Computation of second area moments of cross-section using optical images

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Abstract – The paper proposes a method for automated calculation of second area moments of random complex cross-sections using classical photography. The method uses representation of the image pixel as geometric figure with its own second area moments. The results from second area moments calculation using the proposed method of real cross-section with known values of the second area moments are presented.

Keywords – second area moment; cross-section; computational; optical image.

I. INTRODUCTION

Some building elements are in hard-to-reach places, which makes it difficult to determine the second area moment, whose exact determination is essential for the required constructive calculations. The capabilities provided by the optical digital image and the data obtained should be developed in the engineering-practical direction. Capturing the surface of an object under study can give us a lot of information. The determination of the area of the surfaces is already performed with high accuracy and sufficient high resolution of the image [1]. The use of optical digital images in the analysis and classification of materials is well studied [2-8]. A vast majority of construction engineers use consumer cameras with a resolution of over 15 megapixels, which means that if the surface surveyed occupies much of the image, the surface image will be over 10 million points. A significant advantage of photography is the ability to capture the studied object with high optical zoom (in consumer cameras - more than 40 times), which is a very good opportunity for using such an instrument for measurement and subsequent calculation of second area moment.

II. SECOND AREA MOMENTS OF CROSS-SECTION

Second area moment, or area moment of inertia, is a defined term and represents a geometric characteristic of a plane figure (i.e. cross-section) [10]. The axial second area moments are a type of geometric characteristic of a given section [10]. If we have a random complex cross-section with surface A (Fig. 1), composed of elementary figures (dA) or simpler figures (Ai), the position of the centre of gravity of the section in Cartesian coordinates can be determined by Eq. 1 and Eq. 2 [10].

\[ Z_c = \frac{\iint_A zdA}{\int dA} = \frac{\sum_{i=0}^{n} z_i A_i}{\sum_{i=0}^{n} A_i}, \]  \( (1) \)

\[ Y_c = \frac{\iint_A ydA}{\int dA} = \frac{\sum_{i=0}^{n} y_i A_i}{\sum_{i=0}^{n} A_i}, \]  \( (2) \)

where \( Z_c \) and \( Y_c \) are the Cartesian coordinates of the centre of gravity, \( A_i \) are the areas of the composing figures, \( n \) – the number of \( A_i \) figures.

Now, if we denote by \( z \) and \( y \) the coordinates of any square area \( dA \) (Fig. 1), the second area moment along the OY axis and the second area moment along the OZ axis are [9, 10] respectively:

\[ I_z = \iint_A y^2 dA; I_y = \iint_A z^2 dA \]  \( (3) \)

\[ \text{Fig. 1. Basic concept for determination of the centre of gravity and second area moments} \]

\[ \text{Fig. 2. Basic concept for translation and rotation of the coordinate system} \]

Let us consider how in the case of translation and rotation of the coordinate system, the second area moments would change (Fig. 2) [9, 10]:

\[ \text{[Translation and Rotation of Coordinates]} \]
The equation for second area moments in the case of a square figure (fig. 3) [9, 10] is:

\[ I_x = I_y + A \cdot Z_c^2 ; \quad I_k = I_z + A \cdot Y_c^2 \]  \hspace{1cm} (4)

The equation for second area moments in the case of a square figure (fig. 3) [9, 10] is:

\[ I_y = \iint_A z^2 dA = \iint_A z^2 dydz = \int_0^a z^2 dz \int_0^a dy = \frac{a^4}{12}, \]  \hspace{1cm} (5)

\[ I_z = \iint_A y^2 dA = \iint_A y^2 dydz = \int_0^a y^2 dy \int_0^a dz = \frac{a^4}{12}, \]  \hspace{1cm} (6)

where \( a \) is the side of the square form figure.

It is known that the second area moment of the cross-section consists of the second area moments of all the geometrical figures that fall into it, [9, 10]:

\[ I_y = I_{y1} \pm I_{y2} \pm I_{y3} \pm \cdots \pm I_{yn}, \]  \hspace{1cm} (7)

\[ I_z = I_{z1} \pm I_{z2} \pm I_{z3} \pm \cdots \pm I_{zn}, \]  \hspace{1cm} (8)

Likewise the “+” sign is used when the composing figure has an area, and the “-” sign - when the figure has no area, i.e. hole in the cross section. The same rule applies when determining the centre of gravity of the cross-section.

