©Journal of the Technical University at Plovdiv
“Fundamental Sciences and Applications”, Vol. 13(4), 2006
Anniversary Scientific Conference’ 2006
BULGARIA

A PROOF OF ONE PROPERTY OF A CLASS OF LINEAR TIME OPTIMAL CONTROL PROBLEMS

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Abstract. The paper deals with a class of linear time optimal control problems of decreasing order. A new proof of a property considering the state-space relationships between two neighboring problems within this class is presented. This property underlies a new approach for synthesis of time optimal control, requiring no description of the switching hyper surface.
Key words: Optimal control, Time optimal control, State space method, Synthesis, Switching, Linear system, Control system.

1. Introduction

In 1949 Feldbaum [3] for the first time formulated the time optimal control problem. Since then fundamental theoretical results have been obtained and a great number of papers has been published in this field. It may be stated that despite the more than 40-year intensive research, the synthesis of time optimal control for high order systems is still an open problem. Nevertheless, in the last decade the interest towards this problem considerably declines.

An approach to go further in the solution of the time optimal synthesis problem is to refine the well-known state-space method, removing the factors that restrict its application to low order systems only.

The following time-optimal synthesis problem for a linear system of order ***k*** is considered. The system is described by

$$\begin{aligned}
&\dot{\boldsymbol{x}}_k = A_k \boldsymbol{x}_k + B_k u_k,\\
&\boldsymbol{x}_k = [x_1 \quad x_2 \quad . \quad . \quad . \quad x_k]^{\mathrm{T}},\ \boldsymbol{x}_k \in R^k,\\
&A_k = \mathrm{diag}(\lambda_1, \quad \lambda_2, \quad . \quad . \quad . \quad , \lambda_k),\\
&\qquad \lambda_i \in R,\ \lambda_i \le 0,\ i, j = \overline{1,k},\ \lambda_i \ne \lambda_j \ \text{ if } \ i \ne j,\\
&B_k = [b_1 \quad b_2 \quad . \quad . \quad . \quad b_k]^{\mathrm{T}},\ b_i \in R,\ b_i \ne 0,\ i = \overline{1,k},\\
&\qquad \overline{1,k} = 1, 2, \ \ldots, k.
\end{aligned} \tag{1}$$

The initial and the target states of the system are

$$\boldsymbol{x}_k(0) = [x_{10} \quad x_{20} \quad . \quad . \quad . \quad x_{k0}]^{\mathrm{T}} \tag{2}$$

and

$$\boldsymbol{x}_k(t_{kf}) = [\underbrace{0 \quad 0 \quad . \quad . \quad . \quad 0}_{k}]^{\mathrm{T}} \tag{3}$$

where t_{kf} is unspecified. The admissible control $u_k(t)$ is a piecewise continuous function that takes its values from the range

$$-u_0 \le u_k(t) \le u_0, \; u_0 = const > 0\,. \tag{4}$$

We suppose that $u_k(t)$ is continuous on the boundary of the set of allowed values (4) and in the points of discontinuity τ we have

$$u(\tau) = u(\tau + 0)\,. \tag{5}$$

The problem is to find an admissible control $u_k = u_k(\boldsymbol{x}_k)$ that transfers the system (1) from its initial state (2) to the target state (3) in minimum time, i.e. minimizing the performance index

$$J_k = \int_0^{t_{kf}} dt = t_{kf}\,. \tag{6}$$

We shall refer to this problem as **Problem *A(k)*** and to the set {Problem *A(n)*, Problem *A(n-1)*, ..., Problem *A(1)*}, $n \ge 2$, as **class of problems *A(n), A(n-1), …, A(1)***.

The following relations exist between the systems of Problem *A(k)* and Problem *A(k-1)*, $k = \overline{n,2}$:

$$A_k = \begin{bmatrix} A_{k-1} & 0_{((k-1)\times 1)} \\ 0_{(1\times(k-1))} & \lambda_k \end{bmatrix},\; B_k = \begin{bmatrix} B_{k-1} \\ b_k \end{bmatrix},\; \boldsymbol{x}_k(0) = \begin{bmatrix} \boldsymbol{x}_{k-1}(0) \\ x_{k0} \end{bmatrix}. \tag{7}$$

For Problem *A(k)*, $k = \overline{n,1}$, denote:

$u_k^o(t)$ - the optimal control which is a piecewise constant function taking the values $+u_0$ or $-u_0$ and having at most $(k-1)$ discontinuities [1] - [5], [11], [12];

t_{kf}^o - the minimum of the performance index;

L_{kk-1} - the set of all state space points for which the optimal control has no more than $(k-2)$ discontinuities;

S_k - the switching hyper surface. Note that S_k is time-invariant and includes the state space origin. As it is well known, the switching hyper surface S_k is identical with the set L_{kk-1} [4] (chapter 14).

