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Density distribution function of a self-gravitating isothermal compressible turbulent fluid in the context of molecular clouds ensembles – II. Contribution of the turbulent term and the potential of the outer shells

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ABSTRACT

In this paper, we continue to investigate the energy conservation equation obtained in our previous paper. We set ourselves three new goals: (i) to rewrite the main equations in terms of density profile in order to give more physical insight; (ii) to investigate the significance of two new terms in the energy conservation equation that originate from the gravity of the outer shells of cloud and the masses outer to the cloud, respectively; (iii) to investigate the main equation when the kinetic turbulent term scales according to Larson's law and it is independent, formally, of the accretion, in contrast to the previous work. The combination of supersonic turbulence and spherical symmetry raises a caveat that we comment on in our conclusions. We have obtained two solutions for the density profile, which scale with slopes of –2 and –3/2, respectively. The energy balance for the second solution is the same as in our previous paper: this is a free-fall. For the first solution, there are two cases: (1) if the turbulent term does not scale, then it could be important for the energy balance of the cloud; (2) if the turbulent term does scale, then it is not important for the energy balance of the cloud. The two new gravitational terms do not affect the existence of the two solutions, but the gravitation of the outer masses calibrates the energy balance for the first solution.

Key words: hydrodynamics – turbulence – methods: analytical – ISM: clouds – ISM: structure.

1 INTRODUCTION

It is very important to understand the origin of the probability distribution function (PDF) of the mass density of the interstellar medium (ISM) in order to obtain an explanation for the star formation process from first principles (Hennebelle & Falgarone 2012; Krumholz 2014; Klessen & Glover 2016). The PDF of a medium is determined by the physics of the interstellar gas, but there is a link between the PDF and the local star formation process. This is why the PDF of a given star-forming region is a tool for predicting of the initial stellar mass function, the star formation rate and the star formation efficiency in this region (Krumholz 2014; Offner et al. 2014). If we know how the physics of a medium determines the PDF, then we can make a robust link between that physics and the local star formation (Elmegreen 2018).

Our goal is to obtain the PDF from first principles. However, there are different physical regimes in the ISM. Recently, we investigated this task (Donkov & Stefanov 2018, hereafter Paper I) in the case of cold molecular gas with an isothermal equation of state (Ferriere 2001). We studied a gas ball with radial symmetry that accretes material from the outside. The gas entering the cloud (our gas ball) through its boundary (with supersonic velocity) goes through all the scales and finishes in the centre of the ball where a very small and dense core is located (inside which star formation can eventually occur); see Burkert (2017), who discusses this so-called simple bathtub model. We have also assumed that there is supersonic compressible turbulence and that it is locally homogeneous and isotropic in every shell of the gas ball. The whole system is in a steady state, which concerns both the macro-states (the motion of the fluid elements) and the micro-states (the thermal motion of the molecules). We neglected the magnetic fields and the back-reaction from newborn stars. We also neglected the dissipation, assuming that our scales (these are the radii of the gas ball) belong to the inertial range of the turbulent cascade. So the physics of our system consists of gravity, supersonic turbulence and accretion, and thermodynamics (isothermal state). Solving the set of compressible Euler equations in spherical coordinates, after they were ensemble averaged, we obtained two equations. The first equation, coming from the equation of motion of a fluid element, shows that the sum of the kinetic (accretion plus turbulent), the thermal and the

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gravitational energies of a fluid element per unit mass remains a constant when this fluid element moves through the scales (see equation 9 in Paper I). The second equation, coming from the continuity equation, gives a formula for the accretion velocity, expressed through the density and the scale (see equation 15 in Paper I). Giving explicit forms for the energies per unit mass, we solved the equations approximately up to the leading-order term in the series expansion, which assumes that the PDF is a power law, in two cases: when the core is negligible (the fluid element is too far from the core) and when the core is important (the fluid element is near to the core). In the first case, we obtain a solution with a slope of $-3/2$ (which is a counterpart to the density profile with a slope of -2); this presumes a dynamical equilibrium, in the outer shells of the ball, between accretion and gravity. In the second case, we have a free-fall solution with a slope of -2 (which is a counterpart to the density profile with a slope of $-3/2$) and a balance between the accretion and gravity of the core.