It is known that by its very nature, the pixels of the image represent a square form geometric figure and the cross-section image is composed of many pixels. Consequently, let us assume that the second area moment of the cross section will be a sum of the second area moments of all the pixels (squares) that fall into it (Fig. 4). The second area moment of an arbitrary pixel can be calculated using the equations for the second area moments calculation of a square form figure (Eq. 5 and Eq. 6).

Consequently, the second area moments for an arbitrary pixel representing part of the cross-section are:

\[ I_y = I_{y \text{pixel}} + A_{\text{pixel}} \cdot Z_j^2 = \frac{a_{\text{pixel}}^4}{12} + a_{\text{pixel}}^2 \cdot Z_j^2, \]  \hspace{1cm} (9)

\[ I_z = I_{z \text{pixel}} + A_{\text{pixel}} \cdot Y_j^2 = \frac{a_{\text{pixel}}^4}{12} + a_{\text{pixel}}^2 \cdot Y_j^2, \]  \hspace{1cm} (10)

\[ I_{y \text{pixel}} = I_{z \text{pixel}} = \frac{a_{\text{pixel}}^4}{12}, \]  \hspace{1cm} (11)

\[ A_{\text{pixel}} = a_{\text{pixel}}^2, \]  \hspace{1cm} (12)

where \( A_{\text{pixel}} \) is the area of the image pixel and \( a_{\text{pixel}} \) is the side of the image pixel.

This section describes an algorithm that allows for the automation of the second area moments calculation of an arbitrary cross-section by using an optical photograph of the cross section. The substitution described in section 2 is used. According to Eq. 7 and Eq. 8, the pixels of the cross-section image should have only two values. The first represents areas with solid material and the second represents areas without solid material (i.e. holes). In general, the section area occupies part of the picture. It is necessary to distinguish between the pixels forming the cross section and the background. Image processing is performed according to the algorithm shown in Fig. 5.

The contours of the section are defined first. This is necessary to separate the background further.

The conversion of the colour image into grayscale is then performed according to the following equation [7]:

\[ I_g(x, y) = \frac{1}{3} (I_g^R(x, y) + I_g^G(x, y) + I_g^B(x, y)), \]  \hspace{1cm} (13)

where \( I_g(x, y) \) is the value of a pixel with coordinates \((x,y)\) of the grayscale image and \( I_g^R(x, y), I_g^G(x, y), I_g^B(x, y) \) are respectively the values for red, green and blue colour of the colour image corresponding pixel.

The conversion of the grayscale image into a binary image is done by defining a threshold. In this case, all pixels above this threshold are set to 1 (white in our case) and all pixels below this threshold – 0 (black in our case). The choice of this threshold has a key role for the accuracy of the method.
After applying the algorithm described, the image is prepared for calculating the second area moments, for which the algorithm shown in Fig. 6.

In order to obtain absolute values for the centre of gravity and for the second area moments of the cross-section it is sufficient to determine the real size of the pixel. A real measurement is made between two random points A and B of the cross-section. Then it is necessary to determine the location of these points on the image. The side length of the pixel is as follows:

\[ a_{\text{pixel}} = \frac{\text{measured length between A and B}}{\text{number of pixels between A and B}}, \]  

(14)

where \( N_s \) – number of the pixels, representing areas with solid material, \( N_h \) – number of the pixels, representing areas without solid material (i.e. holes).

The next step is translation and rotation of the coordinate system. The new definition is shown in Fig. 8.

The following parameters: \( l, c \) – coordinates of the origin of the image coordinate system; \( l_{\text{max}}, c_{\text{max}} \) – last row, respectively last column of the image; \( l, c \) – coordinates of the origin of the cross-section coordinate system; \( l_{cg}, c_{cg} \) – coordinates of an arbitrary pixel from the cross-section area are used in Fig. 7.

