Generalized Problem of Cauchy for a Definite Class of Singularly Perturbed Systems of Ordinary Differential Equations with Impulse Effects

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Abstract

An initial value problem for linear and almost nonlinear singularly perturbed systems with generalized impulse actions was considered. The asymptotic expansion was constructed by boundary functions using generalized inverse matrix.

1. Linear singularly perturbed system

1.1. Statement of a problem

A singularly perturbed system

$$\varepsilon\frac{dx}{dt} = Ax + \varepsilon A_1(t)x + \varphi(t), \tag{1}$$
$$t \in [a,b],\ t \neq \tau_i,\ i = \overline{1,p},\ 0 < \varepsilon \ll 1$$
$$a \equiv \tau_0 < \tau_1 < \cdots < \tau_p < \tau_{p+1} \equiv b,$$

is considered under the following conditions:

H1: A is $n \times n$ matrix, with constant elements and its eigenvalues have a negative real parts, i.e., $\lambda_i \in \sigma(A)$, $Re\lambda_i < 0, i = \overline{1,n}$.

H2: $A_1(t)$ is $n \times n$ matrix, which elements are continuously differentiable functions of class $C^\infty[a,b]$.

H3: $\varphi(t)\ :\ [a,b] \to \mathbf{R}^n$ is a partially continuous n-dimensional vector-function, which has breaks of first kind in the points τ_i, $i = \overline{1,p}$, i.e.,

$$\varphi(t) = \varphi_i(t),\quad t \in (\tau_{i-1}, \tau_i],\quad i = \overline{1,p+1},$$
$$\varphi(a) = \varphi_1(\tau_0),\qquad \varphi(b) = \varphi_{p+1}(\tau_{p+1}),$$
$$\varphi_{i+1}(\tau_i) = \lim_{t\to\tau_i+0} \varphi(t),\quad i = \overline{1,p}.$$

An n-dimensional vector-function $x(t,\varepsilon)$, which is partially continuous with respect to the first argument and continuous with respect to the second argument, when $\varepsilon \in (0,\varepsilon_0]$, i.e., $x((\cdot),\varepsilon) \in C((a,b]\backslash\{\tau_1,...,\tau_p\})$, $x(t,(\cdot)) \in C(0,\varepsilon_0]$ is sought. The vector-function $x(t,\varepsilon)$ also satisfies the system (1), the generalized condition of Cauchy

$$Dx(a) = v \tag{2}$$

where D is given $s \times n$ matrix with constant elements, v is given column vector with s components and the generalized impulse conditions in the fixed moments of time

$$N_i x(\tau_i + 0) + M_i x(\tau_i - 0) = h_i,\ i = \overline{1,p}. \tag{3}$$

The matrices M_i and N_i, $i = \overline{1,p}$, satisfy the condition:

H4: M_i, N_i, $i = \overline{1,p}$ are known $k_i \times n$-dimensional matrices with constant elements, $h_i \in \mathbf{R}^{k_i}$ are known column vectors.

A degenerate system

$$Ax + \varphi(t) = 0$$

is obtained at $\varepsilon = 0$ in (1). It has an unique solution

$$x_0(t) = -A^{-1}\varphi(t). \tag{4}$$

The vector-function $x(t,\varepsilon)$ will be seeking on this way that a next limit is true $\lim_{\varepsilon\to 0} x(t,\varepsilon) = x_0(t)$, $t \in (a,b]\backslash\{\tau_1,\dots,\tau_p\}$.

The basic methods for research of linear and nonlinear impulse systems for ordinary differential equations are described in the monographies [11], [4].

Singularly perturbed systems of the form

$$\frac{dx}{dt} = f(t,x,y),\quad \varepsilon\frac{dy}{dt} = g(t,x,y) \tag{5}$$

with initial condition $x(a) = v$, $(D \equiv E)$ are considered in [12], [13].

Results from [12] are applied in [1], [2] for the system (5) with impulse conditions of the form

$$\Delta x|_{t=\tau_i} = S_i x + a_i,\ i = \overline{1,p}. \tag{6}$$

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A fundamental matrix of the solutions of impulse system

$$\frac{dx}{dt} = A(t)x, \quad t \neq \tau_i, \quad \Delta x|_{t=\tau_i} = S_i x, \quad i = \overline{1,p}$$

where $det(S_i + E) \neq 0$ is essential in [11]. With its help the solution of the system (5), (6) is constructed in [1].