Our results correspond to those of previous studies. Larson (1969) and Penston (1969) have investigated a collapsing homogeneous gas ball without accretion. The main forces are self-gravity and isothermal gas pressure. They have solved the equations of motion numerically and have obtained a density profile with a slope of -2 in the outer layers. Also, Shu (1977) and Hunter (1977) have treated the problem analytically and Shu (1977) has obtained two density profiles: -2 for the outer layers (but, in static equilibrium, pressure supports against gravity) and $-3/2$ for the free-falling inner layers near to the singularity (Shu's solution describes the so-called inside-out collapse). Using numerical simulations Naranjo-Romero, Vázquez-Semadeni & Loughnane (2015) have investigated a collapsing core embedded in a larger medium (called the cloud) and accreting material from the latter. They have also obtained a density profile of -2 in the outer layers of the core during its collapse in the cloud. Recently, Li (2018) has obtained a density profile with a slope of -2 when gravity, accretion and turbulence interact. He claims that this slope is universal for scale-free gravitational collapse and that an isothermal state is not a necessary condition. There are confirmations also from observations. In systems such as molecular clumps forming star clusters, the radial density profile is very close to -2 (Mueller et al. 2002; Evans 2003; Wyrowski et al. 2012, 2016; Palau et al. 2014; Csengeri et al. 2017; Zhang & Li 2017).

In this paper, we change the model slightly and we set ourselves three main goals.

(i) The first goal is to rewrite the main equations in terms of the density profile. In Paper I, we wrote the equations in the form that asked for the PDF as a unknown quantity. However, this meant that we lost any physical insight, which is why we want to eliminate this disadvantage. In addition, we show more clearly the way in which our two solutions are obtained.

(ii) The second goal is to reconsider our assumptions in Paper I, concerning the gravitational potential caused by the outer shells with respect to the position of the fluid element. In Section 3.2 of Paper I, where we discuss the explicit form of the gravitational term, we argue that only the gravitational potential that originates from the inner shells of the ball with respect to the position of the fluid element should be included in the equation. Our argument is that the outer shells do not contribute to the gravitational force. This argument is right, because the first derivative (taken with a negative sign), with respect to the radius, of the full gravitational potential is the force. The differentiation eliminates the contribution of the outer shells, which means that the potential of the inner shells determines the motion of the fluid element through the scales. However, we use the equation of balance of the energies per unit mass and we have to work with the full gravitational potential. In this present paper, we consider the influence of the outer shells of the cloud and we conclude that the potential caused by them has a negligible effect on the two solutions that we obtain. Adding a constant term, we also account for the gravitation of the outer masses of the cloud (this is a slight change to the model), assuming that they obey radial symmetry. In contrast to the unimportant outer shells of the cloud, the gravitation of the outer masses is significant for the energy budget in equations (12) and (13).

(iii) The third goal is to investigate the main equation when the kinetic turbulent term is independent, formally, of the accretion and scales according to Larson's law. We suggested this equation in the discussion of Paper I (see equation 28 in Paper I). This matters because it is important to see if the two solutions do exist in the general case, not only in the particular case considered in Paper I.

This paper is organized as follows. Section 2 is dedicated to a derivation of the equation for the density profile. In Section 2.1, we give the explicit forms of the terms in the above-mentioned equation. We also account for the potential of the outer shells of the cloud with respect to the position of the fluid element in explicit form and we introduce the potential of the outer masses with respect to the cloud. Then, in Section 2.2, we obtain the equation for the density profile. We continue in Section 3 by analysing the possible solutions of the latter equation and we obtain solutions for two cases: far from the core (Section 3.1) and near to the core (Section 3.2). We discuss our results and give our conclusions in Section 4.