2. Main Result

Based on some new state space properties of the presented class of problems, a new method for synthesis of the time optimal control for the studied class of linear systems requiring no description of the switching hyper surface is proposed [7] - [10]. A property showing the existence of specific relations between the switching hyper surface for the Problem *A(k)* and the state trajectory starting from the initial state for Problem *A(k)* and generated by the optimal control for Problem *A(k-1)* is studied first in [6] and then in [7] and [8]. Here we shall present a new proof of this property, which underlies the synthesis of time optimal control for the considered class of linear systems and makes possible the efficient design and implementation of time optimal control for high order linear systems.

Theorem. *The state trajectory of system (1) starting from the initial point* $\boldsymbol{x}_k(0)$ *and generated by the optimal control* $u_{k-1}^o(t)$, $t \in [0,\ t_{k-1f}^o]$, *either entirely lies on the switching hyper surface* S_k, *or is above or below* S_k, *nowhere intersecting it.*

Proof. For the initial state of Problem *A(k)* we can write

$$\boldsymbol{x}_k(0) = \begin{bmatrix} x_{10} & x_{20} & . & . & . & x_{k0} \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} \boldsymbol{x}_{k-1}(0) \\ x_{k0} \end{bmatrix}. \tag{8}$$

Let us apply the optimal control $u_{k-1}^o(t)$ of Problem *A(k-1)* to the system of Problem *A(k)* with initial state

$$\boldsymbol{x}_k^1(0) = \begin{bmatrix} x_{10} & . & . & . & x_{k-10} & x_{k0}^1 \end{bmatrix}^{\mathrm{T}} = \\ = \begin{bmatrix} \boldsymbol{x}_{k-1}(0) \\ x_{k0}^1 \end{bmatrix}, \quad x_{k0}^1 = -\frac{\int_0^{t_{k-1f}^o} e^{\lambda_k (t_{k-1f}^o - \tau)} b_k u_{k-1}^o(\tau) d\tau}{e^{\lambda_k t_{k-1f}^o}}. \tag{9}$$

We obtain the trajectory

$$\boldsymbol{x}_k^1(t) = e^{A_k t} \boldsymbol{x}_k^1(0) + \int_0^t e^{A_k (t-\tau)} B_k u_{k-1}^o(\tau) d\tau, \ \ t \in [0,\ t_{k-1f}^o]. \tag{10}$$

It follows from (9), (10)

$$\boldsymbol{x}_k^1(t) = e^{A_k t} \boldsymbol{x}_k^1(0) + \int_0^t e^{A_k (t-\tau)} B_k u_{k-1}^o(\tau) d\tau =$$

$$= e^{\begin{bmatrix} A_{k-1} & 0 \\ 0 & \lambda_k \end{bmatrix} t} \begin{bmatrix} \boldsymbol{x}_{k-1}(0) \\ x_{k0}^1 \end{bmatrix} + \int_0^t e^{\begin{bmatrix} A_{k-1} & 0 \\ 0 & \lambda_k \end{bmatrix} (t-\tau)} \begin{bmatrix} B_{k-1} \\ b_k \end{bmatrix} u_{k-1}^o(\tau) d\tau =$$

$$= \begin{bmatrix} e^{A_{k-1} t} \boldsymbol{x}_{k-1}(0) + \int_0^t e^{A_{k-1}(t-\tau)} B_{k-1} u_{k-1}^o(\tau) d\tau \\ e^{\lambda_k t} x_{k0}^1 + \int_0^t e^{\lambda_k (t-\tau)} b_k u_{k-1}^o(\tau) d\tau \end{bmatrix} =$$

$$= \begin{bmatrix} e^{A_{k-1}t}\boldsymbol{x}_{k-1}(0) + \int_0^t e^{A_{k-1}(t-\tau)} B_{k-1} u_{k-1}^o(\tau) d\tau \\ \dfrac{\int_0^{t_{k-1f}^o} e^{\lambda_k (t_{k-1f}^o - \tau)} b_k u_{k-1}^o(\tau) d\tau}{(-1) e^{\lambda_k (t_{k-1f}^o - t)}} + \int_0^t e^{\lambda_k (t-\tau)} b_k u_{k-1}^o(\tau) d\tau \end{bmatrix}, \quad t \in [0,\ t_{k-1f}^o]. \tag{11}$$