In this paper additional conditions about the matrices M_i, N_i are not put, because of the generalized impulse conditions (3). This shows that the fundamental matrix from [11] can not be used in this case.

A boundary functions introduced in [12], [13] are used in this work. A formally asymptotic solution of the problem (1), (2), (3) in the interval $(a,b]\backslash\{\tau_1,\cdots,\tau_p\}$ is constructed by the method using these boundary functions. This is done by means of generalized inverse matrices and projectors [6], [8], [9], [10].

Diverse questions connected to the impulse systems can be seen in [7].

1.2. Formally asymptotic expansion

A formally asymptotic expansion of the solution of the problem (1), (2), (3) is sought in the form

$$x(t,\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k (x_k^i(t) + \Pi_k^i(\nu_i)), \qquad (7)$$

$$\nu_i = \frac{t-\tau_{i-1}}{\varepsilon}, \quad i = \overline{1,p+1}$$

where $x_k(t)$ are elements of the regular series and $\Pi_k^i(\nu_i)$ are boundary functions in a right neighbourhood of the points τ_{i-1}, $i = \overline{1,p+1}$.

Formally (7) is substituted in the system (1), the function $A_1(t)$ is expanded in the Taylor series in the neighbourhood of the points $t = \tau_{i-1}$, $i = \overline{1,p+1}$. The variables are separated with respect to t and ν_i and the coefficients before the same powers of ε are equalized, then the elements of the regular series are obtained

$$x_k^i(t) = \begin{cases} -A^{-1}\varphi_i(t), & k=0 \\ A^{-1}L[x_{k-1}^i(t)], & k=1,2,... \end{cases} \qquad (8)$$

where L is operator of the form $L[x] = \dfrac{dx}{dt} - A_1(t)x$, $i = \overline{1,p+1}$.

For the coefficients of the singular series the following systems are obtained

$$\frac{d\Pi_k^i(\nu_i)}{d\nu_i} = A\Pi_k^i(\nu_i) + f_k^i(\nu_i), \qquad (9)$$

$$k = 0,1,2,\cdots, \quad i = \overline{1,p+1}$$

where the functions $f_k^i(\nu_i)$ have the form

$$f_k^i(\nu_i) = \begin{cases} 0, & k=0 \\ \sum_{j=0}^{k-1} \dfrac{A_1^{(k-j-1)}(\tau_{i-1})}{(k-j-1)!}\nu_i^{k-j-1}\Pi_j^i(\nu_i), & k=1,2,... \end{cases} \qquad (10)$$

The solutions of the systems (9) are sought in the form

$$\Pi_k^i(\nu_i) = X(\nu_i)c_k^i + \int_0^{\nu_i} X(\nu_i)X^{-1}(s)f_k^i(s)ds, \qquad (11)$$

$$i = \overline{1,p+1}, \quad k = 0,1,\ldots,$$

c_k^i are unknown constant vectors from $\mathbf{R}^n$, $X(\nu_i)$ is normal fundamental matrix of the system $\frac{dX}{dt} = AX$, $X(0) = E_n$.

Let the following condition is fulfilled:

H5: $rankD = m \leq min(s,n)$.

By D^+ is denoted an unique Moore-Penrose inverse matrix of the matrix D, by P_D and P_{D^*} – matrices orthoprojectors $P_D : \mathbf{R}^n \to \ker(D)$, $P_{D^*} : \mathbf{R}^n \to \ker(D^*)$ where $D^* \equiv D^T$.

From (2) is found

$$x(a) = P_D\xi + D^+v, \quad \xi \in \mathbf{R}^n, \qquad (12)$$

if and only if

H6: $P_{D^*}v = 0$.

From (10) and (11) when $\nu_1 = 0$ is obtained $\Pi_k^1(0) = c_k^1$. Keeping in mind (7) and (12) when $t \in (\tau_0, \tau_1]$ is found

$$\sum_{k=0}^{\infty} \varepsilon^k [x_k^1(a) + c_k^1] = P_D\xi + D^+v, \quad \xi \in \mathbf{R}^n. \qquad (13)$$

In (13) the coefficients before the same powers of ε are equalized. Then the constant vectors c_k^1, $k = 0,1,2,\cdots$ are obtained

$$c_0^1 = P_D\xi + D^+v - x_0^1(a), \quad c_k^1 = -x_k^1(a), \quad k = 1,2,...$$

Thus when $t \in (\tau_0, \tau_1]$ the boundary functions have the form

$$\begin{aligned} \Pi_0^1(\nu_1) &= X(\nu_1)[P_D\xi + D^+v - x_0^1(a)], \\ \Pi_k^1(\nu_1) &= -X(\nu_1)x_k^1(a) \\ &+ \int_0^{\nu_1} X(\nu_1)X^{-1}(s)f_k^1(s)ds. \end{aligned} \qquad (14)$$

On this way the solution in the first interval depends on an arbitrary n-dimensional vector ξ and this dependence is done by the functions $f_k^1(\nu_1)$ from (10).

