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ANALYTICAL SOLUTIONS OF MULTIDIMENSIONAL FRACTIONAL-INTEGRO-DIFFERENTIAL EQUATIONS OF FEL (FREE ELECTRON LASER) TYPE

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Abstract

In the present paper analytical solutions of the fractional-integro-differential equations of FEL type are found and analysed when the three-parameter special functions of Mittag–Leffer are used as kernels. The two- and three-dimensional versions of the above equations are considered.

Key words: fractional integrals and derivatives, integro-differential equation, Kummer function, Mittag–Leffer functions, Riemann-Liouville fractional integrals and derivatives

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Introduction. The fractional order differential equations have nowadays an increasing range of applications in mathematical physics (see [5], as well as [1-3]). One of the main priorities of the fractional-differential generalization is to give an adequate way to mathematical description of the inheritance properties of various materials and processes. Such processes are described in [1] and [3] for the free electron laser equation (FEL). The kernel-functions of these integro-differential equations are the exponential and Kummer's functions respectively, while the authors in [2] use the three-parameter functions of Mittag–Leffler. By means of such kernels in the present paper the integro-differential equations of FEL type are considered and solved analytically in the two- and three-dimensional cases.

Preliminaries. The two-dimensional version of integro-differential equation of FEL type with the three-parameter function of Mittag–Leffler as kernel is represented below

$$(1)\qquad D_\tau^\alpha a(x,\tau) = \lambda \int_0^\tau (\tau-t)^{\mu-1} a(x,t) E_{\rho,\mu}^{\beta}(\omega(\tau-t)^\rho)dt + \delta f(x)\tau^{\varepsilon-1}E_{\rho,\varepsilon}^{\gamma}(\omega\tau^\rho),$$

$$(2)\qquad D_\tau^{\alpha-k}a(x,\tau)|_{\tau=0} = a_k f(x),\quad k=1,2,\dots,N,\ N-1\le\alpha<N,\ N\in N,$$

where $\lambda = -i\nu g_0$ is a complex coefficient of amplification, $0\le\tau\le1$, $\mathrm{Re}\,(\rho,\mu,\beta,\varepsilon,\gamma)>0$, $\delta\in\mathbb{R}$, $\omega=i\nu$, $a_k=\mathrm{const}\in R$, $a(x,\tau)$ – a complex amplitude of the field, $f(x)$ is given absolutely integrable function in $[0,\infty)$, and $E_{\rho,\mu}^{\beta}(z)$ is the three-parameter Mittag–Leffler function [2]

$$E_{\rho,\mu}^{\beta}(z) = \sum_{n=0}^{\infty} \frac{(\beta)_n}{\Gamma(\rho n+\mu)}.\frac{z^n}{n!},\quad z\in C,$$

and $(\beta)_n = \beta(\beta+1)\cdots(\beta+n-1)$, $(\beta)_0 = 1$ is Pochhammer symbol. Here $D_\tau^\alpha a(x,\tau)$ is Riemann–Liouville differential operator of fractional order α, defined for every $t \in [0,\tau]$ as follows:

$$D_\tau^\alpha a(x,t) = \frac{d^n}{dt^n}\left[I_t^{n-\alpha} a(x,t)\right] \;\; n = 1+[\alpha],$$

$$I_t^\alpha a(x,t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{a(x,\xi)}{(t-\xi)^{1-\alpha}} d\xi$$

and $\Gamma(\alpha)$ is Euler integral of second kind.

The three-dimensional version of the initial value problem (1)–(2) is

$$(3) \quad \sum_{k=0}^{\infty} \binom{\alpha}{k} \left(\frac{iS\nabla^2}{4}\right)^k D_\tau^\alpha b(x,y,\tau) = \lambda \int_0^\tau e^{-iS\xi\nabla^2/4} b(x,y,\tau-\xi)\xi^{\mu-1} E_{\rho,\mu}^\beta(\omega\xi^\rho) d\xi,$$

$$(4) \quad D_\tau^{\alpha-k} b(x,y,\tau)|_{\tau=0} = \sum_{j=0}^{\infty} \binom{\alpha-k}{j} b_{j+k} \left(\frac{iS}{4}\nabla^2\right) b_0(x,y), \;\; k = 1,2,\ldots,N,$$

where $S = \text{const} \in R$, $b_{j+k} = \text{const}$, $j = 0,1,\ldots,$ $k = 1,2,\ldots,N$, $b_0(x,y)$ is given absolutely integrable function in a rectangular range, $b(x,y,\tau)$ is the field amplitude, and the restrictions for the parameters α, β, ρ, μ are the same as in the two-dimensional version:

$$\nabla^2 \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2.$$

The following two theorems are important in finding the solution of the initial value problems (1)–(2) and (3)–(4).

