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Exact periodic solutions of the sixth-order generalized Boussinesq equation

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Abstract

This paper examines a class of nonlinear sixth-order generalized Boussinesq-like equations (SGBE): $u_{tt} = u_{xx}+3(u^2)_{xx}+u_{xxxx}+\alpha u_{xxxxxx}$, $\alpha \in R$, depending on the positive parameter α. Hirota's bilinear transformation method is applied to the above class of non-integrable equations and exact periodic solutions have been obtained. The results confirmed the well-known nonlinear superposition principle.

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1. Introduction

In the present paper, the bilinear transformation method is used to obtain exact periodic solutions of a class of nonlinear sixth-order generalized Boussinesq-like equations (SGBE), which are not completely integrable. This is a well-known analytical technique that has been developed by Hirota [1] and subsequently applied by other authors, such as Matsuno [2] and Nakamura [3], for finding exact solutions of completely integrable nonlinear evolution equations. Apart from Hirota's direct (or bilinear transformation) method, other known methods of finding such solutions are the inverse scattering transform (IST) method [4] and the Whitham's method of representation of periodic solutions by sums of equally spaced solitons, expressed by sech-function [5]. As an alternative to the IST method, the bilinear transformation method offers a more direct approach to solving nonlinear integrable equations, avoiding the complexity of the IST method. The same technique has been successfully applied by Parker [6] to the regularized long-wave (RLW) equation, which is partially integrable. For the larger class of nonlinear equations which are non-integrable, other analytic tools have been developed as discussed in [7]: Stokes' expansions/Padé approximants, the imbricate-soliton series, as well as various numerical methods. In some recent theoretical developments for non-integrable physical systems, analytic solutions of the modulation problem arising in the time evolution of solitons, bores and shocks [8] have been obtained.

According to Ablowitz and Segur [4], if a nonlinear equation in (1+1) dimensions has a Bäcklund transformation, or a non-Abelian pseudopotential, or an N-soliton solution (perhaps $N \geqslant 3$ is sufficient), then it should be integrable. Furthermore the requirement, that a nonlinear partial differential equation (PDE) has the Painleve property (i.e. it does not possess 'movable' critical singularities), has been proposed as a necessary condition for integrability. In the context of the bilinear transformation method, if a nonlinear PDE can be reduced to the simple form of single bilinear equation $F(D_t, D_x) f \cdot f = 0$, where F is a polynomial or exponential function and D_t, D_x are the bilinear differential operators of Hirota, then this nonlinear equation has an N-soliton solution, i.e. is completely integrable.

In the case considered here, the bilinear transformation method is applied to the following class of nonlinear sixth-order partial differential equations, which are non-integrable:

$$u_{tt} = u_{xx} + 3(u^2)_{xx} + u_{xxxx} + \alpha u_{xxxxxx}, \qquad \alpha \in R, \tag{1.1}$$

where α is a positive parameter. Some generalized Boussinesq-like equations, studied by a number of authors, can be reduced to the above class of equations, using appropriate transformations of the variables $x, t, u(x, t)$.

Setting $\alpha = 0$ in (1.1), we obtain the Boussinesq equation (BE) [9]. The Boussinesq equations appear in the study of the dynamics of shallow fluid layers, the wave propagation in elastic rods and in other problems in physics. As the original BE was structurally unstable, i.e. it was linearly unstable with respect to short wavelength disturbances, predictions could not be made for large intervals of variation of its parameters. Christov *et al* [10, 11] showed that a way to make the BE mathematically correct is to retain the term containing the sixth-order spatial derivative in the approximating expansion. They derived the following linearly stable nonlinear generalized BE of sixth-order, called 6GBE, describing nonlinear atomic chains:

$$u_{tt} = \gamma^2 u_{xx} - \left(\frac{ab^2}{2}\right)(u^2)_{xx} + \beta u_{xxxx} + u_{xxxxxx}, \tag{1.2}$$

where a, b, γ and β are real parameters and $\beta > 0$. The transformations

$$x \to \frac{\sqrt{\beta}}{\gamma} x; \qquad t \to \frac{\sqrt{\beta}}{\gamma^2} t; \qquad u \to \left(-\frac{6\gamma^2}{ab^2}\right) u$$

reduce equation (1.2) to the above equation (1.1) with the parameter $\alpha = \gamma^2/\beta^2$. In the same work [10] the authors derived the nonlinear generalized equation (6GBE):

$$u_{tt} = u_{xx} + (u^2)_{xx} + \frac{\beta}{3} u_{xxxx} + \frac{2\beta^2}{15} u_{xxxxxx}, \tag{1.3}$$

which describes the dynamics of inviscid flow in shallow fluid layer. Here β is a small, positive parameter. Applying the transformations

$$x \to \sqrt{\frac{\beta}{3}} x; \qquad t \to \sqrt{\frac{\beta}{3}} t; \qquad u \to 3u$$

to equation (1.3), we again obtain equation (1.1), but with the parameter $\alpha = 6/5$.