Let $t \in (\tau_{i-1}, \tau_i]$ and the next condition is satisfied:

H7: $rankN_i = m_i \le min(k_i, n), \quad i = \overline{1,p}$.

By N_i^+ is denoted an unique Moore-Penrose inverse matrix of the matrix $N_i, i = \overline{1,p}$, by P_{N_i} and $P_{N_i^*}$ –matrices orthoprojectors $P_{N_i} : \mathbf{R}^n \to \ker(N_i)$, $P_{N_i^*} : \mathbf{R}^{k_i} \to \ker(N_i^*)$ where $N_i^* \equiv N_i^T$.

The series (7) is substituted in the impulse conditions (3), when $t \in (\tau_{i-1}, \tau_i], \quad i = \overline{2, p+1}$

$$M_{i-1}x_{i-1}(\tau_{i-1}, \varepsilon) + N_{i-1}x_i(\tau_{i-1}, \varepsilon) = h_{i-1}, \; i = \overline{2, p+1}. \quad (15)$$

The functions $\Pi_k^{i-1}\Big(\frac{\tau_{i-1} - \tau_{i-2}}{\varepsilon}\Big)$ are exponentially small and it is possible to reject, because of vanishing of the boundary functions, $\Pi_k^i(0) = c_k^i$. Then (15) obtains the form

$$M_{i-1}\sum_{k=0}^{\infty}\varepsilon^k x_k^{i-1}(\tau_{i-1}) + N_{i-1}\sum_{k=0}^{\infty}\varepsilon^k(x_k^i(\tau_{i-1}) + c_k^i) = h_{i-1}, \; i = \overline{2, p+1}. \quad (16)$$

The system (16) is solved with respect to $\sum_{k=0}^{\infty}\varepsilon^k(x_k^i(\tau_{i-1}) + c_k^i)$

$$\sum_{k=0}^{\infty}\varepsilon^k(x_k^i(\tau_{i-1}) + c_k^i) = P_{N_{i-1}}\eta_i + N_{i-1}^+\Big(h_{i-1} - M_{i-1}\sum_{k=0}^{\infty}\varepsilon^k x_k^{i-1}(\tau_{i-1})\Big), \; i = \overline{2, p+1}, \quad (17)$$

if and only if

H8:

$$P_{N_{i-1}^*}\Big(h_{i-1} - M_{i-1}\sum_{k=0}^{\infty}\varepsilon^k x_k^{i-1}(\tau_{i-1})\Big) = 0, \; i = \overline{2, p+1}$$

$$\Rightarrow \begin{cases} P_{N_{i-1}^*}(h_{i-1} - M_{i-1}x_0^{i-1}(\tau_{i-1})) = 0, & k = 0 \\ P_{N_{i-1}^*}M_{i-1}x_k^{i-1}(\tau_{i-1})) = 0, & k = 1, 2, \ldots \end{cases}$$

The coefficients before identical powers of ε are equalized in (17), then

$$\begin{aligned} c_0^i =& P_{N_{i-1}}\eta_i + N_{i-1}^+(h_{i-1} - M_{i-1}x_0^{i-1}(\tau_{i-1})) \\ & - x_0^i(\tau_{i-1}), \; \eta_i \in \mathbf{R}^n, \\ c_k^i =& - N_{i-1}^+M_{i-1}x_k^{i-1}(\tau_{i-1}) - x_k^i(\tau_{i-1}), \\ & k = 1, 2, \ldots, \; i = \overline{2, p+1}. \end{aligned}$$

The boundary functions when $t \in (\tau_{i-1}, \tau_i]$ are the following

$$\Pi_0^i(\nu_i) = X(\nu_i)[P_{N_{i-1}}\eta_i + N_{i-1}^+(h_{i-1} - M_{i-1}x_0^{i-1}(\tau_{i-1})) - x_0^i(\tau_{i-1})],$$

$$\Pi_k^i(\nu_i) = -X\Big(\nu_i)(N_{i-1}^+M_{i-1}x_k^{i-1}(\tau_{i-1}) + x_k^i(\tau_{i-1})|Big) + \int_0^{\nu_i} X(\nu_i)X^{-1}(s)f_k^i(s)ds, \quad (18)$$

$$k = 1, 2, \ldots, \; i = \overline{2, p+1}.$$

Thus the solution $x_i(t, \varepsilon)$, when $t \in (\tau_{i-1}, \tau_i]$, $i = \overline{2, p+1}$ is defined completely and it depends on an arbitrary n-dimensional constant vector η_i.