Using the procedure described in section 2, Eq. 15 and Eq. 16 for coordinates’ calculation of the cross-section’s centre of gravity are obtained.

\[ Z_C = \frac{\sum_{i=1}^{N_s} A_{\text{pixel}} x_i - \sum_{i=1}^{N_h} A_{\text{pixel}} x_i}{(N_s - N_h) A_{\text{pixel}}}, \]  

(15)

\[ Y_C = \frac{\sum_{i=1}^{N_s} A_{\text{pixel}} y_i - \sum_{i=1}^{N_h} A_{\text{pixel}} y_i}{(N_s - N_h) A_{\text{pixel}}}, \]  

(16)

\[ z_i = (l_c - l) a_{\text{pixel}}, \]  

(17)

\[ y_i = (c_i - c) a_{\text{pixel}}, \]  

(18)

where \( N_s \) – number of the pixels, representing areas with solid material, \( N_h \) – number of the pixels, representing areas without solid material (i.e. holes).

Fig. 8. Definition of the origin and direction of the translated and rotated coordinate system.

By using the procedure described in section 2, Eq. 19 and Eq. 20 for second area moments calculation are obtained.

\[ I_y = \sum_{i=1}^{N_s} \left( \frac{a_{\text{pixel}}^2}{12} + A_{\text{pixel}} z_i^2 \right) - \sum_{i=1}^{N_h} \left( \frac{a_{\text{pixel}}^2}{12} + A_{\text{pixel}} z_i^2 \right), \]  

(19)

\[ I_z = \sum_{i=1}^{N_s} \left( \frac{a_{\text{pixel}}^2}{12} + A_{\text{pixel}} y_i^2 \right) - \sum_{i=1}^{N_h} \left( \frac{a_{\text{pixel}}^2}{12} + A_{\text{pixel}} y_i^2 \right), \]  

(20)

\[ y_i = (c_{cg} - c_i) a_{\text{pixel}}, \]  

(21)

\[ z_i = (l_c - l) a_{\text{pixel}}, \]  

(22)

\[ l_{cg} = l_c - \frac{z_c}{a_{\text{pixel}}}; \quad c_{cg} = c_c + \frac{y_c}{a_{\text{pixel}}}. \]  

(23)
IV. VERIFICATION OF THE PROPOSED METHOD

The verification of the algorithm proposed here is performed by using a cross-section with holes (Fig. 9) with the same diameter. We make a calculation of the second area moments on the Y and Z axis by both methods: manual (by hand) using the classical theory of strength of materials and the methodology presented here through a photograph of the cross-section. The methodology is realized in the Matlab environment. The cross-section dimensions are shown in Fig. 9. Fig. 10a shows the photograph of the cross-section, used for the second area moments calculation. Fig. 10b shows the generated binary image with a separated background after using the algorithm shown in Fig. 5.

![Fig. 9. Dimensions of the cross-section, used for method verification](image)

![Fig. 10. (a) Photograph of the cross-section; (b) Binary image of the cross-section after background (gray area) separation](image)

TABLE 1. RESULTS COMPARISON

<table>
<thead>
<tr>
<th></th>
<th>manual (on hand) calculation</th>
<th>proposed method calculation</th>
<th>deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Y</td>
<td>3.993 cm</td>
<td>3.974 cm</td>
<td>0.48 %</td>
</tr>
<tr>
<td>Z</td>
<td>2.487 cm</td>
<td>2.509 cm</td>
<td>0.88 %</td>
</tr>
<tr>
<td>I_y</td>
<td>82.053 cm$^4$</td>
<td>81.246 cm$^4$</td>
<td>0.99 %</td>
</tr>
<tr>
<td>I_z</td>
<td>207.299 cm$^4$</td>
<td>201.745 cm$^4$</td>
<td>2.75 %</td>
</tr>
<tr>
<td>A</td>
<td>40 cm$^2$</td>
<td>40.057 cm$^2$</td>
<td>0.14 %</td>
</tr>
</tbody>
</table>

A comparison between the results obtained with manual calculation and with proposed method is shown in Table 1.

V. CONCLUSION

The results obtained show that the proposed method can be used as simple, fast and relatively accurate solution for second area moments calculation of a random cross-section. The error in calculation is mainly due to the representation of smooth curves with a square pixel, especially in images with low resolution. Another factor influencing correct measurement is the presence of shadows in the picture, which may lead to their mistaken determination as areas with solid material.

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