2 EQUATION FOR THE DENSITY PROFILE

In this section, we aim to rewrite the main equations in terms of the density profile $\varrho(\ell)$, an intrinsic characteristic of our cloud, and to obtain an equation that determines the latter quantity as an unknown function. In Paper I, we derived equation (20), which determines the quantity $Q(s)$, where $s = \ln(\rho/\rho_c)$ is the log-density and ρ_c is the mass density at the outer boundary of the cloud. $Q(s)$ is simply the dimensionless cloud radius. In the present paper, we denote the latter as ℓ and it takes values in the range $\ell_0 \le \ell \le 1$, where the lower limit ℓ_0 is the size of the small and dense core in the centre of our cloud, and the upper limit 1 is a counterpart to the outer boundary of the cloud. For simplicity, we use the dimensionless density profile $\varrho(\ell) = \rho(\ell)/\rho_c$, which is a function of the dimensionless radius ℓ, and is obviously the inverse function of $Q[s(\varrho)] \equiv \ell(\varrho)$. As we are also interested in obtaining of an expression for the PDF, equation (9) gives the link between the latter and $\ell(\varrho)$.

Starting from the equations of the medium (see Section 3.1 in Paper I) under the assumption of a steady state, we obtain the equation for conservation of the total energy of a fluid element, per unit mass, during its motion through the cloud scales. This means that the sum of the averaged kinetic, thermal and gravitational energies, per unit mass, is a constant with respect to ℓ, or

$$\frac{\mathrm{d}}{\mathrm{d}\ell}(\langle v^2/2\rangle + \langle s\rangle + \langle\phi\rangle) = 0. \quad (1)$$

2.1 Explicit form of the terms in equation (1)

In this subsection, we derive the explicit form of the terms in equation (1), taking into account the model presented in Paper I (Section 2) and also briefly mentioned in Section 1. We start with the kinetic energy term,

$$\langle v^2\rangle = \langle v_\mathrm{t}^2\rangle + \langle v_\mathrm{a}^2\rangle, \tag{2}$$

where $\langle v_\mathrm{t}^2\rangle$ is the turbulent kinetic energy per unit mass and $\langle v_\mathrm{a}^2\rangle$ is the accretion kinetic energy per unit mass. The proof that equation (2) is satisfied is given in Paper I (Section 3.2).

Our spherically symmetric cloud is ensemble averaged, which is why we choose to apply a standard scaling relation for $\langle v_\mathrm{t}^2\rangle$:

$$\langle v_\mathrm{t}^2\rangle = \frac{u_0^2}{c_\mathrm{s}^2}\left(\frac{l_\mathrm{c}}{pc}\right)^{2\beta}\ell^{2\beta} = T_0\ell^{2\beta}. \tag{3}$$

Here, u_0 and $0 \leq \beta \leq 1$ are, respectively, the normalizing factor and the scaling exponent of the turbulent velocity fluctuations in the standard law $u = u_0 L^\beta$ (Larson 1981; Padoan et al. 2006; Kritsuk et al. 2007; Federrath et al. 2010). $T_0 \equiv (u_0^2/c_\mathrm{s}^2)(l_\mathrm{c}/\mathrm{pc})^{2\beta}$ is the ratio of the turbulent kinetic energy per unit mass of the fluid element at the boundary of the cloud to the thermal energy per unit mass. This form of the turbulent kinetic energy per unit mass is different from the expression used in Paper I (equation 12 there), which determines the dependence of the turbulence from the accretion. In contrast, in this paper, we presuppose that the turbulence is formally independent of the accretion. The explicit form of the accretion kinetic term has been obtained from the continuity equation in Section 3.3 of Paper I, written as

$$\langle v_\mathrm{a}^2\rangle = A_0\varrho(\ell)^{-2}\ell^{-4}. \tag{4}$$

From the considerations in Section 3.3 of Paper I, we find that $\ell^4\varrho(\ell)^2\langle v_\mathrm{a}^2\rangle = \mathrm{const}(\ell) = A_0$. Taking into account that the quantities are dimensionless and are normalized to the cloud size (for the scale), to the cloud edge density (for the density) and to the sound velocity (for the accretion velocity), we can obtain A_0 if we take $\ell = 1$ (i.e. at the cloud boundary). Then, $\varrho = 1$ and $\langle v_\mathrm{a}^2\rangle = u_{\mathrm{a,c}}^2/c_\mathrm{s}^2$, which is the ratio of the accretion kinetic energy term at the boundary of the cloud and the thermal kinetic energy per unit mass.

