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FRACTIONAL-DIFFERENTIAL MODEL OF CONVECTION STABILITY

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Abstract

In the present paper the increments characterizing the stability of a horizontal fluid layer are compared to those obtained in the fractional-differential analogue of the Oberbeck–Boussinesq system. It has been found that if the fractional parameter $\alpha \in (0, 1]$, the heated fluid layer shows a higher degree of instability than in the classic case, i.e. less stable modes are generated in the fractional-differential analogue of the convection stability than in the classic stability problem.

Key words: convective stability, Oberbeck–Boussinesq approximation, fractional derivative, Mittag–Leffler special function

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1. Introduction. The thermal convection develops as a rule in an unevenly heated fluid. The way and conditions of its heating have an essential influence on the fluid stability. It has been found that a necessary condition for the mechanical equilibrium is the linear temperature variation in z to be a linear function [1], namely

$$T = -Az + B, \quad A = \text{const}, \quad B = \text{const},$$

as usual it is assumed that $A = 1$. In the present paper a comparative analysis is made between the increments obtained in a more generalized model of the horizontal fluid layer stability problem and the increments in a fractional-differential analogue of the Oberbeck–Boussinesq system [2,3]. A basic method for studying the horizontal fluid layer stability is the method of normal perturbations where the flow can be represented as a sum of a basic (stationary) flow and a perturbed one (variation), i.e.

$$U = V_0 + V, \quad \Theta = T_0 + T,$$

where $U(U_x, U_y, U_z)$ is the velocity vector, and $\Theta(x, y, z, t)$ is the fluid temperature in the point (x, y, z) at the moment t.

2. Classical convection stability – generalized model. The Oberbeck–Boussinesq system in its classical version is given below

$$(1)\qquad \begin{cases} \dfrac{\partial}{\partial t}U = -\nabla p + \Delta U + \mathrm{Ra}\Theta k, \\ \mathrm{Pr}\,\dfrac{\partial}{\partial t}\Theta = \Delta\Theta - U\nabla\Theta, \quad \Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2, \\ \operatorname{div} U = 0, \end{cases}$$

where p is the pressure, $\mathrm{Pr} = \nu/a\tau$ – the Prandtl number, $\mathrm{Ra} = \mathrm{Pr}\, g\beta \times (T_w - T_r)d^3/\nu^2$ – the Raleigh number, $(T_w - T_r)$ – the thermal coefficient. After linearization of (1) in the vicinity of the stationary flow $(V_0 = 0, T_0)$ application of the operator

$$\operatorname{rot}\operatorname{rot} F = \nabla \times (\nabla \times F) = \operatorname{grad}(\operatorname{div} F) - \Delta F$$

to both sides of the system (1) and projection on the axis Oz, we get the following system for the bounded velocity variations $V = U_z$ and the temperature $T(x, y, z, t)$:

$$(2)\qquad \begin{cases} \dfrac{\partial}{\partial t}(\Delta V) = \Delta^2 V + \mathrm{Ra}\Delta_2 T, \quad \Delta_2 = \partial^2/\partial x^2 + \partial^2/\partial y^2, \\ \mathrm{Pr}\,\dfrac{\partial T}{\partial t} = \Delta T + V. \end{cases}$$

In the horizontal fluid layer we consider $(0 \le z \le 1)$ the initial and the boundary conditions as usual

$$(3)\qquad V = 0,\ T = \psi(x, y, z),\ \text{for } t = 0,$$

$$(4)\qquad V = 0,\ T = 0,\ \text{for } z = 0 \text{ and } z = 1.$$

To the boundary condition (4) we add the condition

$$(5)\qquad \left.\frac{\partial^2 V}{\partial z^2}\right|_{z=0} = \varphi_0(x, y, t), \quad \left.\frac{\partial^2 V}{\partial z^2}\right|_{z=1} = \varphi_1(x, y, t)$$

and both of them, as we shall see further, have a structure-determining role for the stability of the horizontal layer. Let us notice that the homogeneous version used in [4] is physically inadequate even though it simplifies the solution of the problem.

