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Synthesis of Four Bar Mechanisms as Function Generators by Freudenstein - Chebyshev

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Abstract

This paper describes new method for the synthesis of four-bar linkages for generating a required input-to-output motion. The synthesis method is based on the direct application of Chebyshev's Alternation Theorem for the Equation of Freudenstein. By this approach the maximum structural error which corresponds to the best approximation can be estimated in advance. Two comparative examples are herewith used to illustrate some of the main features of the method. The innovation in this paper is the presentation of the target function as exact satisfied equation. On substituting the solution of this equation in the Equation of Freudenstein a generalized polynomial of Chebyshev is obtained. This polynomial is minimized by the Chebyshev's alternation theorem. This method does not require Chebyshev's spacing of the structural error of the mechanisms. One of the advantages of the so proposed approach is the possibility to predict the peculiarities of the mechanism with respect to the synthesis problem in the beginning of the solution. The so proposed method combines the power of the Freudenstein's equation and the Chebyshev's theorem comprehensiveness. The method of Freudenstein-Chebyshev presented here shows that for every structural error which could be presented as generalized polynomial of Chebyshev can be found the best approximation.

Keywords: Mechanism Synthesis; Function Generator; Best Approximation, Singular Configuration; Freudenstein Equation; Chebyshev's Theorem.

Introduction

The first numerical problem of mechanism synthesis was solved by the eminent mathematician P. L. Chebyshev. In 1854 he solved the problem with the dimensions of the four-bar mechanism with a coupler path, which is best approximated to a straight line for a specified interval of the input link items [1]. By solving this problem, he stated the so-called Chebyshev's Fundamental Theorem, which is also known as the Chebyshev's Alternation Theorem. This theorem is a powerful tool for solving different types of approximation problems, and not only in the field of the theory of mechanisms.

Different approaches have been used to solve synthesis related tasks. Graphical methods allow obtaining the most meaningful concept for the various types of mechanism syntheses, which describe the nature of the problems with synthesis in general [2]. The accuracy of these methods was not satisfactory enough for some engineering problems. These shortcomings were overcome by producing graphic designs of analytical expressions, which are compatible with computer calculations. Nowadays the use of CAD software provides opportunities for improving the accuracy of graphic methods to the level of the analytical ones.

Freudenstein's equation [3] can be considered as the simplest position function expression of the four-bar mechanisms. This simplification is the basis of its widespread application for finding solutions to different kinds of problems in the mechanism synthesis.

Contemporary methods of synthesis are based on recent developments in mathematics, using finite element analysis, neural networks, fussy sets, genetic and evolution algorithms. Penunuri et al indicated the application of the evolution method in solving a wide range of synthesis problems [4]. Evolution algorithms and simplex techniques are at the core of the modules for the optimization of the package for interactive PC-software for synthesis and analysis of mechanisms (SAM) which is a powerful tool for design and optimization of arbitrary planar mechanisms [5]. Regarding the four-bar mechanism, Shpli uses the genetic algorithm to solve a function generator problem [6]. The simplicity and feasibility of the evolution and genetic algorithms used for solving a wide range of problems is described in publications [7-9]. Angeles F. et al present synthesis approaches which do not involve genetic algorithms and which yield satisfactory results [10].

It is well known that the structural error of the function generation mechanisms can be minimized by applying Chebyshev's fundamental theorem. This theorem can be directed towards mechanisms, as follows: If n independent adjustable parameters are involved in the design of a mechanism, which will generate a prescribed path or function, then the largest absolute value of the structural error will be minimized where there are n precision points, so positioned that the $(n-1)$ maximum values of the structural error between each pair of adjacent precision points, as well as between the ends and the nearest precision points x_i are numerically equal with successive alterations in sign [11].

Chebyshev noted that the best linkage approximation of a given mechanism to any function occurs when the absolute value of the maximum structural error between the precision points and both ends of the range are equalized. Chebyshev's spacing of precision points is employed to minimize the structural error [12]. The Freudenstein equation technique, used for dimensional synthesis, can also be relied upon to secure the minimized structural error [13].

One possible method for solving complex synthesis is presented by the author of this paper in [14]. The presented method in this paper leads to a relative complex conversion of the structural error into equation the Equation of Freudenstein.

