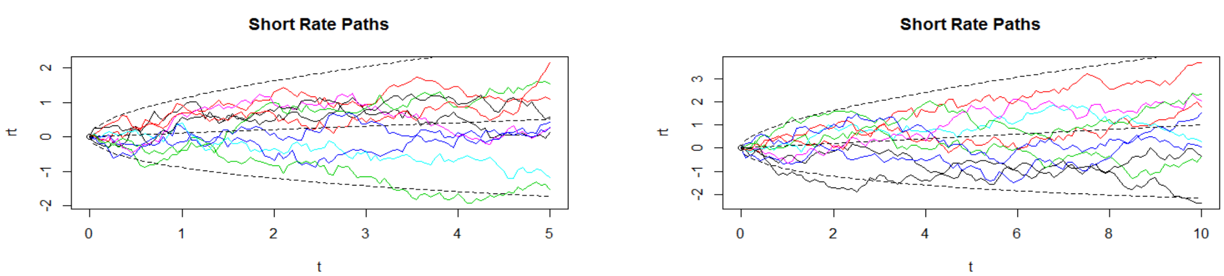

FIGURE 1. Simulation of the process path (1) with numerical values $r_0 = 0.01, \sigma = 0.50$ and with constant drift $a = 0.10$, but with different maturity time- A) $T = 5$ years (left) and B) $T = 10$ years (right)

given on Figures 1 and 2. Using the Euler discretization method in Figure 1 we give ten different simulated paths of the stochastic process. In the graphics there are three dashed lines which highlight the expected value of the process and the confidence bands which represent the two standard deviations. A problem with the process (1) arises when the drift is a constant. This means that some of the simulation's paths describing the interest rate $r(t)$ are below 0 per cents. This is not realistic to happen in real life. That's why many authors searched and created other one-short rate models. An example for such model is the Extended Merton's model, see [6]. Two variations of the of the process (1) used in the Extended Merton's model are given on Figure 2. The difference between these two graphics come from the different values of the standard deviation σ. The price that the investor is willing to pay for a given bond depends on the prevailing interest rate. On the other hand, if the interest rate gets higher then the bond's price will fall down. If the interest rate declines then the bond's price will increase. Usually the investors are willing to pay less for a bond

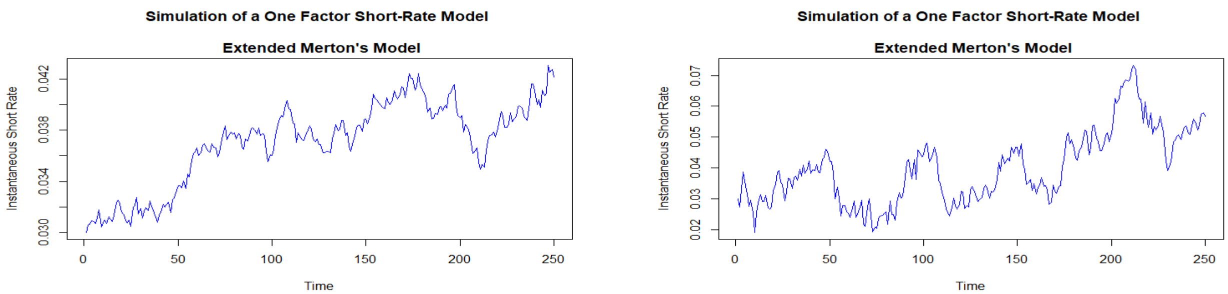

FIGURE 2. Simulation of the process path (1) with fixed values for σ- A) $\sigma = 0.009$ (left) and B) $\sigma = 0.050$ (right)

whose coupon's rate is lower than the prevailing interest rate. Such relationship can be found between the bond's price and the yield to maturity (YTM).

The yield to maturity $R(t,T)$ gives the internal bond's rate of return at time t. It is determined by several factors - a coupon rate, a price and a remaining year. The YTM is given by the following formula:

$$R(t,T) = -\frac{1}{T-t}\ln P(t,T), \tag{2}$$

where $T - t$ is the structure of the interest rates.

In the Extended Merton's model we denote by $P(r,t,T)$ the price value of a zero-coupon bond at time t and maturity T. This price depends on the interest rate movement and it can change over time. Using the Ito's lemma (see [1]) we obtain the following movement in the price of a zero coupon bond:

$$dP(r,t,T) = P_r dr + \frac{1}{2}P_{rr}(dr)^2 + P_t dt, \tag{3}$$

where $P_r = \dfrac{\partial P(r,t,T)}{\partial r}$, $P_t = \dfrac{\partial P(r,t,T)}{\partial t}$ and $P_{rr} = \dfrac{\partial^2 P(r,t,T)}{\partial r^2}$.