Theorem 1 *Let the conditions H1-H5 and H7 are fulfilled. Initial value problem with impulse effects (1),(2), (3) has formally asymptotic solution of the form (7). The coefficients of the regular series have the representation (8) for each interval $(\tau_{i-1}, \tau_i]$, $i = \overline{1, p+1}$. The boundary functions depend on an arbitrary n-dimensional constant vector and have the representation (14) when $t \in (\tau_0, \tau_1]$ and (18) when $t \in (\tau_{i-1}, \tau_i]$, $i = \overline{2, p+1}$, if and only if v, h_i, $i = \overline{1,p}$ and $\varphi(t)$ satisfy H6 and H8. An inequalities*

$$\|\Pi_k^i(\nu_i)\| \le \sigma\exp(-\kappa\nu_i), \; i = \overline{1, p+1}, \; k = 0, 1, \ldots$$

are fulfilled for the boundary functions where σ and κ are positive constants.

The above exposition shows that the coefficients of the expansion (7) really have the form (8), (14) and (18). The exponentially decreasing of the boundary functions is proved analogously to the same proof in [12].

Remark 1: Let $rankD = n$ $(n < s)$. Then the coefficients of the expansion (7) in the first interval $t \in (\tau_0, \tau_1]$ have an unique representation (8), (14), and in the other intervals - parametric solution of the form (8), (18).

Remark 2: Let for some $i = l$ $rankN_l = n$ $(n < k_l)$. Then the coefficients of the expansion (7) in the interval $t \in (\tau_{l-1}, \tau_l]$ have an unique representation (8), (18) and in the other intervals - parametric solution of the form (8), (14) and (8), (18).

Remark 3: If D is square matrix, which has a full rank, then the problem (1), (2), (3) has an unique solution in the first interval and the condition H6 is always real. If $detD = 0$, then the solution of the problem is constructed analogously to the way when D is rectangular matrix.

Remark 4: If some of the matrices N_i is square and has a full rank, then the problem (1), (2), (3) has an unique solution in the corresponding interval $(\tau_{i-1}, \tau_i]$ and the condition H8 is always real. If some of the matrices N_i is square, but $det\, N_i = 0$ then this case is analogous to this when N_i is rectangular matrix.

Remark 5: The solution in each interval depends on an arbitrary n-dimensional constant vector. These vectors do not depend on each of other as a problem considered in [5]. Therefore it is not necessary to do modification of the problem done in [5].

1.3. Bound of the remainder term of the asymptotic series

Let

$$u(t,\varepsilon) = x(t,\varepsilon) - X_n(t,\varepsilon) \qquad (19)$$

where $x(t,\varepsilon)$ is the solution of (1), (2), (3) and

$$X_n(t,\varepsilon) = \sum_{k=0}^{n} \varepsilon^k (x_k^i(t) + \Pi_k^i x(\nu_i)),\ i = \overline{1,p+1}$$

is its asymptotic expansion. It will be showed that $\|u(t,\varepsilon)\| \le c\varepsilon^{n+1}$, under some conditions. It will be used a scheme of proof from [12], [1], but will be done some changes, which are enforced because of generalized initial and impulse conditions.

It is substituted (19) in (1–3) and it is found, that the remainder term $u(t,\varepsilon)$ satisfies initial value problem with impulses

$$\varepsilon\frac{du}{dt} = Au + G(t,u,\varepsilon),\ u(a,\varepsilon) = 0,$$
$$N_i u(\tau_i+0) + M_i u(\tau_i - 0) = 0,\ i = \overline{1,p} \qquad (20)$$

where $u(t,\varepsilon) = u_i(t,\varepsilon) = x_i(t,\varepsilon) - X_n^i(t,\varepsilon),\quad t \in (\tau_{i-1},\tau_i],\quad i = \overline{1,p+1}$. The function $G(t,u,\varepsilon)$ has the form $G(t,u,\varepsilon) = AX_n(t,\varepsilon) + \varepsilon A_1(t)[u + X_n(t,\varepsilon)] + \varphi(t) - \varepsilon\dfrac{dX_n(t,\varepsilon)}{dt}$ and poses next two properties:

(1): $\|G(t,0,\varepsilon)\| \le c_1\varepsilon^{n+1}$ where c_1 is a positive constant.