The purpose of this paper is to show a new method that can be useful for the approximation of simple functions by combining Chebyshev's Fundamental Theorem and Freudenstein's Equation. This method does not require Chebyshev's precision point spacing in the beginning of the solution. Furthermore, the method provides for a problem to be analyzed for the best approximation, and also for the opportunity to establish the existence of such approximation as a whole.

Theory Background

Let us consider a generalized polynomial of the $(n+1)^{th}$ order

$$P(x) = p_0 f_0(x) + p_1 f_1(x) + \ldots + p_n f_n(x) \tag{1}$$

of linearly independent continuous functions $f_i(x)$, where $p_0, p_1, \ldots, p_n$ are constant coefficients. The expression (1) is called polynomial of Chebyshev's systems functions when each polynomial of the functions $f_i(x)$ has no more than n roots in the interval $[a,b]$ [15]. The set of functions $f_i(x), (i = 0, 1, \ldots, n)$ is called Chebyshev's system functions of the order n.

The condition for the existence of the functions of Chebyshev system is equivalent to the condition

$$\Delta_f = \begin{vmatrix} f_0(x_0) & f_0(x_1) & \ldots & f_0(x_n) \\ f_1(x_0) & f_1(x_1) & \ldots & f_1(x_n) \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ f_n(x_1) & f_n(x_2) & \ldots & f_n(x_n) \end{vmatrix} \neq 0 \tag{2}$$

for every different $n+1$ point, where $x_i \in [a,b]$.

A continuous function $F(x)$ is the one least deviating from Chebyshev's system polynomial of the order n of the type (1) if $\Delta_{\max} = \max\limits_{[a,b]} |P(x) - F(x)|$ is minimum. Such kind of function approximation is also called the Best Approximation or Chebyshev's Approximation.

Chebyshev's Alternation Theorem is formulated as follows:

In order for polynomial (1) to deviate as small as possible from a given continuous function $F(x)$ within the interval $[a,b]$, it is necessary and sufficient that the difference

$$\Delta(x) = P(x) - F(x) \tag{3}$$

reaches consecutively its extreme values $\pm L$ with alternating characters at least $(n+2)$ times.

For the four-bar mechanism shown in Figure 1, Freudenstein's equation [16] (F. Freudenstein, 1954) can be written in the view

$$p_1 + p_2 \cos\varphi - p_3 \cos\psi - \cos(\psi - \varphi) = 0 \tag{4}$$

where

$$p_1 = \frac{d^2 + r^2 + R^2 - l^2}{2rR} \tag{5}$$

$$p_2 = \frac{d}{R} \tag{6}$$

and

$$p_3 = \frac{d}{r} \tag{7}$$

are constant coefficients, which depend on the dimensions of the links r, R, l and d as shown in Figure 1.

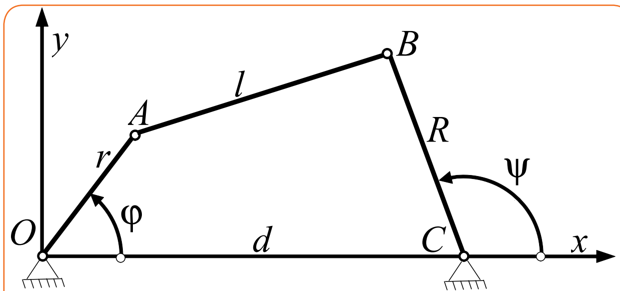

Figure 1: Kinematic scheme of four-bar mechanism.

In the function generation synthesis problems, the output angle has to be approximated to a required function $F = F(\varphi)$ of the input angle φ. Most approximate methods lead to the creation of a target function

$$\Delta = \psi(\varphi) - F(\varphi) \tag{8}$$

and to the search of its minimum within a limited interval for

$\varphi \in [\varphi_1, \varphi_m]$, where $m \le n+1$. The function (8) is also called a structural error.