Taking into account that $dr(t) = a(t)dt + \sigma dW_t$ and substituting in (3) we obtain the following equation for the zero-coupon bond's price:

$$dP = P_r[a(t)dt + \sigma W_t] + \frac{1}{2}P_{rr}[a(t)dt + \sigma dW_t]^2 + P_t dt \tag{4}$$

which leads to

$$dP = \frac{1}{2}\sigma^2 P_{rr}dt + a(t)P_r dt + P_t dt + \sigma P_r dW_t, \tag{5}$$

The calculations above are obtained thanks to [8] where $(dW_t)^2 = dt, dW_t.dt = 0$ and $dt.dt = 0$.

In the further exposition of the Extended Merton's model is very important to find a formula for the zero-coupon bond's price $P(r,t,T)$. The realization of this task is achieved by the assumption of that the bond market is non-arbitrary. Let on this market an investor forms a portfolio with a quantity W which includes a unit quantity of a zero-coupon bond 1 with a maturity T_1 and a quantity w of a zero-coupon bond 2 with a maturity T_2. Then the portfolio's value is given by:

$$W = P_1 + w.P_2, \tag{6}$$

where P_1 is the zero-coupon bond's price of the first portfolio's bond, P_2 is the zero-coupon bond's price of the second portfolio's bond and w is a hedge ratio. From Ito's lemma the change in the value of this hedged portfolio is given by:

$$dW = dP_1 + w.dP_2 \tag{7}$$

We write the Ito's formula separately for the differentials dP_1 and dP_2 and obtain the following partial differential equations:

$$dP_1 = \frac{1}{2}\sigma^2 P_{1_{rr}}dt + a(t)P_{1_r}dt + P_{1_t}dt + \sigma P_{1_r}dW_t,$$

$$dP_2 = \frac{1}{2}\sigma^2 P_{2_{rr}} dt + a(t)P_{2_r} dt + P_{2_t} dt + \sigma P_{2_r} dW_t.$$

Substituting the obtained values of dP_1 and dP_2 in the stochastic differential equation (7) and obtain the following equation:

$$dW = \left[\frac{1}{2}\sigma^2 P_{1_{rr}} + a(t)P_{1_r} + P_{1_t}\right]dt + w\left[\frac{1}{2}\sigma^2 P_{2_{rr}} + a(t)P_{2_r} + P_{2_t}\right]dt + \sigma(P_{1_r} + wP_{2_r})dW_t \tag{8}$$

If the coefficient in front of the differential dW_t is zero i.e $P_1 r + w.P_2 r = 0$ or $w = -\frac{P_{1_r}}{P_{2_r}}$ then we say that on the market is forming a riskless portfolio. If we use this hedge ratio, then the instantaneous return on the portfolio should exactly equal the short-term rate of interest times the value of the portfolio:

$$dW = rWdt = (rP_1 + rwP_2)dt \tag{9}$$

Equating the right sides of the equations (8) and (9), substituting by $w = -\frac{P_{1_r}}{P_{2_r}}$ and dividing by the interest rate volatility $\sigma P_{1_r}, \sigma P_{2_r}$ we obtain the following equality:

$$\frac{P_{1_r}a(t) + \frac{1}{2}P_{1_{rr}}\sigma^2 + P_{1_t} - rP_1}{\sigma P_{1_r}} = \frac{P_{2_r}a(t) + \frac{1}{2}P_{2_{rr}}\sigma^2 + P_{2_t} - rP_2}{\sigma P_{2_r}} = -\lambda \tag{10}$$

For any two maturities T_1 and T_2 the no-arbitrage condition requires that this ratio be equal. The negative of this ratio λ is called a market price of risk. For normal risk aversion the market price of risk should be positive because in a risk-averse market the riskier (longer-maturity) bonds should have a higher expected return than the short rate r and because the rate sensitivity of all bonds P_r is negative. Since the choice of T_1 and T_2 is arbitrary, the market price of risk ratio must be constant for all maturities.