We shall suppose that the dynamic and the thermal fluid characteristics are periodic in x and y, i.e.

$$(6)\qquad V(x, y, z, t) = \zeta(r, t)\cos k_1 x \cos k_2 y, \quad T(x, y, z, t) = \Theta(z, t)\cos k_1 x \cos k_2 y,$$

$$(7)\qquad \Psi(x, y, z) = \Psi(z)\cos k_1 x \cos k_2 y, \quad \varphi_j(x, y, t) = \chi_j(t)\cos k_1 x \cos k_2 y,\ j = 0, 1,$$

where $k_1 = \text{const}$, $k_2 = \text{const}$. Taking into account (6) and (7) the boundary problem (2)–(5) reduces to the following one:

$$(8)\qquad \begin{cases} \dfrac{\partial}{\partial t}\left(\dfrac{\partial^2}{\partial z^2} - k^2\right)\zeta = \left(\dfrac{\partial^2}{\partial z^2} - k^2\right)^2 \zeta - \mathrm{Ra}k^2\Theta, \\ \mathrm{Pr}\dfrac{\partial\Theta}{\partial t} - \zeta = \left(\dfrac{\partial^2}{\partial z^2} - k^2\right)\Theta, \end{cases} \qquad \text{where } k = \sqrt{k_1^2 + k_2^2};$$

$$\zeta(r,t)|_{t=0}=0, \quad \Theta(z,t)|_{t=0}=\Psi(z), \tag{9}$$

$$\zeta=0, \quad \Theta=0, \text{ for } z=0 \text{ and } z=1, \tag{10}$$

$$\left.\frac{\partial^2\zeta}{\partial z^2}\right|_{z=0}=\chi_0(t), \quad \left.\frac{\partial^2\zeta}{\partial z^2}\right|_{z=1}=\chi_1(t). \tag{11}$$

We apply consecutively two integral transformations to the system (8)–(11): the Laplace transformation and the finite sin – Fourier transformation (in this order). For that purpose, if we put

$$Z_s(n,p)=2\int_0^1 Z(z,p)\sin n\pi z\,dz, \quad Z(z,p)=\int_0^\infty e^{-pt}\zeta(z,t)\,dt, \quad n=1,2,\ldots$$

$$\Theta_s(n,p)=2\int_0^1 \Theta_L(z,p)\sin n\pi z\,dz, \quad \Theta_L(z,p)=\int_0^\infty e^{-pt}\Theta(z,t)\,dt, \quad n=1,2,\ldots,$$

where $Rep\geqslant\sigma\geqslant\sigma_0 : |\chi_j(t)|\leqslant|M|e^{\sigma_0 t}$, $j=0,1$ then the boundary value problem (8)–(11) reduces to the nonhomogeneous algebraic system

$$\begin{cases} (\omega_n^2+k^2)(\omega_n^2+k^2+p)Z_s(n,p)-\mathrm{Ra}k^2\Theta_s=\omega_n X(n,p), \\ -Z_s+(\omega_n^2+k^2+p.\,\mathrm{Pr})\Theta_s=\mathrm{Pr}\,.\Psi_s(n), \end{cases}$$

where $\omega_n=n\pi$, $n=1,2,\ldots$

$$X(n,p)=2\left[(-1)^n L(\chi_1)-L(\chi_0)\right], \quad \Psi_\delta(n)=2\int_0^1\Psi(\xi)\sin(n\pi\xi)\,d\xi. \tag{12}$$

The solution of this algebraic system is as follows:

$$\begin{aligned} Z_s(n,p)&=\frac{\omega_n(\omega_n^2+k^2+p.\,\mathrm{Pr})X(n,p)+k^2\mathrm{Ra}\Psi_s(n)}{\mathrm{Pr}(\omega_n^2+k^2)(p-p_1)(p-p_2)}, \\ \Theta_s(n,p)&=\frac{\mathrm{Pr}(\omega_n^2+k^2)(\omega_n^2+k^2+p)\Psi_s(n)+\omega_n X(n,p)}{\mathrm{Pr}(\omega_n^2+k^2)(p-p_1)(p-p_2)}. \end{aligned} \tag{13}$$