The so called here method of Freudenstein-Chebyshev can be described by the following steps:

Let us assume that the target function has been completely satisfied, e.g.

$$\Delta = \psi(\varphi) - F(\varphi) = 0 \tag{9}$$

An explicit solution of (9) can be found for a very wide class of optimization functions with respect to the output variable

$$\psi = \Phi_\Delta(\varphi) \tag{10}$$

By substituting formula (10) in Freudenstein's equation (4) it is found

$$p_1 + p_2 \cos\varphi - p_3 \cos\Phi_\Delta(\varphi) - \cos[\Phi_\Delta(\varphi) - \varphi] = 0 \tag{11}$$

The left hand side of Equation (11) can be presented as

$$\Delta_\psi(\varphi) = P(\varphi) - F_\psi(\varphi) \tag{12}$$

where

$$P(\varphi) = p_1 + p_2 \cos\varphi - p_3 \cos\Phi_\Delta(\varphi) \tag{13}$$

$$F_\psi(\varphi) = \cos[\Phi_\Delta(\varphi) - \varphi] \tag{14}$$

If the set of the functions $\{1, \cos\varphi, -\cos\Phi_\Delta(\varphi)\}$ forms Chebyshev's 3rd order system in the prescribed approximation interval of $\varphi_1 \le \varphi \le \varphi_4$, then Chebyshev's Theorem can be applied in order for the unknown coefficients p_1, p_2 and p_3 to be found. After applying the theorem to the difference (12), the following system of equations pursues:

$$\begin{cases} \Delta_\psi(\varphi_i) = (-1)^i L & i = 1,2,3,4 \\ \Delta'_\psi(\varphi_j) = 0 & j = 2,3 \end{cases} \tag{15}$$

where $\Delta'_\psi(\varphi_j) = \left.\dfrac{d\Delta_\psi(\varphi)}{d\varphi}\right|_{\varphi = \varphi_j}$

The determination of the coefficients p_1, p_2 and p_3 allows the three dimensions of the four-bar mechanism to be calculated as a function of the fourth one.

System (15) consists of six equations. The number of unknowns p_1, p_2, p_3, φ_2, φ_3, and L is six, which means that the system is determinate. Here it is assumed that the ends of the approximation interval φ_1 and φ_4 are known beforehand. It is possible for the synthesis task to become more complicated if the end angles φ_1 and φ_4 are assumed as unknowns and the optimization function at these points, also has extreme values. This leads to the system

$$\begin{cases} \Delta_\psi(\varphi_i) = (-1)^i L \\ \Delta'_\psi = 0 & i = 1,2,3,4 \end{cases} \tag{16}$$

In this case the new two unknowns φ_1 and φ_4 can be considered as extreme points of the range of approximation.

The so described synthesis method can be expanded by the introduction of the initial input and output angles. With the help of these two new constants, it is possible to find similar solutions of synthesis problems for precision points 4 and 5.

The classical synthesis method of Freudenstein is based on the spacing of Chebyshev's precision points, which are the roots of Chebyshev's power polynomials [17]. According to this method, the best solution of the function generation problem can be found when assuming that the structural error at the precision points is zero.

This requires the solution of a system of Freudenstein equations where the pair of the known parameters $(\varphi_{0_i}, \psi_{0_i})$ has been calculated with the help of the function $\psi_{0i} = F(\varphi_{0_i})$. This method of synthesis problem involves a linear system of three equations

$$p_1 + p_2 \cos\phi_{0i} - p_3 \cos\psi_{0i} - \cos(\psi_{0i} - \phi_{0i}) = 0 \quad i = 1,2,3 \tag{17}$$

This method allows for a maximum of 5 precision points to be used if the initial input and output angles are assumed as unknown. In this case, system (17) becomes nonlinear.

Numerical Examples

Suppose a four-bar mechanism has to generate the simple function

$$F = \frac{1}{5}\varphi \tag{18}$$

for the interval of the input angle of $\varphi \in [220^\circ, 280^\circ]$. It follows from formula (9) that the structural error has the following form:

$$\Delta = \psi(\varphi) - \frac{1}{5}\varphi \tag{19}$$

According to assumption (9), it is found that

$$\Phi_\Delta(\varphi) = \frac{1}{5}\varphi \tag{20}$$

Polynomial (13) and function (14) can be rewritten in the form

$$P(\varphi) = p_1 + p_2 \cos\varphi - p_3 \cos\frac{\varphi}{5}, \text{ and } F_\psi(\varphi) = \cos\frac{4\varphi}{5} \tag{21}$$

Difference (12) is now an even function, which is

$$\Delta_\psi(\varphi) = p_1 + p_2\cos\varphi - p_3\cos\frac{\varphi}{5} - \cos\frac{4\varphi}{5} \qquad (22)$$

The set of functions $\{1, \cos\varphi, -\cos\frac{\varphi}{5}\}$ forms Chebyshev's system for $200^\circ \le \varphi \le 280^\circ$ and Chebyshev's theorem can be applied for diminishing the structure error.