In modeling the nonlinear lattice dynamics in elastic crystals, Maugin [12] proposed the nonlinear partial differential SGBE by considering a sixth-order term in the approximating expansion:

$$u_{tt} = u_{xx} + 6(u^2)_{xx} + u_{xxxx} + \tfrac{2}{5} u_{xxxxxx}. \tag{1.4}$$

The simple transformations, $x \to x$; $t \to t$ and $u \to u/2$, reduce equation (1.4) to (1.1) with the parameter $\alpha = 2/5$. Kawahara *et al* [13] studied the solitary waves and their interactions for the above equation (1.4). They seek traveling wave solutions numerically and find that in contrast to the BE, the SGBE admits solitary-wave solutions for a narrow range of variation of the phase velocity (i.e. very close to unity within 5%), but give no reason for this.

In the present paper we apply the bilinear transformation method to SGBE (1.1) and reduce it (see section 2) to a superposition of two equations—a bilinear equation and a residual one. Therefore, SGBE (1.1) cannot be reduced only to a single bilinear equation and the existence of such a residual equation suggests the non-integrability of (1.1). The sixth-order dispersion term is the reason for the loss of integrability which is associated with the loss of a number of properties (for example it does not possess a soliton-type solutions). The theoretical results showed that the non-integrability of SGBE (1.1) does not preclude it to possess exact periodic solutions, represented as an infinite superposition of solitary-wave profiles. Considerable technical difficulties arise in seeking analytical solutions (periodic or solitary) of the non-integrable SGBE. They are due to the need to satisfy the residual equation in the bilinear variant of SGBE. In the present work this is overcome by an appropriate choice of the displacement constant ζ_0.

2. Periodic solutions

We consider here the following nonlinear partial differential SGBE, depending on the real, positive parameter α:

$$u_{tt} - u_{xx} - 3(u^2)_{xx} - u_{xxxx} - \alpha u_{xxxxxx} = 0. \tag{2.1}$$

This equation differs from the BE by the sixth-order dispersion term. On the one hand, this term 'improves' the structural instability of BE, but on the other hand the integrability of BE (or the analytical form of the solutions) is lost. The presence of even partial derivatives in the SGBE shows that it is a two-way equation, namely, it is invariant under the transformation $t \to -t, x \to -x$. It is easily established that if $U(\theta)$, $\theta = x - ct$ is a traveling-wave solution of SGBE (2.1) with constant phase velocity c, then the function

$$u(x,t) = u_0 + U\big[x \pm \big(\sqrt{c^2 + 6u_0}\big)t\big],$$

where $u_0 =$ const, is also a solution with different phase velocity, i.e. the SGBE is invariant under the Galilean transformation. Let us seek a solution of the SGBE in the form

$$u(x,t) = \zeta_0 + 2(\mathrm{In}\,\zeta)_{xx}, \tag{2.2}$$

where $\zeta_0 =$ const, while $\zeta = \zeta(x,t) \in C^6(\Omega)$, where

$$\Omega = \{(x,t) \in R^2 : x \in R, t \geqslant 0\}.$$

As discussed in [3], Hirota has shown that through a transformation similar to the solution form (2.2), many types of nonlinear integrable equations can be reduced to a single bilinear equation (the Korteweg-de Vries (KdV) equation, the Boussinesq equation, the Model equations for shallow water waves, the Toda lattice equation, etc.)