(2): For every $\eta > 0$ exist $\delta = \delta(\eta)$ and $\varepsilon_0 = \varepsilon_0(\eta)$ such that if

$\|u'\| \le \delta,\ \ \|u''\| \le \delta$, then it is fulfilled $\|G(t,u',\varepsilon) - G(t,u'',\varepsilon)\| \le \eta\|u' - u''\|$.

Let initial value problem with impulses

$$\varepsilon\frac{dx}{dt} = Ax + f(t),\ t\in[a,b],\ t \ne \tau_i,\ 0<\varepsilon \ll 1 \qquad (21)$$

$$x(a) = x_0 \qquad (22)$$

$$M_i x(\tau_i - 0) + N_i x(\tau_i + 0) = h_i,\ i = \overline{1,p}, \qquad (23)$$

$$a \equiv \tau_0 < \tau_1 < \cdots < \tau_p < \tau_{p+1} \equiv b$$

be given where the coefficients of the problem satisfy the conditions H1, H3, H4.

The impulse condition (23) is rewritten in the form [5]:

$$\sum_{i=1}^{p+1} l_i x_i(\cdot,\varepsilon) = H \qquad (24)$$

where

$$l_1x_1(\cdot,\varepsilon) = [M_1\Theta_2\cdots\Theta_p]^T x_1(\tau_1,\varepsilon),$$
$$l_ix_i(\cdot,\varepsilon) = [\Theta_1\cdots\Theta_{i-1}M_i\Theta_{i+1}\cdots\Theta_p]^T x_i(\tau_i,\varepsilon)$$
$$+ [\Theta_1\cdots\Theta_{i-2}N_{i-1}\Theta_i\cdots\Theta_p]^T x_i(\tau_{i-1},\varepsilon),$$
$$i = 2,\dots,p,$$
$$l_{p+1}x_{p+1}(\cdot,\varepsilon) = [\Theta_1\cdots\Theta_{p-1}N_p]^T x_{p+1}(\tau_p,\varepsilon),$$
$$H = [h_1h_2\cdots h_p]^T,$$

and $\Theta_i,\ \ i = 2,\cdots,p$ are $k_i \times n$ matrices with zero elements and matrices M_i, N_{i-1} take up i-th and $(i-1)$-th blocks, respectively. Let the following denotations be introduced $Q_i(\varepsilon) = l_i(W(\cdot,\tau_{i-1},\varepsilon)) - k_i \times n$-matrices, $i = \overline{2,p+1}$,

$$Q(\varepsilon) = [Q_2(\varepsilon)Q_3(\varepsilon)\cdots Q_{p+1}(\varepsilon)]_{(\nu\times np)},$$
$$\nu = (k_1 + k_2 + \cdots + k_p),$$
$$\overline{x} = \Big[x_2(\tau_1,\varepsilon)x_3(\tau_2,\varepsilon)\cdots x_{p+1}(\tau_p,\varepsilon)\Big]^T_{np\times 1},$$
$$\overline{h}(\varepsilon) = H - \sum_{i=1}^{p+1} l_i\Big(\int_{\tau_{i-1}}^{(\cdot)} W(\cdot,s,\varepsilon)\frac{1}{\varepsilon}f(s)ds\Big) \qquad (25)$$

where $W(t,s,\varepsilon)$ is fundamental matrix of the solutions of the system

$$\varepsilon\frac{dW}{dt} = AW,\ \ W(s,s,\varepsilon) = E_n. \qquad (26)$$

Let the matrix $Q(\varepsilon)$ has the following structure $Q(\varepsilon) = Q_0 + O(\varepsilon^q\exp(-\frac{\alpha}{\varepsilon})),\ \ Q_{0(\nu\times np)} = const,\ q\in \mathbf{N}$ and the next conditions are fulfilled:

H9: $rank Q_0 = q_0 < min(\nu, np)$,

H10: $P_{Q_0^*}\overline{h}(\varepsilon) = 0$.