The thermal potential is $\langle s\rangle = s$, because in our model the logarithmic density is averaged by assumption. The same is valid for the density: $\varrho = \langle\varrho\rangle$.

The averaged gravitational potential is given by

$$\langle\phi\rangle = -\frac{G}{l_\mathrm{c}c_\mathrm{s}^2}\frac{M(\ell)}{\ell} - \frac{G}{l_\mathrm{c}c_\mathrm{s}^2}\frac{M_0}{\ell} + \langle\phi^\mathrm{ext}\rangle, \tag{5}$$

where ℓ is the radius at which the fluid element resides at the given moment. Also, $M(\ell) = 3M_\mathrm{c}^*\int_{\ell_0}^{\ell}\ell'^2\varrho(\ell')\,\mathrm{d}\ell'$ is the mass of the inner shells corresponding to ℓ, where $M_\mathrm{c}^* = (4/3)\pi l_\mathrm{c}^3\rho_\mathrm{c}$ is a normalizing coefficient whose physical interpretation is given in Section 3.2 of Paper I. Hence, the first term in equation (5) is the gravitational potential caused by the shells that are inner with respect to the fluid element. M_0 is the mass of the dense core at the centre of the cloud and the second term in equation (5) is its gravitational potential at scale ℓ. The last term in equation (5) can be written as

$$\langle\phi^\mathrm{ext}\rangle = -\left(\frac{3GM_\mathrm{c}^*}{l_\mathrm{c}c_\mathrm{s}^2}\right)\int_\ell^1\ell'\varrho(\ell')\,\mathrm{d}\ell' + \frac{\psi^\mathrm{ext}}{c_\mathrm{s}^2},$$

where the first addend is the gravitational potential caused by the outer shells corresponding to ℓ and the second addend is the potential caused by the masses outside the cloud. For the second added, we assume that all the masses outside the cloud give rise to potential ψ^ext in the volume of our cloud. Also, ψ^ext does not depend on the position of the fluid element during its motion through the scales. Of course, this is a simplification. Our assumptions are valid as long as the material outside the cloud obeys a radial symmetry. In reality, this is not the case. However, this is in agreement with the spirit of our model. Finally, $\langle\phi\rangle$ can be expressed by the density profile $\varrho(\ell)$:

$$\langle\phi\rangle = -\frac{3G}{c_\mathrm{s}^2}\frac{M_\mathrm{c}^*}{l_\mathrm{c}}\frac{\int_{\ell_0}^{\ell}\ell'^2\varrho(\ell')\,\mathrm{d}\ell'}{\ell} - \frac{3G}{c_\mathrm{s}^2}\frac{M_\mathrm{c}^*}{l_\mathrm{c}}\int_\ell^1\ell'\varrho(\ell')\,\mathrm{d}\ell'$$
$$-\frac{G}{c_\mathrm{s}^2}\frac{M_0}{l_\mathrm{c}}\frac{1}{\ell} + \frac{\psi^\mathrm{ext}}{c_\mathrm{s}^2}. \tag{6}$$

2.2 Derivation of the equation for $\rho(\ell)$

With this preparation, equation (1) can be written as

$$\frac{\mathrm{d}}{\mathrm{d}\ell}\Bigg\{A_0\varrho(\ell)^{-2}\ell^{-4} + T_0\ell^{2\beta} + 2\ln[\varrho(\ell)] - 3G_0\frac{\int_{\ell_0}^{\ell}\ell'^2\varrho(\ell')\,\mathrm{d}\ell'}{\ell}$$
$$-3G_0\int_\ell^1\ell'\varrho(\ell')\,\mathrm{d}\ell' - \frac{G_1}{\ell}\Bigg\} = 0, \tag{7}$$

where $G_0 = (2G/c_\mathrm{s}^2)(M_\mathrm{c}^*/l_\mathrm{c})$ and $G_1 = (2G/c_\mathrm{s}^2)(M_0/l_\mathrm{c})$ are dimensionless coefficients whose physical meanings are clarified in Section 3.4 of Paper I.