After applying consecutively the inverse finite sin – Fourier transformation and then the inverse Laplace transformation to (13), we obtain the following solutions of (2)–(5):

$$\begin{aligned} V(x,y,z,t)={}&\cos k_1x\cos k_2y\sum_{n=1}^{\infty}\Bigg\{\frac{2k^2\mathrm{Ra}\Psi_s(n)}{(\omega_n^2+k^2)(p_1-p_2)}\left(e^{p_1t}-e^{p_2t}\right) \\ &+\frac{\omega_n(\omega_n^2+k^2+p_1\,\mathrm{Pr})}{\mathrm{Pr}(\omega_n^2+k^2)(p_1-p_2)}\int_0^t e^{p_1\tau}\chi(n,t-\tau)\,d\tau \\ &-\frac{\omega_n(\omega_n^2+k^2+p_2\,\mathrm{Pr})}{\mathrm{Pr}(\omega_n^2+k^2)(p_1-p_2)}\int_0^t e^{p_2\tau}\chi(n,t-\tau)\,d\tau\Bigg\}\sin(n\pi z), \end{aligned} \tag{14}$$

$$T(x,y,z,t) = \cos k_1 x \cos k_2 y \sum_{n=1}^{\infty} \left\{ \frac{\Psi_s(n)}{p_1 - p_2} \left[(\omega_n^2 + k^2 + p_1) e^{p_1 t} - (\omega_n^2 + k^2 + p_2) e^{p_2 t} \right] \right. \tag{15}$$
$$\left. + \frac{\omega_n}{\Pr(\omega_n^2 + k^2)(p_1 - p_2)} \int_0^t (e^{p_1 \tau} - e^{p_2 t}) \chi(n, t-\tau)\, d\tau \right\} \sin(n\pi z),$$

where $\chi(n,t)$ is the Laplace archetype of $X(n,t)$, and also

$$\left\{ \begin{array}{c} p_1 \\ p_2 \end{array} \right\} = -\frac{\Pr + 1}{\Pr} (\omega_n^2 + k^2) \pm \frac{1}{2\Pr} \sqrt{(\Pr - 1)^2 (\omega_n^2 + k^2) + \frac{4k^2 \Pr . \mathrm{Ra}}{\omega_n^2 + k^2}}. \tag{16}$$

3. Fractional-differential model. Let α is a real number so that $n-1 < \alpha \leqslant n$. The fractional-differential analogue of the boundary problem (2)–(5) is of the following type:

$$\left\{ \begin{array}{l} D_t^{\alpha}(\Delta V_{\alpha}) = \Delta^2 V_{\alpha} + \mathrm{Ra} \Delta_2 T_{\alpha}, \\ \Pr . D_t^{\alpha} T_{\alpha} - V_{\alpha} = \Delta T_{\alpha}, \end{array} \right. \tag{17}$$

$$D_t^{\alpha-1} V_{\alpha}|_{t=0} = 0, \quad D_t^{\alpha-1} T_{\alpha}|_{t=0} = \Psi_{\alpha}(x,y,z), \tag{18}$$

$$V_{\alpha} = 0, \quad T_{\alpha} = 0, \text{ for } z = 0,\ z = 1, \tag{19}$$

$$\left. \frac{\partial^2 V_{\alpha}}{\partial z^2} \right|_{z=0} = \varphi_0(x,y,t), \quad \left. \frac{\partial^2 V_{\alpha}}{\partial z^2} \right|_{z=1} = \varphi_1(x,y,t), \tag{20}$$

where $D_t^{\alpha} f(t)$ is the fractional-differential operator of CAPUTO [5]

$$D_t^{\alpha} f(t) = \left\{ \begin{array}{ll} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}}\, d\tau, & n-1 < \alpha < n, \\ \dfrac{d^n f}{dt^n}, & \alpha = n. \end{array} \right.$$