By Q_0^+ is denoted an unique Moore-Penrose inverse matrix of the matrix Q_0, and by P_{Q_0}, $P_{Q_0^*}$ – orthoprojectors $P_{Q_0} : \mathbf{R}^{pn} \to \ker(Q_0)$, $P_{Q_0^*} : \mathbf{R}^{\nu} \to \ker(Q_0^*)$, $Q_0^* \equiv Q_0^T$.

Lemma 1 *Initial value problem with impulse effects (21 – 23) under condition H9 has solution of the form*

$$x_1(t,\varepsilon) = W(t,\tau_0,\varepsilon)x_0 + \int_{\tau_0}^{t} W(t,s,\varepsilon)\frac{1}{\varepsilon}f(s)ds,$$
$$t\in(\tau_0,\tau_1]$$
$$x_i(t,\varepsilon) = W(t,\tau_{i-1},\varepsilon)\left[[P_{Q_0}]_{n_{i-1}}\overline{\eta} + [Q_0^+\overline{h}(\varepsilon)]_{n_{i-1}}\right]$$
$$+ \int_{\tau_{i-1}}^{t} W(t,s,\varepsilon)\frac{1}{\varepsilon}f(s)ds, \qquad (27)$$

$t\in(\tau_{i-1},\tau_i]$ *if and only if the conditions H10 is fulfilled.*

Proof: Let $t \in (\tau_0, \tau_1]$. The solution $x_1(t,\varepsilon)$ of (21)–(23) satisfies the next system

$$\varepsilon \frac{dx_1}{dt} = Ax_1 + f(t), \;\; x_1(\tau_0, \varepsilon) = x_0,$$

which using the Cauchy formula has the form

$$x_1(t,\varepsilon) = W(t,\tau_0,\varepsilon)x_0 + \int_{\tau_0}^{t} W(t,s,\varepsilon)\frac{1}{\varepsilon}f(s)ds.$$

When $t \in (\tau_{i-1}, \tau_i]$, $i = \overline{2, p+1}$ the solution $x_i(t,\varepsilon)$ of (21)–(23) is sought in the form

$$x_i(t,\varepsilon) = W(t,\tau_{i-1},\varepsilon)x_i(\tau_{i-1},\varepsilon) + \int_{\tau_{i-1}}^{t} W(t,s,\varepsilon)\frac{1}{\varepsilon}f(s)ds. \quad (28)$$

The unknown constant vectors $x_i(\tau_{i-1},\varepsilon)$ are solutions of the system

$$\sum_{i=2}^{p+1} l_i(W(\cdot,\tau_{i-1},\varepsilon))x_i(\tau_{i-1},\varepsilon) = H - \sum_{i=1}^{p+1} l_i\left(\int_{\tau_{i-1}}^{(\cdot)} W(\cdot,s,\varepsilon)\frac{1}{\varepsilon}f(s)ds\right),$$

which is obtained when (28) is substituted in (24). According to (25) the last equality takes the form

$$Q(\varepsilon)\overline{x} = \overline{h}(\varepsilon). \quad (29)$$

The exponentially small elements in the matrix $Q(\varepsilon)$ are rejected. Then the system (29) takes the form

$$Q_0\overline{x} = \overline{h}(\varepsilon). \quad (30)$$

The solution of (30) under the condition H9 is

$$\overline{x} = P_{Q_0}\overline{\eta} + Q_0^{+}\overline{h}(\varepsilon),$$

if and only if H10. Then from (25) it is obtained

$$x_i(\tau_{i-1},\varepsilon) = [P_{Q_0}]_{n_{i-1}}\overline{\eta} + [Q_0^{+}\overline{h}(\varepsilon)]_{n_{i-1}}, \quad i = \overline{2, p+1}. \quad (31)$$

When $i = 2$ the first n rows from the matrix P_{Q_0} are taken, respectively from the vector $Q_0^{+}\overline{h}(\varepsilon)$, when $i = 3$ – the second n rows and etc. The equality (31) is substituted in (28) and the equalities (27) are obtained.

