# MULTIPLE REFERENCE FRAMES STATE SPACE MODELS OF BRUSHLESS DC MOTOR FOR FEEDBACK LINEARIZATION

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**Abstract**: Multiple reference frames orientated state space nonlinear mathematical models for brushless DC motors are derived in the paper using the power invariant dqo transformation in rotating and stationary form. The nonlinear state space models obtained are strictly orientated, decoupled, and linear with respect to the model parameters. This allows convenient application of advanced control theory differential geometric design approaches including multi-input multi-output feedback linearization and nonlinear adaptive systems design. Moreover the decoupling of the currents permits closed-loop nonlinear systems synthesis accomplishing simultaneous elimination of the torque pulsations and copper losses minimization. The nonlinear state space models derived are simulated to investigate their properties.

**Key words**: Nonlinear systems, Nonlinear transformations, Feedback linearization, Brushless DC motor, Power invariance.

## INTRODUCTION

The interest in permanent-magnet motors during the last thirty years is due to the multiple advantages they offer. The brushless DC motors (BDCM) are part of this motor class. They are the latest choice of researches due to their high efficiency, silent operation, compact size, high reliability and low maintenance requirements. The BDCM have trapezoidal back electro-motive force and require rectangular-shaped stator phase currents to produce constant torque [16, 17]. The difference between the conventional brushed DC motor and the BDCM is that the latter has no brushes, commutator, and field winding. The excitation of these motors is achieved by strong rare-earth permanent magnets in the rotor providing very powerful field without the need of excitation current and no electrical losses in the rotor [6, 7]. As a result they are very efficient energetically while generating comparatively high torque. The lack of brushes leads to longer life span of the BDCM. These motors have maximal ratio between generated torque and motor inertia or total mass conditioned by the high power density of the BDCM. The BDCM models in original coordinates are commonly used in practice for analysis and design with various types of control methods [1, 2, 3, 10, 11, 15], because the coordinate transformations of these motors [4] do not reduce the order of the system as in the synchronous motor case. The advantage of the dqo transformation and its partial case the $\alpha\beta o$ transformation presented in this article is the decomposition into two virtual motors: a single-phase motor capable to provide pulsating torque only (the homopolar part) and a two-phase motor with rotating field (the dq part). The mathematical equation of the homopolar current $i_o$, which does not exist in original coordinates, is very useful for the torque pulsation elimination. This is achieved by controlling the homopolar current $i_o$ to converge asymptotically to zero. Another advantage is the easy synthesis of copper minimization control via the current $i_d$.

The above mentioned peculiarities prove that the BDCM are highly nonlinear systems and the coefficients of their nonlinear mathematical models are time-variable and some of them are also unknown. These two real circumstances require the use of advanced nonlinear control methods for a high performance closed-loop control system design. Suitable approaches in this context are the Lie algebra multivariable feedback linearization based on differential geometry, adaptive feedback linearization, nonlinear observers synthesis, and adaptive nonlinear control design methods based on Lyapunov stability theory [5, 8, 9, 13, 12, 14]. The joint approach of nonlinear adaptive control design with nonlinear observers can yield the highest performance closed-loop systems implementation if some unsolved yet problems of this design approach are worked out in the near future.

The paper presents a derivation of multiple reference frames nonlinear state space mathematical models for BDCM, using the power invariant dqo transformation and its partial case the $\alpha\beta o$ transformation. The models derived are in strictly orientated state space form suitable for multivariable feedback linearization, nonlinear adaptive control and nonlinear observer design. The mathematical equivalence of the nonlinear models is confirmed by simulation.