In the particular case of classical differentiation, $n = 1$. As in the classical problem (8)–(11) we suppose periodicity in x and y, i.e. that equalities (6) and (7) hold for $V_{\alpha}(x,y,z,t)$, $T_{\alpha}(x,y,z,t)$ and $\Psi_{\alpha}(x,y,z)$ and after the consecutive Laplace and finite sin – Fourier integral transformations

$$Z_{\alpha,s}(n,p) = 2 \int_0^1 Z_{\alpha}(z,p) \sin(n\pi z)\, dz, \quad Z_{\alpha}(z,p) = \int_0^{\infty} e^{-pt} \zeta_{\alpha}(z,t)\, dt,\ n = 1, 2, \ldots$$

$$\Theta_{\alpha,s}(n,p) = 2 \int_0^1 \Theta_{\alpha,L}(z,p) \sin(n\pi z)\, dz, \quad \Theta_{\alpha,L}(z,p) = \int_0^{\infty} e^{-pt} \Theta_{\alpha}(z,t)\, dt,\ n = 1, 2, \ldots$$

the system (17)–(20) reduces to the following algebraic system:

$$\begin{cases} (\omega_n^2+k^2)(\omega_n^2+k^2+p^\alpha)Z_{\alpha,s} - k^2\mathrm{Ra}\Theta_{\alpha,s} = X(n,p)\omega_n, \\ -Z_{\alpha,s} + (\omega_n^2+k^2+p^\alpha\,\mathrm{Pr})\Theta_{\alpha,s} = \mathrm{Pr}\,\Psi_{\alpha,s}(n), \end{cases}$$

the solutions of which for $p_1 \neq p^\alpha$, $p_2 \neq p^\alpha$ are

$$Z_{\alpha,s}(n,p) = \frac{\omega_n(\omega_n^2+k^2+p^\alpha\,\mathrm{Pr})X(n,p)+\mathrm{Ra}k^2\Psi_{\alpha,s}}{\mathrm{Pr}(\omega_n^2+k^2)(p^\alpha-p_1)(p^\alpha-p_2)},$$

$$\Theta_{\alpha,s}(n,p) = \frac{\mathrm{Pr}(\omega_n^2+k^2)(\omega_n^2+k^2+p^\alpha)\Psi_{\alpha,s}+\omega_n X(n,p)}{\mathrm{Pr}(\omega_n^2+k^2)(p^\alpha-p_1)(p^\alpha-p_2)}.$$

Applying the inverse finite sin – Fourier transformation and the inverse Laplace transformation, we obtain the solutions

(21)

$$V_\alpha(x,y,z,t) = \cos k_1x\cos k_2y\sum_{n=1}^{\infty}\Bigg\{\frac{2k^2\mathrm{Ra}\Psi_{\alpha,s}(n)}{(\omega_n^2+k^2)(p_1-p_2)}t^{\alpha-1}[E_{\alpha,\alpha}(p_1t^\alpha)-E_{\alpha,\alpha}(p_2t^\alpha)]$$
$$+\frac{\omega_n(\omega_n^2+k^2+p_1)}{\mathrm{Pr}(\omega_n^2+k^2)(p_1-p_2)}\int_0^t\tau^{\alpha-1}E_{\alpha,\alpha}(p_1\tau^\alpha)\chi(n,t-\tau)\,d\tau$$
$$-\frac{\omega_n(\omega_n^2+k^2+p_2)}{\mathrm{Pr}(\omega_n^2+k^2)(p_1-p_2)}\int_0^t\tau^{\alpha-1}E_{\alpha,\alpha}(p_1\tau^\alpha)\chi(n,t-\tau)\,d\tau\Bigg\}\sin(n\pi z)$$