Lemma 2 *[12]: For the fundamental matrix of the solutions of the system (26) –* $W(t,s,\varepsilon)$*, when* $a \le s \le t \le b$*, is fulfilled the inequality*

$$\|W(t,s,\varepsilon)\| \le \beta \exp\Big(-\frac{\alpha(t-s)}{\varepsilon}\Big)$$

where $\alpha > 0$*,* $\beta > 0$ *are constants.*

Theorem 2 *Let the conditions of Theorem1 are fulfilled and the condition H9 is fulfilled, too. Let* α*,* β*,* η*,* c_2*,* c_3*,* c_4*,* c_5*,* c_6*,* c_7*,* e_i*,* $i = \overline{1, p+1}$*,* δ*,* $\overline{\beta}_1$ *are positive constants and the following inequalities are satisfied*

$$\|W(t,s,\varepsilon)\| \le \beta \exp\Big(-\alpha\frac{(t-s)}{\varepsilon}\Big), \; \beta \le \frac{\alpha}{4},$$
$$\|G(t,0,\varepsilon\| \le c_2\varepsilon^{n+1},$$
$$\|W(t,\tau_{i-1},\varepsilon)u_i(\tau_{i-1},\varepsilon)\| \le \beta_1, \; \beta_1 \le \frac{\delta}{4},$$
$$\|l_i(\psi)\| \le e_i\|\psi\|, \; \|A_1(t)\| \le c_3,$$
$$\sum_{i=1}^{p+1} e_i = c_7, \;\; \|Q_0^{+}\| \le c_4, \;\; c_4c_7(1+c_3) > 2,$$
$$\|\overline{\eta}\| \le c_5, \;\; \|P_{Q_0}\| \le c_6, \;\; c_5 \le \frac{\delta}{2c_6}, \;\; \eta \le 2.$$

Then if $\varepsilon_0 \le \frac{1}{c_3}\Big(\frac{2}{c_4c_7} - 1\Big)$*, the remainder term of the asymptotic series satisfies the next inequality* $\|u(t,\varepsilon)\| \le c\varepsilon^{n+1}$*,* $c > 0$*,* $c = const$*,* $\varepsilon \in (0, \varepsilon_0]$*, if and only if the condition H10 is fulfilled.*

Proof: The solution of the problem (20) is sought in the form

$$u_i(t,\varepsilon) = W(t,\tau_{i-1},\varepsilon)u_i(\tau_{i-1},\varepsilon) + \int_{\tau_{i-1}}^{t} W(t,s,\varepsilon)\frac{1}{\varepsilon}G(s,u_i,\varepsilon)ds, \quad (32)$$
$$t \in (\tau_{i-1}, \tau_i], \; i = \overline{1, p+1},$$
$$G(s,u_i,\varepsilon) = \varepsilon A_1(t)u_i(t,\varepsilon) + G(t,0,\varepsilon).$$

For the integral equation (32) the method of successive approximations is applied. It is assumed that

$$\|u_i(\tau_{i-1},\varepsilon)\| \le \delta \;\; i = \overline{2, p+1}, \quad (33)$$

From the properties **(1)**, **(2)** of the function $G(t,u,\varepsilon)$ and Lemma2 is obtained

$$\|u_i(t,\varepsilon)\| \le c\varepsilon^{n+1},$$
$$t \in (\tau_{i-1}, \tau_i], \; i = \overline{1, p+1}, \;\; \varepsilon \in (0, \varepsilon_0].$$

The assumption (33) will be checked as follows.

According to Lemma1 the vectors $u_i(\tau_{i-1},\varepsilon)$ are determined from the system $Q(\varepsilon)\overline{u} = \overline{h}(\varepsilon)$ where

$$Q(\varepsilon) = [Q_2(\varepsilon)Q_3(\varepsilon)\cdots Q_{p+1}(\varepsilon)]_{(\nu\times np)},$$
$$Q_i(\varepsilon) = l_i(W(\cdot,\tau_{i-1},\varepsilon)), \; i = \overline{2, p+1},$$
$$\overline{u} = [u_2(\tau_1,\varepsilon)u_3(\tau_2,\varepsilon)\cdots u_{p+1}(\tau_p,\varepsilon)]^T_{np\times 1},$$
$$\overline{h}(\varepsilon) = -\sum_{i=1}^{p+1} l_i\Big(\int_{\tau_{i-1}}^{(\cdot)} W(\cdot,s,\varepsilon)\frac{1}{\varepsilon}G(s,u_i,\varepsilon)ds\Big),$$

$$Q(\varepsilon) = Q_0 + O(\varepsilon^q \exp(-\frac{\alpha}{\varepsilon})),$$
$$Q_{0(\nu\times np)} = \text{const}, \ q \in \mathbf{N}.$$

The exponentially small elements in $Q(\varepsilon)$ are rejected. From the system

$$Q_0\overline{x} = \overline{h}(\varepsilon),$$

according to Lemma1 is obtained

$$u_i(\tau_{i-1},\varepsilon) = [P_{Q_0}]_{n_{i-1}}\overline{\eta} + [Q_0^+\overline{h}(\varepsilon)]_{n_{i-1}}.$$