## STATE SPACE MODEL IN ORIGINAL COORDINATES

The derivation of this model is based on the assumptions that the induced by the harmonic fields of the stator currents in the rotor, the iron losses, and the leakage losses are neglected. The initial equation in matrix-vector form is

$$\mathbf{V}_{abc} = \mathbf{R}\mathbf{I}_{abc} + \mathbf{L}\frac{d\mathbf{I}_{abc}}{dt} + \mathbf{E}_{abc}, \tag{1}$$

$\mathbf{V}_{abc} = [v_a, v_b, v_c]^T$, $\mathbf{I}_{abc} = [i_a, i_b, i_c]^T$, $\mathbf{E}_{abc} = [e_a, e_b, e_c]^T$ are the vectors of the stator voltages, currents, and EMF respectively. The matrix of the stator resistances and the matrix of the inductances are

$$\mathbf{R} = \begin{bmatrix} R & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & R \end{bmatrix}, \ \mathbf{L} = \begin{bmatrix} L-M & 0 & 0 \\ 0 & L-M & 0 \\ 0 & 0 & L-M \end{bmatrix}. \tag{2}$$

The phase voltage equation is identical with the armature voltage equation of the DC motor. The resemblance to a DC machine and the lack of brushes and commutator are the reasons for which this machine is called permanent magnet brushless DC machine. The electromagnetic torque is

$$T_e = [e_a i_a + e_b i_b + e_c i_c]\frac{1}{\omega_m}. \tag{3}$$

The instantaneous induced EMF can be written as

$$e_a = \lambda_p f_a(\theta_r)\omega_m \,,\; e_b = \lambda_p f_b(\theta_r)\omega_m \,,\; e_c = \lambda_p f_c(\theta_r)\omega_m \quad (4)$$

where the functions $\mathbf{f}_{abc}^T = [f_a(\theta_r), f_b(\theta_r), f_c(\theta_r)]$ have the same trapezoidal shape as $e_a$, $e_b$, and $e_c$ with a maximum magnitude $\pm 1$. The induced EMF edges are rounded because the EMF are derivatives of the flux linkages, which are continuous functions. The fringing also makes the flux density functions smooth with no abrupt edges. The electromagnetic torque can be rewritten as

$$T_e = \lambda_p \mathbf{f}_{abc}^T \mathbf{I}_{abc} \,. \quad (5)$$

The motion equation for system with inertia J, friction coefficient B, and load torque $T_L$ is

$$\frac{d\omega_m}{dt} = \frac{T_e}{J} - \frac{T_L}{J} - \frac{B}{J}\omega_m \,. \quad (6)$$

The electrical speed and position are related by

$$\frac{d\theta_r}{dt} = P\omega_m \,, \quad (7)$$

where P is the number of pole pairs, $\omega_m$ is the mechanical rotor speed, and $\theta_r$ is the electrical rotor position. Considering the BDCM dynamics according to equations (1), (5), (6), and (7) after choosing the state space vector

$$\mathbf{x} = [x_1, x_2, x_3, x_4, x_5]^T = [\theta_r, \omega_m, i_a, i_b, i_c]^T \,, \quad (8)$$

and control vector

$$\mathbf{u} = [u_1, u_2, u_3]^T = [v_a, v_b, v_c]^T \quad (9)$$

leads to the first state space model of the BDCM in original abc coordinates

$$\begin{aligned}
\dot{x}_1 &= c_1 x_2 \\
\dot{x}_2 &= c_2(f_a(x_1)x_3 + f_b(x_1)x_4 + f_c(x_1)x_5)c_3^{-1} - c_4 - c_5 x_2 \\
\dot{x}_3 &= -c_6 x_3 - c_7 f_a(x_1)x_2 + c_8 u_1 \qquad (10)\\
\dot{x}_4 &= -c_6 x_4 - c_7 f_b(x_1)x_2 + c_8 u_2 \\
\dot{x}_5 &= -c_6 x_5 - c_7 f_c(x_1)x_2 + c_8 u_3
\end{aligned}$$

with $c_1 = P$, $c_2 = \lambda_p$, $c_3 = J$, $c_4 = T_L / J$, $c_5 = B/J$, $c_6 = R/(L-M)$, $c_7 = \lambda_p/(L-M)$, $c_8 = 1/(L-M)$.