Let us denote the expression within the curly brackets in equation (7) by E_0. This is the total energy per unit mass of the fluid element. It is clear that $\psi^\mathrm{ext}/c_\mathrm{s}^2$ contributes to E_0 and calibrates the total energy. If we compare the total energy of the fluid element in the present work (E_0^II) and in Paper I (E_0^I), we have the relation: $E_0^\mathrm{II} = E_0^\mathrm{I} - \psi^\mathrm{ext}/c_\mathrm{s}^2$ (for the role of the gravitational potential, caused by the outer masses, in the cloud's energy balance, see Ballesteros-Paredes et al. 2018).

Then we have

$$A_0\varrho(\ell)^{-2}\ell^{-4} + T_0\ell^{2\beta} + 2\ln[\varrho(\ell)] - 3G_0\frac{\int_{\ell_0}^{\ell}\ell'^2\varrho(\ell')\,\mathrm{d}\ell'}{\ell}$$
$$-3G_0\int_\ell^1\ell'\varrho(\ell')\,\mathrm{d}\ell' - \frac{G_1}{\ell} = E_0. \tag{8}$$

which is a non-linear integral equation for the function $\varrho(\ell)$. A solution for $\varrho(\ell)$ would allow us to find the PDF of mass density [if we know the inverse function $\ell(\varrho)$]:

$$\mathrm{PDF}(\varrho) = -3\ell(\varrho)^2\frac{\mathrm{d}\ell(\varrho)}{\mathrm{d}\ln(\varrho)}. \tag{9}$$

3 STUDY OF THE EQUATION FOR THE DENSITY PROFILE

We search for a solution of the form $\varrho(\ell) = \ell^{-p}$, which corresponds to a power-law PDF, $p(s) \propto \exp(qs)$ with $q = -3/p$ (see equation 9). The motivation for this ansatz is the same as in Paper I. A solution of this type is the simplest possible and, besides, the star formation process occurs in the power-law tails of the PDFs. A more general approach would be to ask for a solution in the form of a series of increasing exponents – with a small parameter $(1 - \ell)$ – but this is not our goal in the current work.

Making this substitution in equation (8) after some algebra, we obtain

$$A_0\ell^{2p-4} + T_0\ell^{2\beta} + 2(-p)\ln\ell - 3G_0\frac{\ell^{2-p}}{3-p}\left[1-\left(\frac{\ell_0}{\ell}\right)^{3-p}\right]$$
$$-3G_0\frac{1-\ell^{2-p}}{2-p} - G_1\ell^{-1} = E_0. \tag{10}$$

The expression on the left-hand side of the equation depends on ℓ, which means that the equation can be satisfied only approximately. Different assumptions and approximations yield different solutions for the parameter p, which is the slope of the density profile.

We study the following two cases. In the first case, the core can be neglected (i.e. A_0, T_0, $G_0 \gg G_1$), and we search for a solution when the fluid element is far from the core ($1 \gtrsim \ell \gg \ell_0$). In the second case, the core has a significant contribution or A_0, T_0, $G_0 \sim G_1$, and the fluid element is near to the core ($\ell \sim \ell_0$).