According to the case considered in the system of equations (15), it follows that:

$$p_1 + p_2\cos\varphi_i - p_3\cos\frac{\varphi_i}{5} - \cos\frac{4\varphi_i}{5} = (-1)^i L \quad i = 1,2,\ldots 4 \qquad (23)$$

$$-p_2\sin\varphi_i + \frac{1}{5}p_3\sin\frac{\varphi_i}{5} + \frac{4}{5}\sin\frac{4\varphi_i}{5} = 0, \qquad j = 2,3$$

where $\varphi_1 = 200^\circ$ and $\varphi_4 = 280^\circ$. This system leads to the following numerical solution:

$p_1 =$ -5.838535436, $p_2 =$ -1.757488964, $p_3 =$ 8.57533417, $\varphi_{2,} =$ 3.852698185, $\varphi_3 =$ 4.545660767 and $L =$ 0.01874486286. Assuming that $R = 1$, the lengths of the rest of the links are calculated as follows: $d =$ -1.757488964, $l =$ 1.318176828, and $r =$ -0.2049469944.

The same problem is solved by the classical method of Freudenstein. The spacing of precision points is calculated by the formula

$$\varphi_{0i} = \frac{\varphi_4 + \varphi_1}{2} + \frac{\varphi_4 - \varphi_1}{2}\cos\frac{(2i-1)\pi}{2n} \qquad i = 1,2,3 \qquad (24)$$

The above-described system (18) is obtained and its solution $p_1 =$ -5.838535436, $p_2 =$ -1.757488964, $p_3 =$ 8.57533417 has been found.

The four-bar configurations for the end input angular positions are shown in Figure 2. The error distribution of both methods is shown in Figure 3. The dashed line shows the solution by the method of Freudenstein-Chebyshev and the solid line describes the classical solution obtained by Chebyshev's precision points spacing.

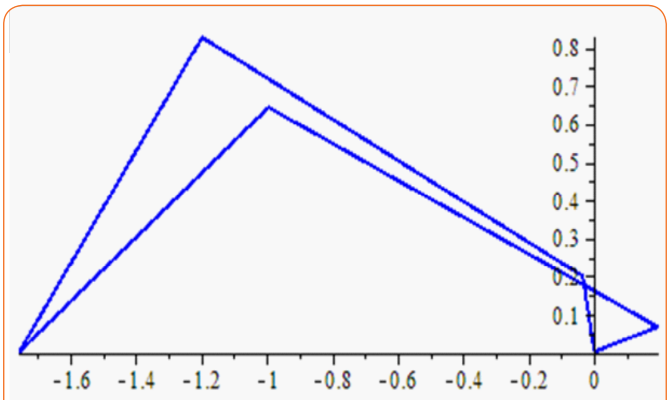

Figure 2: Two end configurations of the synthesized four-bar mechanism.

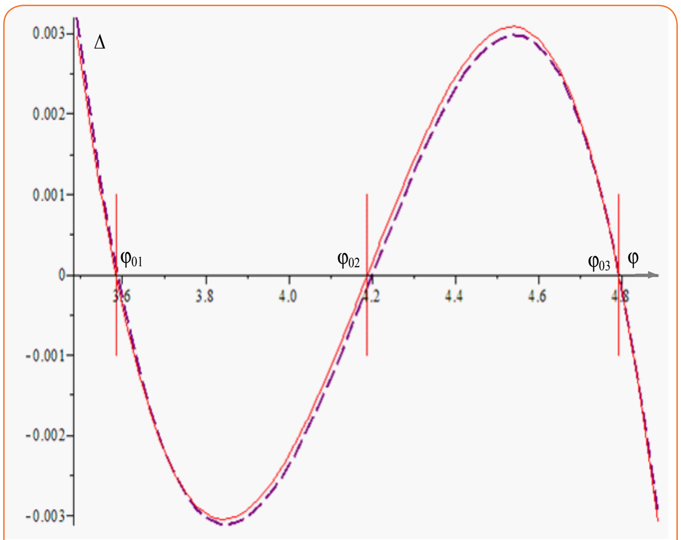

Figure 3: Comparison of the error distributions of the two synthesis methods. The dashed line is the Freudenstein-Chebyshev's distribution and the solid line is the Freudenstein's precision point spacing method. The precision points are ϕ01, ϕ02, and ϕ03.