After substituting (2.2) into (2.1) and double integration on x, we obtain the bilinear reduction of SGBE:

$$\zeta^4\big[D_t^2 - (1 + c_1(t)x)D_x^2 - D_x^4 - \alpha D_x^6 + c_1(t)x + c_2(t)\big]\zeta\cdot\zeta \\ + 15\alpha D_x^2\zeta\cdot\zeta\left[\zeta^2 D_x^4\zeta\cdot\zeta - 2\big(D_x^2\zeta\cdot\zeta\big)^2 - \frac{2\zeta_0}{5\alpha}\zeta^4 + \frac{c_1(t)x}{15\alpha}\zeta^4\right] = 0, \tag{2.3}$$

where D_t and D_x denote the bilinear Hirota operators [1], defined by the equality

$$D_t^m D_x^n \varphi(x,t)\psi(x,t) = (\partial_t - \partial_{t'})^m(\partial_x - \partial_{x'})^n \varphi(x,t)\cdot\psi(x',t')|_{t=t'}^{x=x'}$$

(see appendix A). In (2.3), $c_1(t)$ and $c_2(t)$ are unknown integration constants (which will be clarified later in the text).

It is obvious that if $\zeta(x, t)$ is a solution of the following two equations:

$$\left[D_t^2 - (1 + c_1(t)x)D_x^2 - D_x^4 - \alpha D_x^6 + c_1(t)x + c_2(t))\right] \zeta \cdot \zeta = 0 \tag{2.4}$$

and

$$\zeta^2 D_x^4 \zeta \cdot \zeta - 2\left(D_x^2 \zeta \cdot \zeta\right)^2 - \frac{\zeta^4}{5\alpha}\left[2\zeta_0 - \frac{1}{3}c_1(t)x\right] = 0, \tag{2.5}$$

then $\zeta(x, t)$ is also a solution of (2.3). The first of these two equations is called bilinear and the second is called residual.

Let us note that in the bilinear reduction of nonlinear integrable equations there are no residual equations. This particular aspect of SGBE is a complicating factor for finding analytic solution. We will seek a solution of the bilinear equation (2.4) as follows:

$$\zeta(x, t) = \theta_3(\xi, q) = \sum_{n=-\infty}^{\infty} q^{n^2} \mathrm{e}^{2\mathrm{i}\pi\xi n}, \qquad q = \mathrm{e}^{\mathrm{i}\pi\tau}, \tag{2.6}$$

where $\theta_3\,(\xi, q)$ is the third Jacobi theta function, given in appendix B, with periods 1 and π, provided that $\mathrm{Im}\tau > 0$ (i.e. $0 < |q| < 1$).

Here $\xi = kx + \omega t + \delta$ is the phase variable, with k, ω, δ arbitrary (possibly complex) parameters. If we replace ζ by $\theta_3\,(\xi, q)$ in the residual equation (2.5), it is easily found that (2.5) is fulfilled only if $c_1(t) \equiv 0$ (see [6]), since the variables x and t in the rest of the equation cannot be separated. Therefore, $c_2(t) = B$ is a constant with respect to ξ and q, and equations (2.4) and (2.5) take the final forms:

$$\left[D_t^2 - D_x^2 - D_x^4 - \alpha D_x^6 + B\right] \zeta \cdot \zeta = 0 \tag{2.7}$$

and

$$\zeta^2 D_x^4 \zeta \cdot \zeta - 2\left(D_x^2 \zeta \cdot \zeta\right)^2 - \frac{2\zeta_0}{5\alpha}\zeta^4 = 0. \tag{2.8}$$

Substituting $\zeta(x, t)$ from (2.6) into (2.7), we obtain the following equation:

$$\sum_{m=-\infty}^{\infty} G(m)\,\mathrm{e}^{2\mathrm{i}\pi\xi m} = 0, \tag{2.9}$$

where

$$G(m) = \sum_{n=-\infty}^{\infty} \{-(\pi\omega)^2(4n - m)^2 + (k\pi)^2(4n - 2m)^2 - (k\pi)^4(4n - 2m)^4 + \alpha(k\pi)^6(4n - 2m)^6 + B\}q^{n^2+(m-n)^2}. \tag{2.10}$$

Let us note that G depends only on the integer m by two intermediate arguments [3]. If in the above equality we change $n \to s + 1$, where $s \in \mathbb{Z}$, then we easily obtain the index parities:

$$G(m) = G(m-2)q^{2(m-1)} = G(m-4)q^{2(2m-4)} = \cdots = G(0)q^{m^2/2}$$

if $m \in \mathbb{Z}$ is an even number or

$$G(m) = G(m-2)q^{2(m-1)} = G(m-4)q^{2(2m-4)} = \cdots = G(1)q^{(m^2-1)/2}$$

if $m \in \mathbb{Z}$ is an odd number, which means that the equation $G(m) = 0$ is equivalent to the system $G(0) = G(1) = 0$, for each $m \in \mathbb{Z}$.