3.1 Solution far from the core

When the core is neglected, equation (10) takes the form:

$$A_0\ell^{2p-4} + T_0\ell^{2\beta} + 2(-p)\ln\ell - 3G_0\frac{\ell^{2-p}}{3-p}\left[1-\left(\frac{\ell_0}{\ell}\right)^{3-p}\right]$$
$$-3G_0\frac{1-\ell^{2-p}}{2-p} = E_0\ell^0. \tag{11}$$

Before we continue, we should comment on the thermal term and the second addend in the parentheses of the gravitational term resulting from the inner shells. Regarding the thermal term, the turbulent and accretion velocities are supersonic by assumption, which means that the pressure term in the equation of motion (see equation 4 of Paper I) is negligible compared with the kinetic terms. The thermal term in our equation comes from the pressure term, so it can be neglected. It might be important only if the obtained solution for p leads to exponents for accretion and turbulent terms, which are positive. The second addend in the parentheses of the gravitational term is also negligible, because according to observations and simulations, typically $1 \le p \le 2$, and far from the core $\ell_0/\ell \ll 1$, and then $(\ell_0/\ell)^{3-p} \ll 1$.

The exponents of the main terms, obtained with our ansatz, are $2p - 4$, 2β, $2 - p$ and 0, respectively. An approximate solution of equation (11) can be obtained in the following way. With the approximations commented on in the previous paragraph, equation (11) contains only terms that have power-law dependence on ℓ. If the exponents of all the terms are equal, then the powers of ℓ factor out and only constants remain. The questions that arise then are which terms have equal exponents and whether they dominate over the rest of the terms. Because $0 < \ell \le 1$, the lower powers dominate over the higher powers. A non-trivial solution of equation (11) can be found only if the number of leading terms is at least two. If just one term dominates, it remains unbalanced and the only solution is a trivial solution. In order to find a solution for p, we do the following. We choose a pair of terms and hypothesize that their exponents are equal and that the remaining terms are inferior or, at most, equal to them. Equating the two exponents, we obtain a simple equation for p. We solve this, evaluating the exponents of all the terms with the obtained value, and we check if our hypothesis is confirmed. The same method is applied to all possible pairs of terms (Zhivkov 1999; Riley, Hobson & Bence 2006).

Let us, for example, assume that the turbulent term and the accretion term have equal exponents and that they dominate over the other terms in equation (11). This assumption results in the following simple equation for p: $2\beta = 2p - 4$. Its root, $\beta + 2$, is given in the first row and second column of Table 1.

Table 1. Comparison of the exponents of the main terms in the equation (11) and the corresponding roots for p.

Exponents	2β	$2p-4$	$2-p$	0
2β	–	$\beta+2$	$2(1-\beta)$	–
$2p-4$	$\beta+2$	–	2	2
$2-p$	$2(1-\beta)$	2	–	2
0	–	2	2	–

Table 2. Values of the exponents of the main terms in equation (11), according to every root obtained in Table 1.

Roots	Exponents			
	2β	$2p-4$	$2-p$	0
2	2β	0	0	0
$\beta+2$	2β	2β	$-\beta$	0
$2(1-\beta)$	2β	-4β	2β	0

As a next step, we use the obtained root for p and evaluate the values of the exponents of all the terms in equation (11). The results are given in the second line of Table 2. With this root just one of the terms, the one whose exponent is $-\beta$ dominates over the others. As it appears, the assumption is not justified. Besides, the dominant term remains unbalanced. Hence, this root does not allow us to find a non-trivial solution and we have to check the other possible pairs of exponents.

The roots for p that we obtain with the procedure described above are 2, $\beta + 2$ and $2(1 - \beta)$. This is made clear in Table 1. In Table 2, the values of the exponents for every root are given. To make a conclusion about the existence of a solution of equation (11), we have to remember the range of β: $0 \le \beta \le 1$. If $\beta = 0$, the three cases are equivalent and there is only one solution, $p = 2$ ($q = -3/2$), and the energy balance is

$$A_0 + T_0 - 3G_0 \approx E_0.$$

If $\beta > 0$, then there exists a solution only if $p = 2$ ($q = -3/2$), and the energy balance is

$$A_0 - 3G_0 \approx E_0.$$

These approximate equalities express the balance of the energy components. They are valid only if and as long as the remaining terms can be neglected.