The risk market factor is

$$-\lambda = \frac{\frac{1}{2}\sigma^2 P_{rr} + P_r a(t) + P_t - rP}{\sigma P_r} \tag{11}$$

A Classical Extended Merton's Model With A Risk Market Factor

The Extended classical Merton's model with a risk market factor for estimating the zero-coupon bond's price is given by the following Cauchy problem:

$$\left[\begin{array}{c} \frac{1}{2}\sigma^2 P_{rr} + (a(t) + \lambda\sigma)P_r - rP + P_t = 0 \\ P(r, T, T) = 1 \end{array}\right. \tag{12}$$

with boundary conditions equal to one unit value.

If $\lambda = 0$ then the risk price market factor is eliminated and we obtain a model without a risk factor. The Cauchy problem for the Extended classical Merton's model without a risk factor is given by:

$$\left[\begin{array}{c} \frac{1}{2}\sigma^2 P_{rr} + a(t)P_r - rP + P_t = 0 \\ P(r, T, T) = 1. \end{array}\right. \tag{13}$$

NUMERICAL EXPERIMENTS

For the models given in (12) and (13) a comparative numerical characteristic analysis is made by using the CAS Wolfram Mathematica 11.3. The values of the parameters λ, σ and the maturity time T are fixed. The time value t and the interesting rate r are changing. We take the drift function $a(t)$ to be a constant. The two experiments are comparative for estimating the zero-coupon's bond price based on the Cauchy's problems (12) and (13). For the first experiment we take five years maturity time, *i.e.*, $T = 5$ years and for the second experiment we take ten years maturity time, *i.e.*, $T = 10$. For the both experiments are used one and the same numerical values for the parameters describing the interest drift $a(t) = 0.10$, the standard deviation $\sigma = 0.50$ and the market price of risk $\lambda = 0.015$. Only the values representing the interest rates r and the time t until the maturity year T are varied.

First Numerical Experiment

The first numerical experiment describes the change in the zero-coupon bond's price for five years maturity time. Using the CAS Wolfram Mathematica 11.3 and the data in Table 1 we calculate the bond's value for each year before the maturity time for the both Cauchy's problems (12) and (13). Observing the last column of price we can notice that when there is a risk market factor on the market, see (12) then the zero-coupon bond's price is decreasing. There is a decline in the price until the third year and after that it begins to rise before reaching the maturity year.

TABLE 1. Numerical values for all parameters for the two Extended Merton's models

	σ	a	λ	r	t(years)	T	Price
Extended Merton's model without the market risk factor	0.50	0.10	-	0.005	0	5	0.9660
	0.50	0.10	-	0.010	1	5	0.9530
	0.50	0.10	-	0.015	2	5	0.9500
	0.50	0.10	-	0.020	3	5	0.9569
	0.50	0.10	-	0.025	4	5	0.9735
	0.50	0.10	-	0.030	5	5	1.00
Extended Merton's model with the market risk factor	0.50	0.10	0.015	0.005	0	5	0.9643
	0.50	0.10	0.015	0.010	1	5	0.9490
	0.50	0.10	0.015	0.015	2	5	0.9452
	0.50	0.10	0.015	0.020	3	5	0.9524
	0.50	0.10	0.015	0.025	4	5	0.9707
	0.50	0.10	0.015	0.030	5	5	1.00

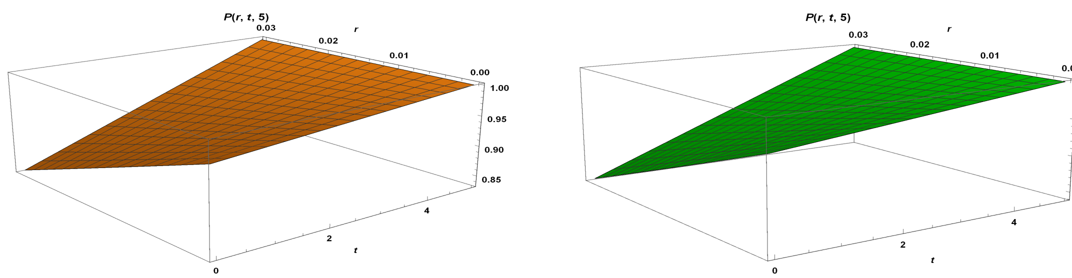

FIGURE 3. 3D graphics describing changes in the zero-coupon bond's price for the absence of the risk market factor A) (in orange surface, left) and when it presence B) (in green, right)