This circumstance reduces the bilinear equation (2.9) to a simple linear algebraic system with respect to the phase frequency ω and the integration const B. Applying the identities, given in appendix C, we obtain from equation (2.9) for $m = 0$ and $m = 1$ the following algebraic system in ω and B:

$$(\pi\omega)^2\theta_3' - B(\theta_3/8q) = 64\alpha(k\pi)^6(q\theta_3' + q^2\theta_3'')' - 8(k\pi)^4(q\theta_3)' + (k\pi)^2\theta_3';$$
$$(\pi\omega)^2\theta_2' - B(\theta_2/8q) = 64\alpha(k\pi)^6(q\theta_2' + q^2\theta_2'')' - 8(k\pi)^4(q\theta_2)' + (k\pi)^2\theta_2', \tag{2.11}$$

where $\theta_j = \theta_j\ (0,q^2)$, $j = 2, 3$, are the Jacobi theta functions, given in appendix B. The obtained system (2.11) is compatible and definite since

$$\Delta = \frac{\pi^2}{8q}(\theta_2'\theta_3 - \theta_2\theta_3') = -\frac{\pi^2}{8q}W(\theta_{2,}\theta_3) \neq 0,$$

where $W(\theta_{2,}\theta_3)$ is the Wronskian of θ_2 and θ_3, and hence its only solutions are

$$\omega^2 = k^2 - 8k^4\pi^2\left(1 + q\frac{\theta_1'^2(0,q)}{2W(\theta_2,\theta_3)}\right) + 64\alpha k^6\pi^4\left(1 + \frac{3q\theta_1'^2(0,q) + 2q^2W'(\theta_2',\theta_3'}{2W(\theta_2,\theta_3)}\right), \tag{2.12}$$

$$B = 64(k\pi)^4q^2\frac{W(\theta_2',\theta_3')}{W(\theta_2,\theta_3)} + 512\alpha(k\pi)^6q^2\left(\frac{3W(\theta_2',\theta_3') + qW'(\theta_2',\theta_3')}{W(\theta_2,\theta_3)}\right). \tag{2.13}$$

In the above representation we have used for convenience the following equality [14]:

$$\theta_2(0,q^2)\theta_3''(0,q^2) - \theta_2''(0,q^2)\theta_3(0,q^2) = \tfrac{1}{2}[\theta_1'(0,q)]^2,$$

where θ_1 is the first Jacobi theta function, given in appendix B. In other words, the function $\zeta = \theta_3(\xi, q)$ is an exact periodic solution of the bilinear equation (2.7) if the phase frequency ω satisfies the dispersion equation (2.12), while the value of the integration const B is chosen according to (2.13). The non-zero values of B have an important role in the periodic nature of the equation, but it will be shown that for $q \to 0$, $B \to 0$ also, which is in fact the solitary-wave limit of the periodic solutions.

The existence of the residual equation (2.8) does not allow us at this stage to identify the function $\zeta = \theta_3(\xi, q)$ as an exact periodic solution of SGBE. To do that, it is sufficient to establish whether there is a condition (or conditions) under which ζ would also satisfy the residual equation (2.8). Under the bilinear transformation procedure the constant ζ_0 from (2.2) would normally be included in the bilinear equation, but here it has been included in the residual equation, expecting it to have a balancing effect in the equation. For this purpose, let us represent ζ_0 as a formal numeric series:

$$\zeta_0 = 16(k\pi)^4\sum_{m=-\infty}^{\infty} a_m, \tag{2.14}$$

where the numbers a_m are unknown for the time being. Substituting ζ and ζ_0 from (2.6) and (2.14), respectively, into (2.8) and applying the Cauchy formula

$$\left(\sum_{\nu=-\infty}^{\infty}\alpha_\nu\right)\left(\sum_{\nu=-\infty}^{\infty}\beta_\nu\right) = \sum_{n,m=-\infty}^{\infty}\alpha_n\beta_{m-n},$$

we obtain the following infinite system of equations:

$$\sum_{m=-\infty}^{\infty}16(k\pi)^4\left\{\sum_{n=-\infty}^{\infty}n^2[n^2 - 2(3n-m)^2]q^{2n^2+(2n-m)^2} - \frac{2}{5\alpha}a_m\sum_{n=-\infty}^{\infty}q^{2n^2+(2n-m)^2}\right\}e^{2i\pi\xi m} = 0. \tag{2.15}$$