The last term on the left-hand side in equation (11) deserves special attention. This term is the gravitational potential caused by the outer shells of the cloud with respect to the fluid element. If $p = 2$, then the denominator of this term equals zero but the numerator also vanishes. Applying the L'Hospital's rule, we can obtain a non-infinite limit:

$$\frac{1-\ell^{2-p}}{2-p} \longrightarrow -\ln(\ell).$$

Then the entire gravitational term is written as $-3G_0[1 - \ln(\ell)] \simeq -3G_0$, because if the fluid element is far from the core, then $1 \gtrsim \ell \gg \ell_0$ and $\ln(\ell) \sim 0$.

Moreover, when $p = 2$, if we take into account the considerations in Section 4.1 of Paper I, concerning the average density of the hole cloud, then $3G_0 = \langle G\rangle$. This is the averaged gravitational energy per unit mass of the fluid element for the entire cloud. The terms for the accretion kinetic and turbulent kinetic energies have a similar property, $A_0 = \langle A\rangle$ and $T_0 = \langle T\rangle$, because accretion does not scale if $p = 2$ and the turbulent term is important only if it does not scale.

Finally, when the core is neglected, there exists only one solution, $\varrho(\ell) = \ell^{-2}$, where the PDF is PDF($s$) ≈ (3/2)exp (− 3$s$/2), but there are two possibilities for energy balance: (i) if the turbulence does not scale ($\beta = 0$), then

$$\langle A\rangle + \langle T\rangle - \langle G\rangle \approx E_0; \tag{12}$$

(ii) if the turbulence scales ($\beta > 0$), then

$$\langle A\rangle - \langle G\rangle \approx E_0. \tag{13}$$

3.2 Solution near to the core

When the core is not negligible (the fluid element is near to the core), equation (10) can be written as

$$A_0\ell^{2p-4} + T_0\ell^{2\beta} + 2(-p)\ln\ell - 3G_0\frac{\ell^{2-p}}{3-p}\left[1-\left(\frac{\ell_0}{\ell}\right)^{3-p}\right]$$
$$-3G_0\frac{1-\ell^{2-p}}{2-p} - G_1\ell^{-1} = E_0\ell^0. \tag{14}$$

According to the same arguments as in the previous section, we can neglect the thermal term. The gravitational term, accounting for the potential of the inner shells, is also ∼0, because near to the core $\ell \sim \ell_0$, and the expression in the parentheses vanishes. Hence, the exponents of the main terms are $2p - 4$, 2β, $2 - p$, -1 and 0. We can apply the same method for obtaining the solutions for p as in the previous section, but there is a simpler physical consideration. If the core is important, then the leading-order exponent must be -1. In this case, there are two possibilities. The first possibility is that the gravitation of the core is balanced by the gravitational term resulting from the outer shells of the cloud and $2 - p = -1$ leads to $p = 3$. However, the energy balance fails because both terms are negative. The second, and only, possibility is that the gravitation of the core is balanced by the accretion term, and it requires $p = 3/2$. Therefore, the only solution in this case is $\varrho(\ell) = \ell^{-3/2}$, where the PDF is PDF($s$) ≈ 2exp (− 2$s$), and the energy balance is

$$A_0 - G_1 \approx 0. \tag{15}$$

This is the well-known free-fall solution from Paper I (see Section 4.2 there).

4 DISCUSSION AND CONCLUSIONS

In the previous sections, we have given the main equations in the terms of density profile and we have obtained expressions with more physical insight, which gives the model more clarity. In addition, using Tables 1 and 2, we have illustrated the method of obtaining the solutions of equation (10), which is also clarified in Section 3.1. With this, we consider our first goal to have been accomplished.

Our second goal was to investigate equation (1) when we account for the gravitational potential caused by the outer shells with respect to the position of the fluid element, in contrast to our previous paper. According to the considerations in Sections 3.1 and 3.2, we can conclude that the gravitation of the outer shells of the cloud is not important for the two cases that we have studied. So our two solutions are not influenced by the new term on the left-hand side of equation (8). This is not the case with the constant gravitational term $\psi^{\rm ext}/c_{\rm s}^2$ caused by the outer masses of the cloud. It contributes to the total energy of the fluid element E_0 and we believe that it is of key importance for the right energy balance in equations (12) and (13) (Ballesteros-Paredes et al. 2018).