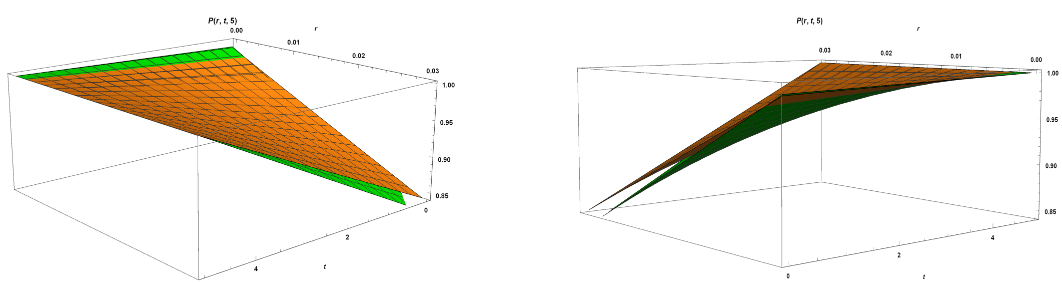

FIGURE 4. Comparison of the two graphs from Figure 3 at different angles

The 3D graphs are made by using these selected numerical values (see, Figure 3 and Figure 4). The zero–coupon bond's price change is given for the changing of the parameters r (interest rate) and t (time before the maturity time). This can be seen in Figure 4. For fixed values of the parameters a and σ and changing the values of r and t we obtain Figure 3. The left graph A) and the right graph B) from Figure 3 are obtained by using the data given in Table 1. It is obvious there is a difference between the prices for this five year period.

In Figure 4 we compare the two graphs A) and B). It is seen that the upper surface given by in orange is the zero-coupon bond's price when there is not a risk factor on the market. The lower surface given in green is the price of the zero-coupon bond when there is a risk factor on the market. From this 3D counterpart graph shown at a different

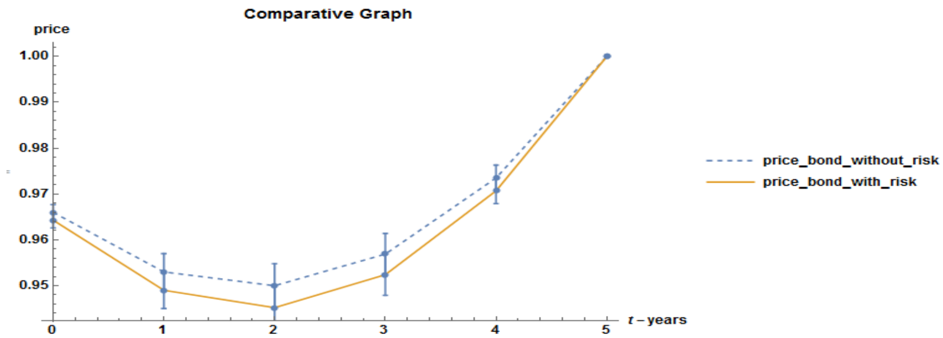

FIGURE 5. Comparison of the contours of the two surfaces

angle (see, Figure 4), it is seen that the surface which describes the bond's price when there is a risk market factor is below the surface which describes the bond's price when there is no a risk market factor. The price difference which is significent and visible on the 3D images, given in Figure 4, is also represented in 2D graph where the contour of the two surfaces from Figure 4 is given (see, Figure 5). The plots in Figure 5 and the data in Table 2 show the bond's price differences for each year to the maturity time $T = 5$.

TABLE 2. The price variation in the individual periods before the maturity date

year	0	1	2	3	4	5
bond price difference	0.017	0.004	0.0048	0.0045	0.0028	0

From the obtained results we can conclude that when there is a risk market factor the zero-coupon bond's price gets lower. This fact is a beneficial one for an investor who intend to buy a large amount of bonds when the prices are lower on the market.

Second Numerical Experiment

In the second numerical experiment the values of parameters σ, λ, a and the maturity time T are fixed and the values of r and t change for the both models, as they are given in Table 3. The periods in which the bond price is computed are the even years. Like the first numerical experiment the zero-coupon bond's price for both models decline until the fourth year and after that it rise until reaching the maturity time.