Theorem 1. If the numbers ρ, μ, β, γ, ε, ω are such that $\mathrm{Re}\,(\rho,\mu,\varepsilon) > 0$, then

$$(5) \quad \int_0^\tau (\tau-\xi)^{\mu-1} E_{\rho,\mu}^\beta(\omega(\tau-\xi)^\rho)\xi^{\varepsilon-1} E_{\rho,\varepsilon}^\gamma(\omega\xi^\rho) d\xi = \tau^{\mu+\varepsilon-1} E_{\rho,\mu+\varepsilon}^{\beta+\gamma}(\omega\tau^\rho).$$

Theorem 2. If, $\mathrm{Re}\,(\rho,\mu,\varepsilon) > 0$ then the following equalities:

(a) $(E_{\rho,\mu,\omega}^\beta . E_{\rho,\varepsilon,\omega}^\gamma \varphi)(\tau) = (E_{\rho,\mu+\varepsilon,\omega}^{\beta+\gamma}\varphi)(\tau)$;

(b) $I_\tau^\alpha(E_{\rho,\mu,\omega}^\beta \varphi)(\tau) = (E_{\rho,\mu,\omega}^\beta I_\tau^\alpha \varphi)(\tau) = (E_{\rho,\mu,\omega}^\beta \varphi)(\tau)$;

(c) $(E_{\rho,\mu,\omega}^1 . E_{\rho,\omega}^1 \varphi)(\tau) = (E_{\rho,\mu+\varepsilon,\omega}^2 \varphi)(\tau)$,

are valid for any function $\varphi(\tau)$ which is absolutely integrable on the interval $(0,\rho]$, where

$$(6) \quad (E_{\rho,\mu,\omega}^\beta \varphi)(\tau) = \int_0^\tau (\tau-\xi)^{\mu-1} E_{\rho,\mu}^\beta(\omega(\tau-\xi)^\rho)\varphi(\xi) d\xi, \;\; \tau > 0.$$

Proofs of the statements of Theorem 1 and Theorem 2 follow directly from the definitions of the three-parameter Mittag–Leffler function and B(p,q) – the beta function (see [6]).

3. Analytical solution in the two-dimensional case. We look for a solution of the initial value problem (1)–(2) of the form

$$a(x,\tau) = A(\tau)f(x)$$

supposing that $A(\tau) \in L(0,1] = \left\{\varphi(\tau): \quad \|\varphi\| = \int\limits_0^1 |\varphi(\xi)|d\xi < \infty\right\}$.

After applying Riemann–Liouville fractional differential operator of order α and Theorem 1 to both sides of equation (1) and the initial condition (2), and using the operator, defined in (6), we get a nonhomogeneous integral equation of Volterra type [4]

$$A(\tau) = A_0(\tau) + \lambda(\mathbf{E}^{\beta}_{\rho,\mu,\omega}A)(\tau) + \delta\tau^{\alpha+\varepsilon-1}E^{\gamma}_{\rho,\alpha+\varepsilon}(\omega\tau^{\rho}), \quad \text{where} \tag{7}$$

$$A_0(\tau) = \sum_{k=1}^{N} \frac{a_k}{\Gamma(\alpha-k+1)}\tau^{\alpha-k}. \tag{8}$$

The general solution of equation (7) is the asymptotic series [4]

$$A(\tau) = A_0(\tau) + \delta\tau^{\alpha+\varepsilon-1}E^{\gamma}_{\rho,\alpha+\varepsilon}(\omega\tau^{\rho}) + \sum_{k=1}^{\infty}\lambda^k A_k(\tau), \tag{9}$$

where after applying Theorem 2, we get the following iterations:

$$\begin{aligned}
A_1(\tau) &= A_0(\tau) + \delta\tau^{\alpha+\varepsilon-1}E^{\gamma}_{\rho,\alpha+\varepsilon}(\omega\tau^{\rho}) + \lambda(\mathbf{E}^{\beta}_{\rho,\mu+\alpha,\omega}A_0(\tau);\\
A_2(\tau) &= A_0(\tau) + \left[\lambda(\mathbf{E}^{\beta}_{\rho,\mu+\alpha,\omega}A_0)(\tau) + \lambda^2(\mathbf{E}^{2\beta}_{\rho,2(\mu+\alpha),\omega}A_0(\tau)\right]\\
&+ \delta\left[\tau^{\alpha+\varepsilon-1}E^{\gamma}_{\rho,\alpha+\varepsilon}(\omega\tau^{\rho}) + \tau^{\mu+\varepsilon+2\alpha-1}E^{\beta+\gamma}_{\rho,\mu+\varepsilon+2\alpha}(\omega\tau^{\rho})\right];\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
A_n(\tau) &= A_0(\tau) + \sum_{k=1}^{n}\lambda^k(\mathbf{E}^{k\beta}_{\rho,k(\mu+\alpha),\omega}A_0)(\tau)\\
&+ \delta\sum_{j=0}^{n-1}\lambda^j\tau^{j\mu+\varepsilon+(j+1)\alpha-1}E^{j\beta+\gamma}_{\rho,j\mu+\varepsilon+(j+1)\alpha}(\omega\tau^{\rho})\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}$$