## STATE SPACE MODEL IN dqo COORDINATES

The model of the BDCM in dqo reference frames is derived by the dqo transformation. The dqo transformation is used when the rotating speed of the reference frames is different from zero. The straight and the inverse dqo transformations are

$$\mathbf{T}_{dqo}(\theta_r) = \sqrt{\frac{2}{3}} \begin{bmatrix} \cos(\theta_r) & \cos(\theta_r - \frac{2\pi}{3}) & \cos(\theta_r + \frac{2\pi}{3}) \\ \sin(\theta_r) & \sin(\theta_r - \frac{2\pi}{3}) & \sin(\theta_r + \frac{2\pi}{3}) \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}, \quad (11a)$$

$$\mathbf{T}_{dqo}^{-1}(\theta_r) = \sqrt{\frac{2}{3}} \begin{bmatrix} \cos(\theta_r) & \sin(\theta_r) & \frac{\sqrt{2}}{2} \\ \cos(\theta_r - \frac{2\pi}{3}) & \sin(\theta_r - \frac{2\pi}{3}) & \frac{\sqrt{2}}{2} \\ \cos(\theta_r + \frac{2\pi}{3}) & \sin(\theta_r + \frac{2\pi}{3}) & \frac{\sqrt{2}}{2} \end{bmatrix}. \quad (11b)$$

for reference frames rotating with the electrical rotor speed $\omega_r$ (rotor reference frames). This transformation is power invariant as $\mathbf{T}_{dqo}^{-1} = [\mathbf{T}_{dqo}]^{-1}$ and it is applicable for signals with arbitrary shape. Applying the dqo transformation for the matrix-vector equation (1) gives the matrix-vector equation of the BDCM electrical dynamics in rotating dqo coordinates written in terms of the currents

$$\frac{d\mathbf{I}_{dqo}}{dt} = -(\mathbf{L}^{-1}\mathbf{R} + \mathbf{T}_{dqo}\frac{d\mathbf{T}_{dqo}^{-1}}{d\theta_r}P\omega_m)\mathbf{I}_{dqo} + \mathbf{L}^{-1}(\mathbf{V}_{dqo} - \mathbf{E}_{dqo}) \quad (12)$$

The transformed electromagnetic torque (5) is given by

$$T_{edqo} = \lambda_p \mathbf{f}_{dqo}^T \mathbf{T}_{dqo}^{-T} \mathbf{T}_{dqo}^{-1} \mathbf{I}_{dqo} = \lambda_p \mathbf{f}_{dqo}^T \mathbf{I}_{dqo} \,. \quad (13)$$

Taking into account $T_{edqo}$, the motion equation (6), the electrical dynamics (12) and choosing the state space vector

$$\mathbf{x} = [x_1, x_2, x_3, x_4, x_5]^T = [\theta_r, \omega_m, i_d, i_q, i_o]^T \quad (14)$$

leads to the first state space model

$$\begin{aligned}
\dot{x}_1 &= c_1 x_2 \\
\dot{x}_2 &= c_2(f_d(x_1)x_3 + f_q(x_1)x_4 + f_o(x_1)x_5)c_3^{-1} - c_4 - c_5 x_2 \\
\dot{x}_3 &= -c_6 x_3 - (c_7 f_d(x_1) + x_{3c}(x_1, x_3, x_4))x_2 + c_8 u_1 \qquad (15)\\
\dot{x}_4 &= -c_6 x_4 - (c_7 f_q(x_1) + x_{4c}(x_1, x_3, x_4))x_2 + c_8 u_2 \\
\dot{x}_5 &= -c_6 x_5 - (c_7 f_o(x_1) + x_{5c}(x_1, x_3, x_4))x_2 + c_8 u_3
\end{aligned}$$

where

$$x_{3c}(x_1, x_3, x_4) = \frac{2}{3}\Bigg[-\Big(\sin(2x_1) + \sin(2x_1 - \frac{4\pi}{3}) + \sin(2x_1 + \frac{4\pi}{3})\Big)\frac{x_3}{2} + \Big(\cos^2(x_1) + \cos^2(x_1 - \frac{2\pi}{3}) + \cos^2(x_1 + \frac{2\pi}{3})\Big)x_4\Bigg],$$