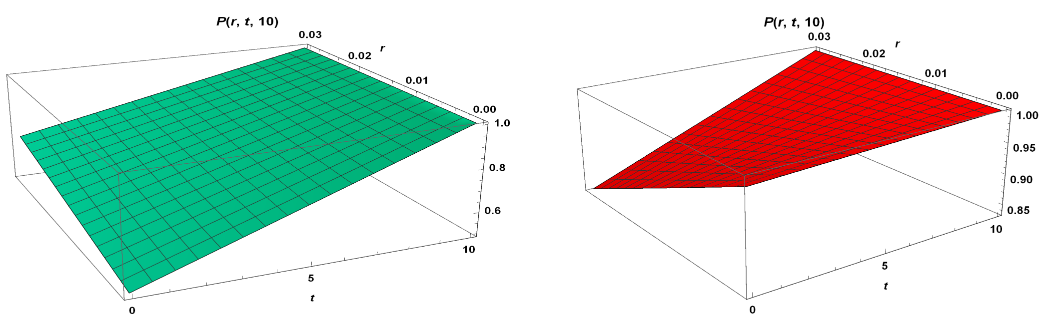

FIGURE 6. 3D graphs which describes the zero-coupon bond's price in the absence of the risk market factor A) (green surface, left) and with its presence - B) (in red, right)

The left graph A) and the right graph B) from Figure 6 are obtained by using the data given in Table 3. From the 3D counterpart graph presented from different angle (see, Figure 7), it is seen that when $r \in [0, 0.010]$ the upper surface given in green is the zero-coupon bond's price when there is not a risk factor on the market and the lower given in red is the surface representing the zero-coupon bond's price with a risk factor. When $r \in [0.010, 0.03]$ the upper surface is the red one and the lower is the green surface.

TABLE 3. Numerical values for all parameters for the two Extended Merton's models

	σ	a	λ	r	t(years)	T	Price
Extended Merton's model without the market risk factor	0.50	0.10	-	0.005	0	10	0.9455
	0.50	0.10	-	0.010	2	10	0.8944
	0.50	0.10	-	0.015	4	10	0.8745
	0.50	0.10	-	0.020	6	10	0.8856
	0.50	0.10	-	0.025	8	10	0.9276
	0.50	0.10	-	0.030	10	10	1.00
Extended Merton's model with the market risk factor	0.50	0.10	0.015	0.005	0	10	0.9249
	0.50	0.10	0.015	0.010	2	10	0.8955
	0.50	0.10	0.015	0.015	4	10	0.8885
	0.50	0.10	0.015	0.020	6	10	0.9037
	0.50	0.10	0.015	0.025	8	10	0.9409
	0.50	0.10	0.015	0.030	10	10	1.00

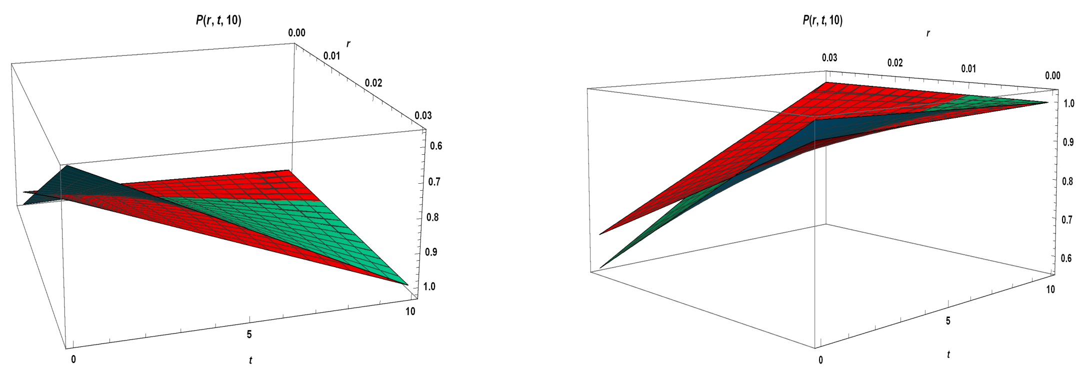

FIGURE 7. Comparison of the two graphs from Figure 6 at different angles

The price difference, that is visible from the 3D images, is given in Figure 7. It is also presented in 2D graph (see, Figure 8). The plot in Figure 8 and the results in Table 4 show the bond's prices differences for each year to the maturity time $T = 10$.

TABLE 4. The price variation in the individual periods before the maturity date

year	0	2	4	6	8	10
bond price difference	0.0206	0.0011	0.0140	0.0181	0.0133	0

From the obtained results we can concluded that when the interested rate is between 0 and 0.01 the zero-coupon bond's price is lower. This is a beneficial fact for a investor who wants to buy a large amount of bonds. Moreover the zero-coupon bond's price with and without risk market factor depend on the period of maturity and it is sensitive on the variables λ and σ.