Regarding our third goal (i.e. to present the kinetic turbulent energy in a more general form), we should note that both solutions we have obtained in this work are the same as the solutions in Paper I. However, there are differences in the equations of energy balance. For the second solution $\varrho = \ell^{-3/2}$, the energy balance has preserved its form as in Paper I. This is a free-fall solution and the energy balance per unit mass for the fluid element is $A_0 - G_1 \approx 0$.

For the first solution $\varrho = \ell^{-2}$, there are two regimes. If $\beta = 0$, then the turbulent kinetic energy could be important: $\langle A \rangle + \langle T \rangle - \langle G \rangle \approx E_0$ (this holds only if $T_0 \sim A_0$). The lack of scaling reminds us of coherent cores (Goodman et al. 1998), whose scales are of order $\ell \sim 0.1$ pc. This phenomenon ($\beta = 0$) is observed, also, at larger scales in Rosette molecular cloud (see Veltchev et al. (2018), Section 5.4). If $\beta > 0$, then the turbulent kinetic energy is not important: $\langle A \rangle - \langle G \rangle \approx E_0$. The latter case strongly supports the idea for hierarchical and chaotic gravitational collapse at all of the cloud scales (Ballesteros-Paredes et al. 2011a,b, 2018; Ibánez-Mejia et al. 2016; Elmegreen 2018).

Briefly, we note that our model is an attempt to give an abstract statistical description of classes of molecular clouds that have the same PDF, size ℓ_c, core size ℓ_0, edge density ρ_c, core density ρ_0 and temperature T. The ball (which is the average representative of the class), obeying radial symmetry, is an idealized object, but this is the simplest model that we could construct. It is clear that we have lost the specific morphology and physics of every cloud from the class, but we believe that we can obtain the main properties of the class members.

The compatibility of supersonic turbulence and spherical symmetry is the major caveat of our model. Large-scale supersonic turbulence gives rise to shocks that might result in substantial departures from the spherical symmetry. Such departures will induce a non-symmetric gravitational potential and hence will have influence over gravitational terms in our equation. It is difficult to say precisely how these departures will affect the energy balance equations, but we consider our approach to be a first step in this task.

Another significant problem is that the second solution, near to the core, is a free-fall solution with a profile $p = 3/2$ ($q = -2$). Some modern simulations (Kritsuk, Norman & Wagner 2011) and observations (Schneider et al. 2015), where two power-law tails occur, report a different profile for the second tail: $p \sim 3$ ($q \sim -1$). This value can be explained, according to the authors of the cited papers, by a decrease of the mass flow rate (fall under the action of gravity) from the larger to the smaller scales of the cloud. Among the possible reasons for such a decrease are non-zero angular momentum of the small dense structures (of the dense core in the centre of the cloud, in our case; Kritsuk et al. 2011; Schneider et al. 2015), large opacity, pressure increase as a consequence of temperature increase (i.e. the system leaves the isothermal regime), the presence of magnetic fields and the back-reaction on the cloud from the newborn stars (Schneider et al. 2015), etc. All this physics is neglected in our model, which is why the inconsistency between the second slopes seems normal. This is a hint of the possible directions for an elaboration of the model. One way is to suggest that near to the core the system leaves the isothermal regime and we have a polytropic equation of state: $p_{\rm th} \propto \rho^{\Gamma}$ and $\Gamma \neq 1$. This change in thermodynamics leads to the following equation near to the core:

$$A_0 \ell^{2p-4} + T_0 \ell^{2\beta} + \frac{\Gamma}{\Gamma - 1} \ell^{p(1-\Gamma)} - 3G_0 \frac{\ell^{2-p}}{3-p} \left[1 - \left(\frac{\ell_0}{\ell} \right)^{3-p} \right]$$

$$-3G_0 \frac{1 - \ell^{2-p}}{2-p} - G_1 \ell^{-1} = E_0 \ell^0. \tag{16}$$

We leave the study of this equation for future work.

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