The results shows that the method of Freudenstein-Chebyshev's is more precise than the classical method of Freudenstein, and the real precision points do not coincide with these ones determined by the Chebyshev's spacing.

The next example considers the approximation of the same target function (18) $F = \frac{1}{5}\varphi$, only that it is for the interval between $0 \le \phi \le 90^\circ$.

The solution of system (23), which presents the method of Freudenstein - Chebyshev for this interval, gives the parameter values $p_1 =$ -4.99081, $p_2 =$ -0.41842, $p_3 =$ 5.5724, $\varphi_{2,} =$ 0.77809, $\varphi_3 =$ 1.3560 and $L =$ 0.00008842. The distribution of Chebyshev's polynomial for these parameters is shown in Figure 4. It is evident that this polynomial has Chebyshev's form, but the real distribution of the structural error shown in Figure 4 does not satisfied Chebyshev's Alternation Theorem. The reason for this result is that the difference (12) is some type of weighted target function (14).

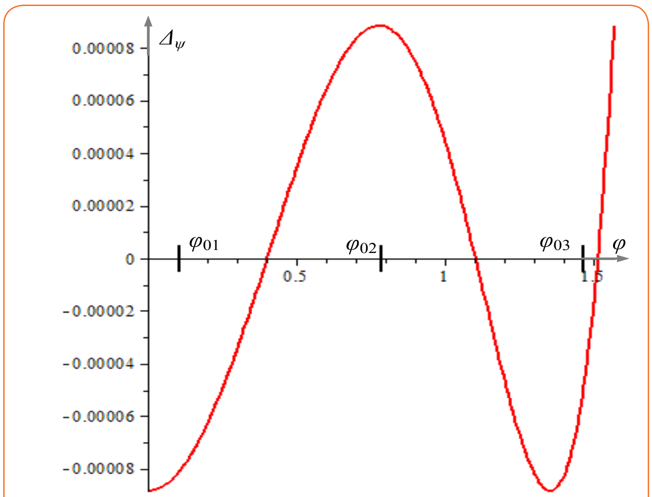

Figure 4: The polynomial of Chebyshev for 0 ≤ϕ ≤90º. The precision points ϕ01, ϕ02 and ϕ03 do not coincide with the roots of the polynomial.

The graph in the Figure 5 shows that in the interval between $0 \le \varphi \le 0.75$ *radians* there is a 100 times bigger error and the precision points for the two graphs shown in Figure 5 and 6 do not coincide.

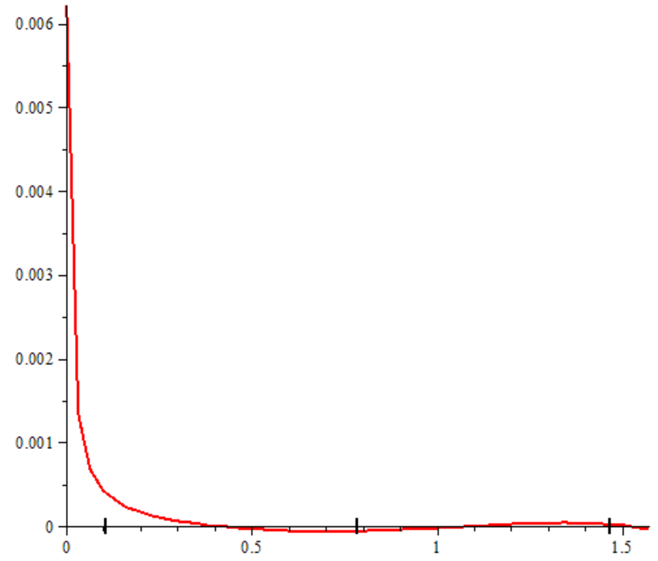

Figure 5: The structural error distribution according to Freudenstein-Chebyshev's method.

The same task, solved by the classical method of precision points spacing method, gives the graphical results shown in Figure 6. A configuration of the four-bar mechanism for the considered range of approximation is shown in Figure 7.