The series within the braces are convergent for the restriction $0 < |q| < 1$. Taking into account the evident identity

$$\sum_{n=-\infty}^{\infty} q^{6n^2-4mn} = \sum_{n=-\infty}^{\infty} q^{6n^2} \mathrm{e}^{2\mathrm{i}\pi(-2\tau m)n} = \theta_3(\eta, q^6),$$

where $\eta = -2\tau m$, we obtain for the vertical displacements

$$a_m = \frac{5\alpha}{4\pi^4\theta_3(\eta, q^6)}\left[3\mathrm{i}\pi m\theta_3^{(3)}(\eta, q^6) - \frac{17}{8}\theta_3^{(4)}(\eta, q^6) + m^2\pi^2\theta_3^{(2)}(\eta, q^6)\right], \tag{2.16}$$

where $m = 0, \pm 1, \pm 2, \ldots, \theta_3^{(\nu)}(\eta, q^6) = \frac{\partial^\nu \theta_3}{\partial \eta^\nu}$, $\nu = 2,3,4$.

Since (2.15) is valid for every $m \in Z$, then it follows that the series (2.14) is convergent with a_m as defined in (2.16).

Now we can get the conclusion that for an adequate definition of the numeric series (2.14) with terms a_m as in (2.16), the function $\zeta = \theta_3(\xi, q)$ can satisfy the residual equation (2.8), i.e. we obtain that the function

$$u(x,t) = 16(\pi k)^4 \sum_{m=-\infty}^{\infty} a_m + 2k^2 \frac{\mathrm{d}}{\mathrm{d}\xi}\left[\frac{\theta_3'(\xi, q)}{\theta_3(\xi, q)}\right] \tag{2.17}$$

is an exact periodic solution of SGBE provided the phase frequency ω satisfies the dispersion relation (2.12), the integration constant B takes its value from (2.13), while the terms a_m in the numeric series (2.14) are presented as in (2.16). By ($'$) in (2.17) we have denoted the derivative by the phase variable ξ.

3. Dispersion relation

Although solutions (2.12) and (2.13) for ω and B, respectively, are presented in quite a compact form, it is more convenient to use their asymptotic representations for the dispersion relation analysis:

$$(\pi\omega)^2 = (k\pi)^2 - 4(k\pi)^4\left(\frac{1 - 30q^2 + 81q^4 + \cdots}{1 - 6q^2 + 9q^4 + \cdots}\right) + 4^2\alpha(k\pi)^6\left(\frac{1 - 126q^2 + 729q^4 + \cdots}{1 - 6q^2 + 9q^4 + \cdots}\right) \tag{3.1}$$

and

$$B = \frac{384(k\pi)^2q^2}{(1 - 6q^2 + 9q^4 + \cdots)^2}[(-1 + 966q^4 - 20q^7 + \cdots) + 4\alpha(k\pi)^2(5 - 219q^4 + 360q^6 + \cdots]. \tag{3.2}$$

We can get some important conclusions about the dispersion relation and the integration constant.

First let us note that the periodic waves in SGBE are diffusive waves, since $\omega''(k) \neq 0$ (as can be seen from (3.1)) for $k \in R$, $\omega(k) \in R$, i.e. the waves' phase velocity is different from their group velocity. The wave packs consist of great number of solitary waves (as is shown in section 4), moving in both directions with group velocity $\omega'(k)$.