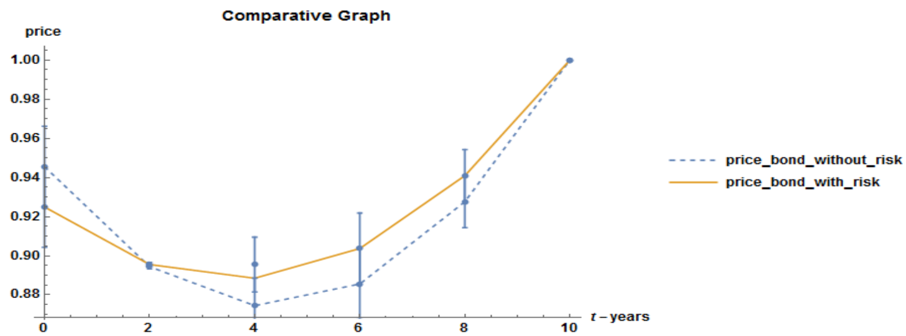

FIGURE 8. Comparison of the contours of the two surfaces

The real zero-coupon bond's price with a risk market factor vs a Monte Carlo method

The exact solution of the Cauchy problem (12) can be given by the formula:

$$P(r, \tau) = \exp\left(rF(\tau) + G(\tau)\right), \tag{14}$$

where $\tau = T - t$ is the period before reaching the maturity time. The functions $F(\tau)$ and $G(\tau)$ have the following forms:

$$F(\tau) = -\tau, \quad G(\tau) = \frac{\sigma^2\tau^3}{6} - \frac{\tau^2}{2}(a(t) + \lambda\sigma).$$

On the other hand, the zero-coupon bond's price at the maturity year T can be estimated by the integral:

$$P(r, \tau, T) = \exp\left(-\int_{T-\tau}^{T} r(u)du\right), \tag{15}$$

where r is a interest rate process from (1). The values of the integral (15) can be estimated by a Monte Carlo (MC) method. For this we define the random variable (r.v.) $\xi = \exp\left(-\int_{T-\tau}^{T} r(u)du\right)$ with a normal distribution depending of the interest rate process r. Thus, the mathematical expectation of the r.v. ξ, *i.e.*, (is equal to $P(r, \tau, T)$, *i.e.*, $E\xi = P(r, \tau, T)$. To compute approximately $E\xi$ we define a MC estimator in the following way:

$$\bar{\xi} = \frac{1}{N}\sum_{i=1}^{N} \xi^{(i)} \xrightarrow{P} P(r, \tau, T), \tag{16}$$

where $\xi^{(1)}, \ldots, \xi^{(N)}$ are independent values of ξ and $\xrightarrow{P}$ means stochastic convergence as $N \longrightarrow \infty$. Thus, the values $P(r, \tau, T)$ of the zero-coupon bond's price for the maturity year T is estimated by the $\bar{\xi}$.

The rate of convergence is evaluated by the "law of the three sigmas", [9]:

$$P\left(|\bar{\xi} - P(r, \tau, T)| < 3\frac{\sqrt{\mathrm{Var}(\xi)}}{\sqrt{N}}\right) \approx 0.997.$$

Here $\mathrm{Var}(\xi) = E\xi^2 - (E\xi)^2$ is the variance of the MC estimator. The peculiarity of any MC estimator is that the result is obtained with a statistical error [9]. As N increases, the statistical error decreases proportionally to $N^{-1/2}$, *i.e.*, we say the stochastic convergence rate is approximately $O(N^{-1/2})$.

The results for the zero-coupon bond's price obtained by the exact solution (14) and by the MC estimator (16) are shown in Table 5. The calculations are made at the following values of the parameters: $r_0 = 0.10$, $\sigma = 0.50$, $\lambda = 0.015$ and maturity $T = 1$ year. The function $a(t)$ is considered to be a constant function with a value $a = 0.10$.

We can see that with increasing the number of N simulations the accuracy of the MC solution is improved, *i.e.*, the MC approach is acceptable for estimating the zero-coupon bond's price.