In such case, even though the graph of error distribution passes through Chebyshev's precision points, the polynomial does coincide with that of Chebyshev. The relative error is small but this is not the best solution in the Chebyshev sense. What are the reasons for this situation? As mentioned above, expression (11) is an even function whereby polynomial (12) is symmetrically distributed with respect to the ordinate. The symmetry imposes 6 precision points in the extended interval -90°≤φ<90°. This means that the precision points have to be calculated according to these 6 points within an extended interval, and not in the same way as it was done for the 3 ones.

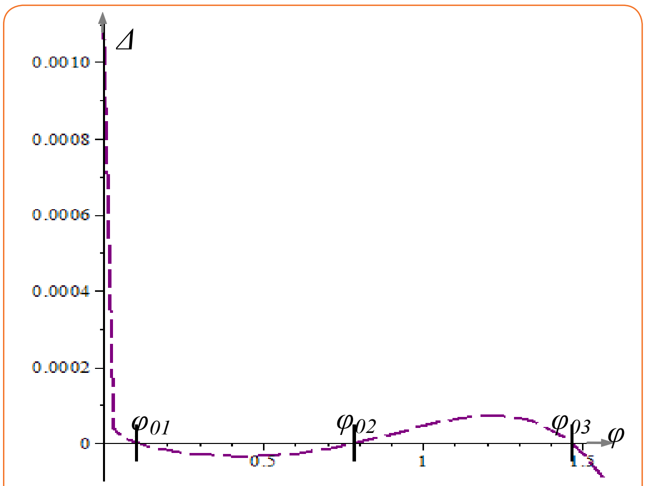

Figure 6: The distribution of the structural error, which corresponds to the Freudenstein's method of precision point spacing.

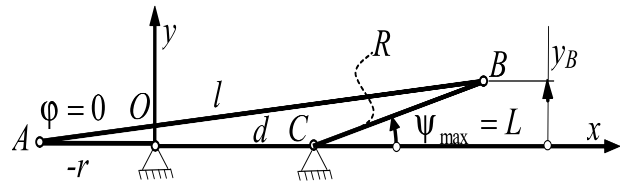

Figure 7: The so-called quasi-singular configuration of four-bar mechanism (the lengths of the links are not scaled). The link CB here almost coincide with axis x because of the very small value of yB.

The next reason that concerns the non-linear transformation of the Freudenstein-Chebyshev method results from the quasi-singular configuration of the mechanism for φ=0. It follows from formula (11) and the symmetry, that the one of maximum errors has to be in the centre of the interval -90°≤φ<90°. Because of the very small relative error for this position, the mechanism is very close to its singular configuration, which is determined by

$$d - r + R \approx l \tag{25}$$

The approximate view of this quasi-singular configuration is shown in figure 8. Because the target function at $\varphi = \upsilon$ has an extreme, it follows that $\psi(\varphi) = \psi_{\max} = L$. If the assumption is made that $R = 1$, then the ordinate of point *B* is $y_R = R \sin L$ or $y_B \approx L$. As the maximum value of the structural error is relatively small it could be assumed that point *B* is very close to the x abscise and from the engineering point of view all links almost lie on the $0x$ axe. In the ideal singular configuration, all links form a straight line. Because of the presence of a slight deviation y_B, this configuration is referred to as quasi-singular.

In the singular configuration, the velocity ratio of the mechanism is not determined and the prescribed synthesis condition cannot be satisfied.

Conclusion

The Freudenstein-Chebyshev method provides a better solution compared to the classical method of spacing of precision points, but for technical applications the difference between the two errors is negligible.

By the Freudenstein-Chebyshev method, the maximum error and the precision points can be estimated during the synthesis process. There is no need of precision point spacing.

Applying the method of Freudenstein - Chebyshev can avoid some peculiarities of synthesis in some special cases, such as in singular configuration, even or odd functions of the structural error.

As it was shown here, using the precise point spacing method may give unexpected results in some specific cases. By the method of Freudenstein-Chebyshev it is seen that the precise point spacing method fails in cases when the structural error in not a Chebyshev's polynomial.

The method can be extended to solve even problems, which involve four and five precision points involving the initial angles of the crank and rocker.

The approach described here can also be applied to other types of mechanisms and synthesis problems. There are almost no restrictions with respect to the type of the target functions as well.

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