In the solitary-wave limit $q \to 0$ of the periodic solutions, the dispersion relation coincides with the corresponding dispersion relation of the linearized equation SGBE, but with doubled phase, i.e. if in the linearized version of equation (2.1) we set $u \sim \exp[2(\mathrm{i}kx + \mathrm{i}\omega t)]$, then we would obtain the following dispersion relation:

$$(\pi\omega)^2 = (k\pi)^2 - 4(k\pi)^4 + 16\alpha(k\pi)^6,$$

which is also generated by (3.1) when $q \to 0$. The constant B, although not having dynamic characteristics, plays a major structural role in the generation of periodic solutions. In the general case, for $0 < |q| < 1$, obviously $B \neq 0$; however, we have to note that for real wave numbers k, we could have values of $q \in (0,1)$, for which the integration constant B would be zero. For example, if we restrict the asymptotic development (3.2) to the terms of $O(q^4)$ (under the assumption that the terms greater than $O(q^5)$ are negligible), then for

$$q = \left\{ \frac{5}{6} \left[\frac{1 - 4\alpha(k\pi)^2}{161 - 146\alpha(k\pi)^2} \right] \right\}^{1/4} < 1,$$

we would have $B = 0$ for those wave numbers k, for which

$$-\frac{1}{2\pi\sqrt{\alpha}} < k < \frac{1}{2\pi\sqrt{\alpha}}.$$

Obviously, this is a prerequisite for the formation of solitary-wave solutions. These solutions are discussed in section 4. The form of (3.2) could also give us grounds to conclude that for $q \to 0$ we will have $B \to 0$ as well, which is the solitary-wave limit of the periodic waves.

4. Real periodic solutions

Generally speaking, the analytic solution of SGBE obtained in (2.17) is a complex analytic function which is periodic in the spatial variable ξ with periods $2\pi/k$ and $2\pi\tau/k$. Evidently, these periods will be real or complex depending on the choice of the arbitrary parameters k and τ. The real solutions generated by (2.17) are of physical interest. Let us choose $\tau = \mathrm{i}\varepsilon$, where, without limiting the generality, it is assumed that $\varepsilon > 0$; hence $q = \mathrm{e}^{-\pi\varepsilon}$ is a real number, such that $0 < q < 1$.

(a) The wave number k is real. The phase frequency ω is also real (see (3.1)) and hence the phase variable ξ is real or complex depending on the phase shift δ. It is obvious from (2.17) that we have a real periodic solution of SGBE, provided that δ is real. This solution can also be represented in a better analytic form by accounting for the logarithmic derivative:

$$\frac{\theta_3'(\xi, q)}{\theta_3(\xi, q)} = 4\sum_{m=1}^{\infty} \frac{(-1)^m q^m}{1 - q^{2m}} \sin(2m\xi) = 2\sum_{m=-\infty}^{\infty} \frac{(-1)^m q^m}{1 - q^{2m}} \sin(2m\xi).$$

Let us note that the Jacobi theta function θ_3 has a grid of simple poles at the points $\xi_{mn} = \left(m + \frac{1}{2}\right) + \left(n + \frac{1}{2}\right)\tau$, m, n being integers. To avoid these simple poles, we impose the condition $|\mathrm{Im}\,\xi\| < \pi\varepsilon$, i.e. $|\mathrm{Im}\,\delta| < \pi\varepsilon$. In this case, we obtain for the periodic solution

$$u(x, t) = 8k^2 \sum_{m=-\infty}^{\infty} \left[\frac{(-1)^m m q^m}{1 - q^{2m}} \cos(2m\xi) + 2\pi^4 k^2 a_m \right]. \tag{4.1}$$

We can estimate from the asymptotic representation (3.1) that the phase frequency ω will take only real values if the discriminant of the following quadratic polynomial (in the square brackets) is negative:

$$(k\pi)^2 \left[16\alpha \left(\frac{1 - 126q^2 + 729q^4}{1 - 6q^2 + 9q^4} \right) (k\pi)^4 - 4\left(\frac{1 - 30q^2 + 81q^4}{1 - 6q^2 + 9q^4} \right) (k\pi)^2 + 1 \right] = (\pi\omega)^2 \geqslant 0, \tag{4.2}$$

and under the assumption that the terms greater than $O(q^4)$ are negligible ($q = \mathrm{e}^{-\pi\varepsilon}$). In the case $\alpha = 2/5$, after some calculations we obtain the following relation for ε:

$$0.3515 < \varepsilon < 0.8740, \tag{4.3}$$

which is the condition for the generation of real periodic waves for each real value of the wave number k (in the case $\alpha = 2/5$).