TABLE 5. Estimation the price of zero-coupon bond through MC simulations

MC simulation trials	Exact Merton's price	MC price	MC standard error	Absolute error
$N = 100$	0.8939698	0.8984495	0.02396860	0.0044797
$N = 1000$	0.8939698	0.8943047	0.00829448	0.0003349
$N = 5000$	0.8939698	0.8936243	0.00372736	0.0003455
$N = 10000$	0.8939698	0.8932649	0.00263476	0.0007049

TABLE 6. Price of the zero-coupon bond evaluated by the exact solution and by the MC estimator for different values of the risk market factor λ and $r_0 = 0.10, a = 0.10, \sigma = 0.50$ and maturity time $T = 1$ year.

MC simulation trials	λ	Exact Merton's price	MC price	MC standard error	Absolute error
$N = 10000$	0.010	0.8950879	0.8955072	0.002604139	0.0004193
$N = 10000$	0.015	0.8939698	0.8950296	0.002638022	0.0010598
$N = 10000$	0.020	0.8928530	0.8940182	0.002652642	0.0011652
$N = 10000$	0.025	0.8917376	0.8946665	0.002616038	0.0029289

In the next Tables 6-8, we present how the parameters: (i) the risk market factor (λ) and (ii) standard deviation (σ) affects on the price of the zero-coupon bond.

In Table 6 the zero-coupon bond's price is estimated at different values of the market risk factor price (λ) but the values of the standard deviation and the drift are fixed, *i.e.*, ($\sigma = 0.50$) and $a = 0.10$.

In Table 7 the zero-coupon bond's price is estimated having a fixed risk market factor $\lambda = 0.020$ and the drift is a constant, *i.e.*, $a = 0.10$. Here, the standard deviation (σ) is changed. In both cases the maturity time is $T = 1$ year.

TABLE 7. Price of the zero-coupon bond evaluated by the exact solution and the Monte Carlo estimator at $r_0 = 0.10, a = 0.10, \lambda = 0.020$ and maturity time $T = 1$ year. The values of σ are changed.

MC simulation trials	σ	Exact Merton's price	MC price	MC standard error	Absolute error
$N = 10000$	0.500	0.8928530	0.8956000	0.002647110	0.002747
$N = 10000$	0.650	0.9175178	0.9236818	0.003611777	0.006164
$N = 10000$	0.800	0.9499620	0.9522273	0.004655753	0.0022653
$N = 10000$	0.950	0.9909578	0.9998949	0.005914831	0.0089371

More interested case is when the product $\lambda\sigma$ is a constant, (see Table 8).

The results in Tables 6, 7 and 8 show that the increase of the risk market factor affects to the price of the zero-coupon bonds. Moreover, when the risk market factor increases, the price of the zero-coupon bond decreases faster in the case when the product $\lambda\sigma$ is a constant (see, Table 8) in comparing with the case when the standard deviation (σ) is a constant (see, Table 7). Finally, we can conclude the MC method can be efficiently applied in estimating the price of the zero-coupon bonds in more complicated cases when the formula (14) is difficult to find after integration.

TABLE 8. : Price of the zero-coupon bond when $\lambda\sigma = 0.01, r_0 = 0.10, a = 0.10$ and maturity time $T = 1$ year

MC simulation trials	λ	σ	Exact Merton's price	MC price	MC standard error	Absolute error
$N = 10000$	0.0125	0.800	0.9528161	0.95802655	0.004690435	0.00521045
$N = 10000$	0.0200	0.500	0.8928530	0.89517830	0.002648170	0.0023253
$N = 10000$	0.0250	0.400	0.8795601	0.88266977	0.002068395	0.00310967

CONCLUDING REMARKS

In this paper we introduced the zero-coupon bond's price price in the short-term interest rate Extended Merton's model. We showed that the movement of the interest rate $r(t)$ is given by a stochastic differential equation and the zero-coupon bond's price is obtained by solving a Cauchy problem with specific boundary condition. We considered the influence of the market price of a risk λ over the bond's price and called this model-a model with a risk factor. Analogously, when the influence of the market price of a risk is eliminated we obtain a model without a risk factor. Taking different values of the risk factor and the other parameters we give numerical experiments which show how the existence of the risk factor affects on the zero-coupon bond's price for the Extended Merton's model. We also considered the more complicated case when the market price of the risk depends on time. In this case it is difficult to calculate exactly the zero-coupon bond's price. The numerical simulations using the Monte Carlo method that we gave have shown that the stochastic numerical methods can be successfully applied for estimating the zero-coupon bond's price. The numerical results showed that the market risk factor λ and the standard deviation σ influenced on the zero-coupon bond's price.

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