and $x_{4c}(x_1, x_3, x_4)$, and $x_{5c}(x_1, x_3, x_4)$ are similar nonlinear functions resulting from the dqo transformation. The control vector $\mathbf{u} = [u_1, u_2, u_3]^T = [v_d, v_q, v_o]^T$ and the parameters $c_1 = P$, $c_2 = \lambda_p$, $c_3 = J$, $c_4 = T_L / J$, $c_5 = B / J$, $c_6 = R/(L-M)$, $c_7 = \lambda_p/(L-M)$, $c_8 = 1/(L-M)$. The advantage of the dqo transformation is that it decomposes the machine in two virtual motors: a single-phase motor only capable to provide a pulsating torque (the homopolar part) and a two-phase motor with a rotating field (the dq part). This advantage gives the facility for synthesis of an asymptotically stable subsystem for zeroing the homopolar current, which would eliminate the torque pulsation. Another convenience is the easy control synthesis minimizing the copper losses. It is achieved by controlling the current $i_d$ to converge to zero.

## STATE SPACE MODEL IN αβo COORDINATES

The straight $\alpha\beta o$ transformation and its inverse are a partial case of the dqo transformation (11)

$$\mathbf{T}_{\alpha\beta o} = \mathbf{T}_{dqo}(0) \,, \quad (16a)$$

$$\mathbf{T}_{\alpha\beta o}^{-1} = \mathbf{T}_{dqo}^{-1}(0) \,. \quad (16b)$$

This transformation converts the model of the BDCM in new stationary coordinates and it is also power invariant as $\mathbf{T}_{\alpha\beta o}^{-1} = [\mathbf{T}_{\alpha\beta o}]^{-1}$. The transformation of the original BDCM model in $\alpha\beta o$ coordinates is easier than in the dqo coordinates. Applying the transformation (16) via the rule $\mathbf{x}_{\alpha\beta o} = \mathbf{T}_{\alpha\beta o}^{-1}\mathbf{x}_{abc}$ to equation (1) leads to

$$\mathbf{V}_{\alpha\beta o} = \mathbf{R}\mathbf{I}_{\alpha\beta o} + \mathbf{L}\frac{d\mathbf{I}_{\alpha\beta o}}{dt} + \mathbf{E}_{\alpha\beta o} \,. \quad (17)$$

The electromagnetic torque (5) in $\alpha\beta o$ coordinates reads

$$T_{e\alpha\beta o} = \lambda_p \mathbf{f}_{\alpha\beta o}^T \mathbf{T}_{\alpha\beta o}^{-T} \mathbf{T}_{\alpha\beta o}^{-1} \mathbf{I}_{\alpha\beta o} = \lambda_p \mathbf{f}_{\alpha\beta o}^T \mathbf{I}_{\alpha\beta o} \,. \quad (18)$$

Considering (17), (18), the mechanical dynamics (6), and (7) with state space vector

$$\mathbf{x} = [x_1, x_2, x_3, x_4, x_5]^T = [\theta_r, \omega_m, i_\alpha, i_\beta, i_o]^T \,, \quad (19)$$

and control vector

$$\mathbf{u}=[u_1,u_2,u_3]^T=[v_\alpha,v_\beta,v_o]^T \qquad (20)$$

gives the third state space model in $\alpha\beta o$ coordinates

$$\begin{aligned}
\dot{x}_1 &= c_1 x_2\\
\dot{x}_2 &= c_2(f_\alpha(x_1)x_3+f_\beta(x_1)x_4+f_o(x_1)x_5)c_3^{-1}-c_4-c_5x_2\\
\dot{x}_3 &= -c_6x_3-c_7f_\alpha(x_1)x_2+c_8u_1\\
\dot{x}_4 &= -c_6x_4-c_7f_\beta(x_1)x_2+c_8u_2\\
\dot{x}_5 &= -c_6x_5-c_7f_o(x_1)x_2+c_8u_3
\end{aligned} \qquad (21)$$

where $c_1=P$, $c_2=\lambda_p$, $c_3=J$, $c_4=T_L/J$, $c_5=B/J$, $c_6=R/(L-M)$, $c_7=\lambda_p/(L-M)$, $c_8=1/(L-M)$. The $\alpha\beta o$ transformation yields a simpler state space model than the dqo transformation which is an advantage. Both transformations are power invariant and decompose the machine into two virtual motors. The main advantage of these transformations is that they provide a separate mathematical equation of the homopolar current $i_o$, which does not exist in original coordinates, but it is useful for the torque pulsation elimination being an essential goal of the control for these motors. All this is possible because the $\alpha\beta o$ current equations are decoupled as in original coordinates which allows independent control of the currents.