(b) The wave number k is purely imaginary. Now, we will assume that $k \to \mathrm{i}k$, and without any limitations, we assume that $k > 0$. If for every choice of such a purely imaginary value of k the phase frequency ω, given by (3.1), is also imaginary, then for a suitable choice of the phase shift δ we can get again real periodic solutions. Indeed, if $\mathrm{i}\xi \to \mathrm{i}\xi + \pi\tau$ ($\tau = \mathrm{i}\varepsilon$), then by using the periodic property of the theta function $\theta_2(\mathrm{i}\xi, q) = q^{1/4}\,\mathrm{e}^{\mathrm{i}\xi}\theta_3(\mathrm{i}\xi + \pi\tau/2)$, we obtain

$$\frac{\theta_3'(\mathrm{i}\xi + \pi\tau/2, \tau)}{\theta_3(\mathrm{i}\xi + \pi\tau/2, \tau)} = \left[\frac{\theta_2'(\mathrm{i}\xi, q)}{\theta_2(\mathrm{i}\xi, q)} - \mathrm{i}\right]. \tag{4.4}$$

Using also the identity (see [15])

$$\frac{\theta_2'(z, q)}{\theta_2(z, q)} = \mathrm{i}\sum_{m=-\infty}^{\infty} \tanh[\mathrm{i}(z - m\pi\varepsilon)],$$

from (2.17) we can write

$$u(x, t) = 2k^2 \sum_{m=-\infty}^{\infty} [\mathrm{sech}^2(\xi - m\pi\varepsilon) + 8\pi^4 k^2 a_m], \tag{4.5}$$

where a_m are given by (2.16). We obtained an exact real periodic solution of SGBE, represented as an infinite sum of spatially located solitary waves, as the location of each wave being determined by the values of a_m from (2.16).

To summarize at this section, we can say that it is possible to obtain exact real periodic solutions $u(x, t)$ of the nonlinear equation SGBE employing two different approaches: by choosing the phase variable ξ to be either real or purely imaginary. In both cases real periodic solutions are generated by an appropriate choice of the phase shift δ. At first glance, the real periodic solutions obtained by the two methods, are completely different. For the solutions (4.1), we have an infinite superposition of cosinusiodal impulses with different amplitudes and increasing frequencies. On the other hand, the analytic solution (4.5) is an infinite superposition of solitary-wave profiles of constant phase, but with varying phase shifts. Actually, these two kinds of waves are dynamically equivalent. This remarkable feature of the nonlinear periodic wave to be represented as an infinite sum of solitary-wave profiles is called 'nonlinear superposition principle'. Toda [16] was the first to show that the cnoidal wave in the KdV equation can be represented as a double infinite sum of reiterating sech^2 solitary-wave profiles. A few authors have subsequently shown that a number of evolution nonlinear equations possess this property. For example, Parker [6, 15, 17] showed that in practice for the intermediate long-wave (ILW) equation there is no superposition of solitary-wave solutions in the traditional sense of the linear theory, but rather superposition of their forms. This can be explained by the circumstance that the periodic wave velocity is different from the velocity of the solitary wave satisfying the more general condition $u \to u_0$ for $\xi \to \infty$ ($u_0 = \mathrm{const}$).

Despite the fact that the SGBE considered here is not completely integrable, the obtained real periodic solution (4.5) gives grounds to consider that the nonlinear superposition principle occurs in this case, i.e. the periodic wave is an infinite superposition of solitary-wave profiles of the type: $\mathrm{sech}^2(\xi - n\pi\varepsilon) + 8\pi^4 k^2 a_n$. In this case each of the profiles stands at a different ambient level depending on a_n.

For larger values of $q \to 1$, from (4.5) and the dispersion relation obtained before, we can make the conclusion that the adjacent solitary-wave profiles have a large overlapping area and their sum forms a cosinusoidal wave with a small amplitude, i.e. for $q \to 1$ the nonlinear effects are weak and in this case linear effects predominate. When $q \to 0$, a regime occurs where nonlinear effects become dominant over the linear ones, while the adjacent solitary-wave forms become more and more differentiated since the length of the wave tends to ∞.