Since $u_{k-1}^o(t)$ is the optimal control of Problem *A(k-1)*, then

$$\boldsymbol{x}_k^1(t_{k-1f}^o) = \begin{bmatrix} \boldsymbol{x}_{k-1}(t_{k-1f}^o) \\ (-1)\int_0^{t_{k-1f}^o} e^{\lambda_k (t_{k-1f}^o - \tau)} b_k u_{k-1}^o(\tau) d\tau + \int_0^{t_{k-1f}^o} e^{\lambda_k (t_{k-1f}^o - \tau)} b_k u_{k-1}^o(\tau) d\tau \end{bmatrix} =$$

$$= \begin{bmatrix} \boldsymbol{x}_{k-1}(t_{k-1f}^o) \\ 0 \end{bmatrix} = [\underbrace{0 \quad 0 \quad . \quad . \quad . \quad 0}_{k}]^{\mathrm{T}}. \tag{12}$$

Hence, the optimal control $u_{k-1}^o(t)$, which is a piecewise constant function with no more than $(k-2)$ discontinuities, transfers the system from the initial state (9) to the state space origin in the moment t_{k-1f}^o. Therefore, the point $\boldsymbol{x}_k^1(0)$ and the trajectory starting from this point and generated by $u_{k-1}^o(t)$, $t \in [0,\ t_{k-1f}^o]$, lie entirely on the switching hyper surface S_k.

Consider now the trajectory $\boldsymbol{x}_k(t)$ with initial point (2) in form (8), generated by the optimal control $u_{k-1}^o(t)$. Taking into account (7) we can write

$$\boldsymbol{x}_k(t) = e^{A_k t}\boldsymbol{x}_k(0) + \int_0^t e^{A_k (t-\tau)} B_k u_{k-1}^o(\tau) d\tau =$$

$$= e^{\begin{bmatrix} A_{k-1} & 0 \\ 0 & \lambda_k \end{bmatrix} t} \begin{bmatrix} \boldsymbol{x}_{k-1}(0) \\ x_{k0} \end{bmatrix} + \int_0^t e^{\begin{bmatrix} A_{k-1} & 0 \\ 0 & \lambda_k \end{bmatrix}(t-\tau)} \begin{bmatrix} B_{k-1} \\ b_k \end{bmatrix} u_{k-1}^o(\tau) d\tau =$$

$$= \begin{bmatrix} e^{A_{k-1}t}\boldsymbol{x}_{k-1}(0) + \int_0^t e^{A_{k-1}(t-\tau)} B_{k-1} u_{k-1}^o(\tau) d\tau \\ e^{\lambda_k t} x_{k0} + \int_0^t e^{\lambda_k (t-\tau)} b_k u_{k-1}^o(\tau) d\tau \end{bmatrix}, \quad t \in [0,\ t_{k-1f}^o]. \tag{13}$$

According to (13) and (10) we have

$$\begin{aligned} \boldsymbol{x}_k(t) - \boldsymbol{x}_k^1(t) = e^{A_k t}\boldsymbol{x}_k(0) + \int_0^t e^{A_k (t-\tau)} B_k u_{k-1}^o(\tau) d\tau - \\ - \left[e^{A_k t}\boldsymbol{x}_k^1(0) + \int_0^t e^{A_k (t-\tau)} B_k u_{k-1}^o(\tau) d\tau \right], \quad t \in [0,\ t_{k-1f}^o]. \end{aligned} \tag{14}$$

From (14), having in mind (11) and (13), we obtain

$$\boldsymbol{x}_k(t) - \boldsymbol{x}_k^1(t) = \begin{bmatrix} e^{A_{k-1}t}\boldsymbol{x}_{k-1}(0) + \int_0^t e^{A_{k-1}(t-\tau)}B_{k-1}u_{k-1}^o(\tau)d\tau \\ e^{\lambda_k t}x_{k0} + \int_0^t e^{\lambda_k(t-\tau)}b_k u_{k-1}^o(\tau)d\tau \end{bmatrix} -$$

$$-\begin{bmatrix} e^{A_{k-1}t}\boldsymbol{x}_{k-1}(0) + \int_0^t e^{A_{k-1}(t-\tau)}B_{k-1}u_{k-1}^o(\tau)d\tau \\ e^{\lambda_k t}x_{k0}^1 + \int_0^t e^{\lambda_k(t-\tau)}b_k u_{k-1}^o(\tau)d\tau \end{bmatrix}, \quad t \in [0,\ t_{k-1f}^o]. \tag{15}$$

Thus,

$$\boldsymbol{x}_k(t) - \boldsymbol{x}_k^1(t) = \begin{bmatrix} 0_{((k-1)\times 1)} \\ e^{\lambda_k t}(x_{k0} - x_{k0}^1) \end{bmatrix}, \quad t \in [0,\ t_{k-1f}^o]. \tag{16}$$