(22)

$$T_\alpha(x,y,z,t) = \cos k_1x\cos k_2y\sum_{n=1}^{\infty}\Bigg\{\frac{\Psi_{\alpha,s}(n)}{p_1-p_2}t^{\alpha-1}\,[(\omega_n^2+k^2+p_1)E_{\alpha,\alpha}(p_1t^\alpha)$$
$$-(\omega_n^2+k^2+p_2)E_{\alpha,\alpha}(p_2t^\alpha)]$$
$$+\frac{\omega_n}{\mathrm{Pr}(\omega_n^2+k^2)(p_1-p_2)}\int_0^t\tau^{\alpha-1}[E_{\alpha,\alpha}(p_1\tau^\alpha)-E_{\alpha,\alpha}(p_2\tau^\alpha)]\chi(n,t-\tau)\,d\tau\Bigg\}\sin(n\pi z),$$

where $\Psi_{\alpha,s}(n) = 2\int_0^1\Psi_\alpha(\xi)\sin n\pi\xi\,d\xi$, $\Psi_\alpha(z) = \Psi_\alpha(x,y,z)/\cos k_1x\cos k_2y$, and $E_{\alpha,\beta}(t)$ is the two-parameter function of Mittag–Leffler [7]

$$E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty}\frac{t^k}{\Gamma(\alpha k+\beta)},\quad \alpha>0,\ \beta>0$$

and $\Gamma(z)$ is the Euler integral of second kind.

4. Conclusions. First, let us note that for $\alpha = 1$ the solutions (14) and (15) of the boundary problem (2)–(5) are identical with the solutions (21) and (22) of the fractional-differential system (17)–(20). This is based on the equalities

$$E_{\alpha,\alpha}(p_j t^\alpha)|_{\alpha=1} = \sum_{n=0}^{\infty} \frac{(p_j t)^n}{\Gamma(n+1)} = \sum_{n=0}^{\infty} \frac{(p_j t)^n}{n!} = e^{p_j t} \quad j = 0, 1$$

as well as on the fact that $\Psi_{1,s}(n) = \Psi_s(n)$. The state of mechanical equilibrium is stable or unstable depending on how the variations $V(x,y,z,t)$ and $T(x,y,z,t)$ behave for $t \to \infty$ (in the classic version) or $V_\alpha(x,y,z,t)$ and $T_\alpha(x,y,z,t)$ in the fractional-differential variant. In the classic version, if the increments are negative and the boundary condition (5) is homogeneous (i.e. $\chi(n,t) = 0$), then the state of the fluid layer is stable obviously since $\Psi_s(n) \sim 0(1/n^2)$ (see [6]). According to (16) the negative increments $Rep_1 < 0$ and $Rep_2 < 0$ arise for Ra < 0 and sufficiently high values of $|\mathrm{Ra}|$, but even in this case a structural change could occur in stability, if $\chi(n,t) \sim e^{bt}$, $b > 0$. The reason is that the terms in (14) and (15) containing integrals, in this case are of order $(e^{bt} - e^{p_j t})/(b - p_j)$, $j = 1, 2$ which shows that the boundary condition (5) is structure-defining for the fluid layer stability (instability).

In the fractional-differential version the role of the increments is taken by the Mittag-Leffler functions. The behaviour of these functions [7,8] is the reason for making the conclusion that for $n - 1 < \alpha \leqslant n$ and $\chi(n,t) \sim e^{bt}$, $b \leqslant Rep_j$ and $Rep_j > 1$ (instability) in the fractional-differential case the fluid layer shows a higher degree of instability than in the classical one, because $Re(p_j^{1/\alpha}) > Re(p_j) > 0$. While in the stability case: $Rep_j < 0$ and $\chi(n,t) \sim e^{bt}$, $b \leqslant Rep_j$ we have $Re(p_j/\alpha) \leqslant Rep_j$, $j = 1, 2$ which means that the fractional-differential version of convection stability generates less stable modes than in the classic case (14) and (15).

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