Finally we get for the amplitude $a(x,\tau)$

$$\begin{aligned}
a(x,\tau) = &\left\{\sum_{k=1}^{N} a_k\tau^{\alpha-k}\left(\sum_{j=0}^{\infty}\lambda^j\tau^{j(\mu+\alpha)}E^{j\beta}_{\rho,j(\mu+\alpha)+\alpha-k+1}(\omega\tau^{\rho})\right)\right.\\
&\left. + \delta\sum_{j=0}^{\infty}\lambda^j\tau^{j\mu+\varepsilon+(j+1)\alpha-1}E^{j\beta+\gamma}_{\rho,j\mu+\varepsilon+(j+1)\varepsilon}(\omega\tau^{\rho})\right\} f(x)
\end{aligned}. \tag{10}$$

4. Analytical solution in the three-dimensional case. Let in the three-dimensional initial value problem (3)–(4) we substitute

$$B(x,y,\tau) = e^{iS\tau\nabla^2/4}\, b(x,y,\tau). \tag{11}$$

Taking into account the fractional differentiation of the product of two functions ([5], p.96)

$$D^{\alpha}_{\tau}\left(\varphi(\tau)\psi(\tau)\right) = \sum_{k=0}^{\infty}\begin{pmatrix}\alpha\\k\end{pmatrix}\varphi^{(k)}(\tau)D^{\alpha-k}_{\tau}\psi(\tau),$$

the initial value problem (3)–(4) reduces to the following problem of the function $B(x,y,\tau)$

$$D^{\alpha}_{\tau}B(x,y,\tau) = \lambda\int\limits_0^{\tau}(\tau-t)^{\mu-1}E^{\beta}_{\rho,\mu}(\omega(\tau-t)^{\rho})B(x,y,t)dt, \tag{12}$$

(13) $$D_\tau^{\alpha-k} B\,(x,y,\tau)|_{\tau=0} = b_k b_0(x,y),\ \ k=1,2,\dots,N.$$

Let us separate the variables in problem (12)–(13), i.e.

$$B(x,y,\tau) = V(\tau) b_0(x,y),$$

where we suppose that $V(\tau) \in L(0,\tau]$, then using operator $(E^\beta_{\rho,\mu,\omega}\varphi)(\tau)$defined by (6), we get the homogeneous equation of Volterra type

(14) $$D_\tau^\alpha V(\tau) = V_0(\tau) + \lambda(E^\beta_{\rho,\mu,\omega}V)(\tau),$$

where

(15) $$V_0(\tau) = \sum_{k=1}^{N} \frac{b_k}{\Gamma(\alpha-k+1)}\tau^{\alpha-k}.$$

It is obvious that problem (14)–(15) coincides with the homogeneous version ($\delta = 0$) of equation (7). Thus, by the same reasons we get the solution of (3)–(4) in the form

(16)

$$b(x,y,\tau) = e^{-iS\tau\nabla^2/4}\left\{\sum_{k=0}^{N} b_k \tau^{\alpha-k}\left(\sum_{j=0}^{\infty}\lambda^j \tau^{j(\mu+\alpha)} E^{j\beta}_{\rho,j(\mu+\alpha)+\alpha-k+1}(\omega\tau^\rho)\right) b_0(x,y)\right\}.$$

5. Conclusions. Let us consider some important particular cases. In the one-dimensional version ($f(x) = 1$) of solution (10) when $\rho = 1$, keeping in mind that

$$E^\beta_{1,\mu}(z) = \frac{1}{\Gamma(\mu)}\Phi(\beta,\mu,z) = \frac{1}{\Gamma(\mu)}\sum_{n=0}^{\infty}\frac{(\beta)_n}{(\mu)_n}.\frac{z^n}{n!} = \Phi*(\beta,\mu,z),$$

and after some uncomplicated transformations we get

$$a(x,\tau)\Big|_{\substack{x=0\\ \rho=1}} = \sum_{k=1}^{N} b_k\tau^{\alpha-k}\left(\sum_{n=1}^{\infty}\lambda^n\tau^{n(\mu+\alpha)}\Phi*(n\beta, n(\mu+\alpha+1)+\alpha-k+1, i\nu\tau)+\right.$$
$$\left. +\ \delta\sum_{n=1}^{\infty}\lambda^n\tau^{n(\mu+1)+(n+1)\alpha+\gamma}\Phi*(n\beta+\gamma, n(\mu+1)+(\gamma+1)(n+2), i\nu\tau)\right)$$

which is the solution obtained by the authors in [3] (see (2.11), p.92). It can be shown directly that the one-dimensional version of (10) coincides with the solution of (3.3), p.386 of [2], derived by solving the fractional-differential model of FEL type equation with the three-parameter Mittag–Leffler function as a kernel.

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