Let us analyze the difference between $\boldsymbol{x}_k(t)$ and $\boldsymbol{x}_k^1(t)$ in (16). For the ***k***th coordinate $e^{\lambda_k t}(x_{k0} - x_{k0}^1)$ we have:

1. If $x_{k0} = x_{k0}^1$, then the initial state $\boldsymbol{x}_k(0)$ (2) coincides with $\boldsymbol{x}_k^1(0)$ (9), i.e.

$$\boldsymbol{x}_k(0) = \boldsymbol{x}_k^1(0). \tag{17}$$

We have already shown, that the point $\boldsymbol{x}_k^1(0)$ (9) and the trajectory starting in $\boldsymbol{x}_k^1(0)$ and generated by the control $u_{k-1}^o(t)$ for $t \in [0,\ t_{k-1f}^o]$ lie entirely on the switching hyper surface S_k. Thus, the theorem is proved in case $\boldsymbol{x}_k(0) = \boldsymbol{x}_k^1(0)$.

2. If $x_{k0} \neq x_{k0}^1$, then the initial state $\boldsymbol{x}_k(0)$ (2) does not coincide with $\boldsymbol{x}_k^1(0)$ (9), i.e.

$$\boldsymbol{x}_k(0) \neq \boldsymbol{x}_k^1(0), \tag{18}$$

and

$$e^{\lambda_k t}(x_{k0} - x_{k0}^1) \tag{19}$$

for $t \in [0,\ t_{k-1f}^o]$ does not change its sign and never equals zero, since t_{k-1f}^o is finite.

Hence, in this case the trajectory (10) with initial point (9), generated by $u_{k-1}^o(t)$, $t \in [0,\ t_{k-1f}^o]$, and lying entirely on S_k, is the projection on S_k of the trajectory $\boldsymbol{x}_k(t)$ with initial point $\boldsymbol{x}_k(0)$, generated by $u_{k-1}^o(t)$, $t \in [0,\ t_{k-1f}^o]$. The direction of this projection is parallel to the axis Ox_k. Moreover, the trajectory $\boldsymbol{x}_k(t)$, $t \in [0,\ t_{k-1f}^o]$, nowhere intersects the switching hyper surface S_k. Thus, the theorem is proved.

REFERENCES

1. **Athans, M., P. L. Falb**, *Optimal Control*, McGraw-Hill, New York, 1966.
2. **Boltyanskii, V. G.**, *Mathematical Methods of Optimal Control*, Holt, Rinehart and Winston, New York, 1971.
3. **Фельдбаум, А. А.**, **"**Простейшие релейные системы автоматического регулирования", Автоматика и телемеханика, т. X, № 4, 1949.
4. **Feldbaum, A. A. and A. G. Butkowsky**, *Methods of the Theory of the Automatic Control*, Nauka, Moscow, 1971 (in Russian).
5. **Leitmann, G.**, *The calculus of variations and optimal control*, Plenum Press, 1981.
6. **Penev, B.**, "A proof of one property of one class of time optimal control problems" (in Russian), Journal of the Technical University at Plovdiv (BULGARIA) "Fundamental Sciences and Applications", Vol. 5, 1997, Series A - Pure and Applied Mathematics, ISSN 1310-8271, pp. 139-147.
7. **Penev, B. G.**, "A Method for Synthesis of Time-Optimal Control of Any Order for a Class of Linear Problems for Time-Optimal Control", Ph.D. Dissertation, Technical University of Sofia, 1999 (in Bulgarian).
8. **Penev, B. G. and N. D. Christov**, "On the Synthesis of Time Optimal Control for a Class of Linear Systems", in Proc. 2002 American Control Conference, Anchorage, May 8-10, 2002, pp. 316-321.
9. **Penev, B. G. and N. D. Christov**, "On the State-Space Analysis in the Synthesis of Time-Optimal Control for a Class of Linear Systems", in Proc. 2004 American Control Conference, Boston, June 30 - July 2, 2004, Vol. 1, pp. 40-45.
10. **Penev, B. G. and N. D. Christov**, **"**A fast time-optimal control synthesis algorithm for a class of linear systems", in Proc. 2005 American Control Conference, Portland, 8-10 June 2005, Vol. 2, pp. 883-888.
11. **Pinch, E. R.**, *Optimal Control and the Calculus of Variations*, Oxford University Press, Oxford, 1993.
12. **Pontryagin, L. S., V. G. Boltyanskii, R. V. Gamkrelidze and E. F. Mischenko**, *The Mathematical Theory of Optimal Processes*, Pergamon Press, Oxford, 1964.

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