## SIMULATION AND SYSTEM TIME RESPONSES

The three state space models derived are simulated as open loop systems from zero initial conditions and with passive load torque created by the friction included in the models. Figure 1 confirms the equivalence of the three state space models displaying the coincidence of the torques and the angular velocities which are outputs of the simulated systems. The evolution of the currents is shown on figures 2, 3, and 4 in original abc, rotating dqo, and stationary $\alpha\beta o$ coordinates respectively. The pulsations of the current $i_o$ create the torque pulsations seen on figure 1a. The trapezoidal electromotive forces of the BDCM in abc and $\alpha\beta o$ coordinates can be seen on figures 5 and 6. The control input voltages evolve on figure 7 in abc coordinates.

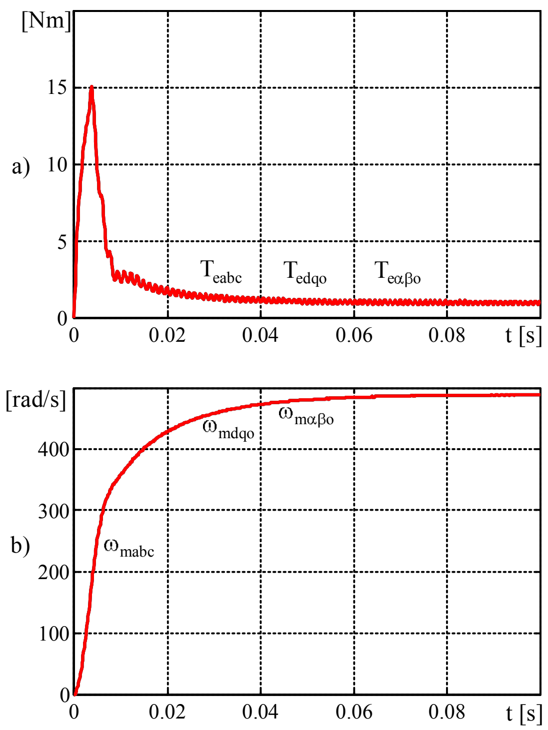

Figure 1: Torque and angular velocity responses

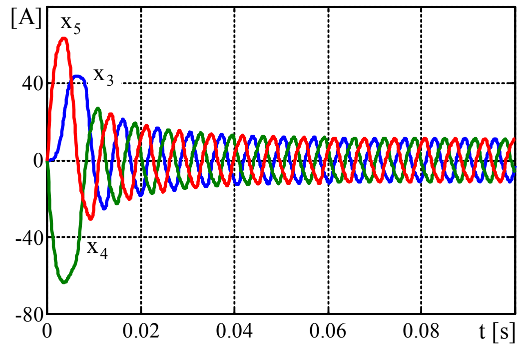

Figure 2: Currents in abc coordinates

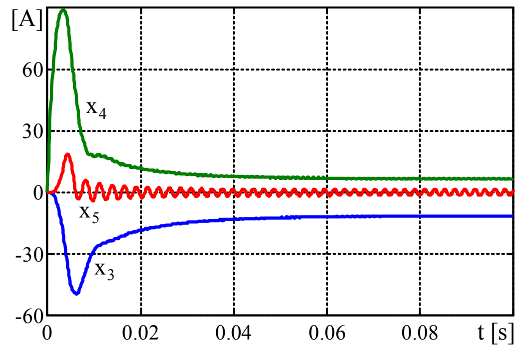

Figure 3: Currents in rotating dqo coordinates

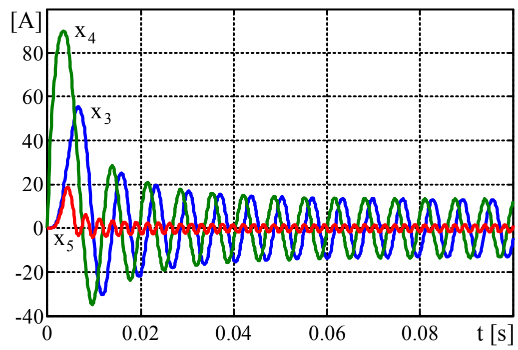

Figure 4: Currents in stationary $\alpha\beta o$ coordinates

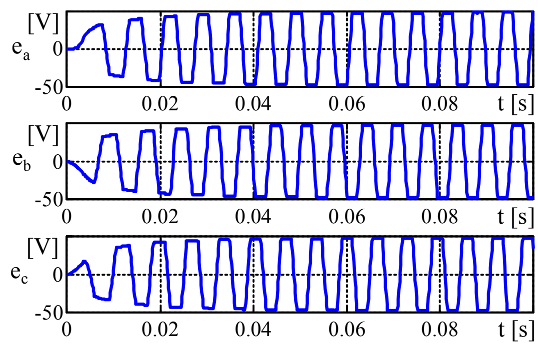

Figure 5: EMF in abc coordinates

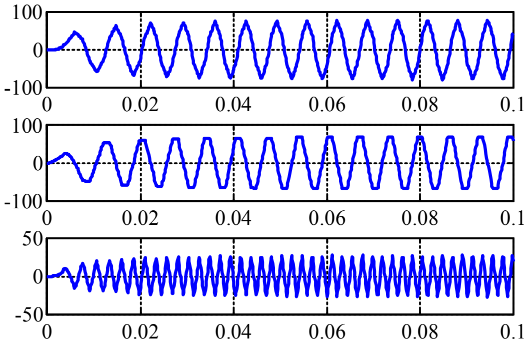


Figure 6: EMF in $\alpha\beta o$ coordinates

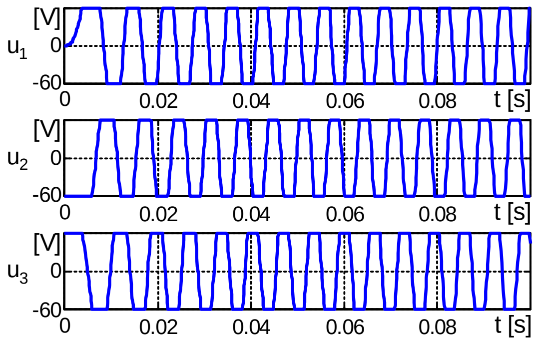


Figure 7: Control voltages in abc coordinates

The torque pulsations caused by the pulsations of the current $i_o$ can be eliminated if this current is controlled to converge asymptotically to zero.