5. Conclusions

The bilinear transformation method has been applied here to a class of nonlinear sixth-order generalized Boussinesq-like equations (SGBE) that are not completely integrable. The exact periodic solutions obtained in this paper show the applicability of this method for some non-integrable partial differential equations. The representation of the constant of vertical displacement ζ_0 in terms of an infinite series allowed satisfying the residual equation (2.8). Furthermore it showed that unlike the obtained periodic solutions of some integrable or partially integrable equations [2, 6, 15], every periodic profile of the analytic solution (2.17) has its own vertical displacement a_m. The displacement is 'upward' for $a_m > 0$ and 'downward' for $a_m < 0$. In the above cited works the vertical displacement of the periodic profiles is the same for each wave. The balance of terms in the residual equation (2.8) by the numbers a_m becomes possible because of the same order of its terms with respect to the linearly independent functions $\exp(2i\pi\xi n)$, $n \in Z$. It is obvious that if there is such uniformity of the terms of a residual equation, then the bilinear transformation method would be applicable. We have to mention here that the above statement is made under the assumption that the residual equation is not bilinear with respect to the operators D_t and D_x. If the residual equation possesses a bilinear structure, as it is the case for the nonlinear equations RLW and RLWBE [6], then the index parity is usually applied to the individual terms.

The solution (4.5) confirms the well-known nonlinear superposition principle in mathematical physics, i.e. it represents an infinite sum of solitary waves with different spatial displacements a_m. Boyd [7] has shown that each solitary-wave profile in the sum (see (4.5)) has the same phase velocity as the periodic wave originating it, but in the general case these velocities are different. Therefore, in practice there is no real superposition in the conventional sense accepted in linear theory, but only 'imbrication' of solitary-wave forms [7, 15].

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Appendix A. Logarithmic derivatives expressed by the Hirota's bilinear differential operators D_t, D_x

$$(\ln\zeta)_{tt} = \frac{D_t^2\zeta\cdot\zeta}{2\zeta^2};\qquad (\ln\zeta)_{xx} = \frac{D_x^2\zeta\cdot\zeta}{2\zeta^2};$$

$$(\ln\zeta)_{xxxx} = \frac{D_x^4\zeta\cdot\zeta}{2\zeta^2} - 6\left(\frac{D_x^2\zeta\cdot\zeta}{2\zeta^2}\right)^2;$$

$$(\ln\zeta)_{xxxxxx} = \frac{D_x^6\zeta\cdot\zeta}{2\zeta^2} - \frac{15}{2\zeta^4}\left(D_x^2\zeta\cdot\zeta\right)\left(D_x^4\zeta\cdot\zeta\right) + \frac{15}{\zeta^6}\left(D_x^2\zeta\cdot\zeta\right)^3.$$

Appendix B. The Jacobi theta functions

$$\theta_1(\xi, q) = \mathrm{i} \sum_{n=-\infty}^{\infty} (-1)^n q^{(n-1/2)^2} \mathrm{e}^{\mathrm{i}\pi\xi(2n-1)}, \qquad \xi \in C, \quad q = \mathrm{e}^{\mathrm{i}\pi\tau}$$

$$\theta_2(\xi, q) = \sum_{n=-\infty}^{\infty} q^{(n-1/2)^2} \,\mathrm{e}^{\mathrm{i}\pi\xi(2n-1)};$$

$$\theta_3(\xi, q) = \sum_{n=-\infty}^{\infty} q^{n^2} \,\mathrm{e}^{\mathrm{i}\pi\xi 2n};$$

$$\theta_4(\xi, q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} \,\mathrm{e}^{\mathrm{i}\pi\xi 2n}.$$

Appendix C. Identities for the Jacobi theta functions

$$\sum_{n=-\infty}^{\infty} q^{2n^2} = \theta_3(0, q^2) = \theta_3; \qquad \sum_{n=-\infty}^{\infty} q^{n^2+(n-1)^2} = q^{1/2}\theta_2(0, q^2) = q^{1/2}\theta_2;$$

$$\sum_{n=-\infty}^{\infty} n^2 q^{2n^2} = q\theta_3'/2; \qquad \sum_{n=-\infty}^{\infty} (2n-1)^2 q^{n^2+(n-1)^2} = 2q^{3/2}\theta_2';$$

$$\sum_{n=-\infty}^{\infty} n^4 q^{2n^2} = q(q\theta_3')'/4; \qquad \sum_{n=-\infty}^{\infty} (2n-1)^4 q^{n^2+(n-1)^2} = 4q^{3/2}(q\theta_2')';$$

$$\sum_{n=-\infty}^{\infty} n^6 q^{2n^2} = q(q\theta_3' + q^2\theta_3'')'/8; \qquad \sum_{n=-\infty}^{\infty} (2n-1)^6 q^{n^2+(n-1)^2} = 8q^{3/2}(q\theta_2' + q^2\theta_2'')'.$$

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