## CONCLUSIONS

The paper has presented the derivation of state space mathematical models of BDCM in original, dqo, and $\alpha\beta o$ coordinates. This is achieved by the dqo transformation and its partial case, when $\theta_r = 0$, the $\alpha\beta o$ transformation. These transformations do not reduce the order of the system as in the synchronous motor case. Their main advantage is the decomposition of the machine into two virtual motors. Both transformations give the mathematical equation of the homopolar current which does not exist in original coordinates. This equation is another advantage that gives the chance for synthesis of asymptotically stable subsystem for elimination of torque pulsation by zeroing the homopolar current $i_o$. The comparison of the dqo and $\alpha\beta o$ transformations leads to the fact that only the $\alpha\beta o$ transformation decouples the currents. That is because in the dqo model the currents $i_d$ and $i_q$ depend on one another. Thus the $\alpha\beta o$ transformation allows minimization of the copper losses by controlling the currents $i_\alpha$ and $i_\beta$. A disadvantage of the dqo transformation is the highly nonlinear dqo model which complicates the control synthesis. There is no particular advantage of transforming the original model of the BDCM in rotating dqo and stationary $\alpha\beta o$ coordinates with respect to the signal periodicity. The three introduced strictly orientated state space models are specially designed for the application of advanced control theory differential geometric design approaches including multi-input multi-output feedback linearization and adaptive systems design. The nonlinear state space models are simulated for illustration of their dynamic properties in the respective reference frames.

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