For the norm of $\overline{h}(\varepsilon)$ is obtained

$$\|\overline{h}(\varepsilon)\| \le \frac{1}{4}c_7(\varepsilon_0 c_3 + 1)\delta\frac{\beta}{\alpha}.$$

Then $\|u_i(\tau_{i-1},\varepsilon)\| \le c_6c_5 + c_4\frac{1}{4}c_7(\varepsilon_0c_3+1)\delta$. Since $\varepsilon_0 \le \frac{1}{c_3}\Big(\frac{2}{c_4c_7} - 1\Big)$, then

$$\|u_i(\tau_{i-1},\varepsilon)\| \le c_6\frac{\delta}{2c_6} + c_4\frac{1}{4}c_7\Big(\frac{2-c_4c_7}{c_4c_7c_3}c_3 + 1)\delta = \delta.$$

Remark 6: If $D \equiv E$ and M_i, N_i are square matrices, then the same result is obtained if it is followed the way described in [1].

1.4. Function impulse conditions

Let instead of the generalized impulse conditions (3) the function $x(t,\varepsilon)$ satisfies the next impulse conditions

$$M_ix(\tau_i - 0) + N_ix(\tau_i + 0) = I_i(x), \ i = \overline{1,p} \quad (34)$$

and the following condition is fulfilled:

H11: M_i, N_i, $i = \overline{1,p}$ are $k_i \times n$ matrices with constant elements and $I_i(x)$ are k_i-dimensional vector functions, which elements are continuously differentiable functions in the neighbourhood of the solution of the degenerate system (4).

The series (7) is substituted in the impulse conditions (34). The function $I_i(x)$ is expanded in the Taylor series in the neighbourhood of $x_0^i(t)$.

$$\begin{aligned} I_i(x) &= I_i(x(\tau_i - 0)) \\ &= I_i\Big(\sum_{k=0}^{\infty}\varepsilon^k\Big(x_k^i(\tau_i) + \Pi_k^i\Big(\frac{\tau_i-\tau_{i-1}}{\varepsilon}\Big)\Big)\Big) \\ &= I_i\Big(\sum_{k=0}^{\infty}\varepsilon^k x_k^i(\tau_i)\Big) = I_i(x_0^i(\tau_i)) \\ &+ I_i'(x_0^i(\tau_i))(\varepsilon x_1^i(\tau_i) + \cdots + \varepsilon^k x_k^i(\tau_i) + \cdots) \\ &+ I_i''(x_0^i(\tau_i))(\varepsilon x_1^i(\tau_i) + \cdots + \varepsilon^k x_k^i(\tau_i) + \cdots)^2 + \cdots \\ &= I_i(x_0^i(\tau_i)) + \varepsilon I_i'(x_0^i(\tau_i))x_1^i(\tau_i) \\ &+ \varepsilon^2(I_i'(x_0^i(\tau_i))x_2^i(\tau_i) + \beta_{i2}) + \cdots \\ &+ \varepsilon^k(I_i'(x_0^i(\tau_i))x_k^i(\tau_i) + \beta_{ik}) + \cdots \end{aligned}$$

where β_{ik} are given by $x_\nu^i(\tau_i)$, $\nu = \overline{1,k-1}$.

From (34) under condition H7 is found

$$\begin{aligned} &\sum_{k=0}^{\infty}\varepsilon^k(x_k^i(\tau_{i-1}) + c_k^i) = P_{N_{i-1}}\eta_i \\ &+N_{i-1}^+(I_{i-1}(x_0^{i-1}(\tau_{i-1}))+\varepsilon I_{i-1}'(x_0^{i-1}(\tau_{i-1}))x_1^{i-1}(\tau_{i-1}) \\ &+\varepsilon^2(I_{i-1}'(x_0^{i-1}(\tau_{i-1}))x_2^{i-1}(\tau_{i-1}) + \beta_{i-12}) + \cdots \\ &+\varepsilon^k(I_{i-1}'(x_0^{i-1}(\tau_{i-1}))x_k^{i-1}(\tau_{i-1}) + \beta_{i-1k}) + \cdots \\ &- M_{i-1}\sum_{k=0}^{\infty}\varepsilon^k x_k^{i-1}(\tau_{i-1})), \eta_i \in \mathbf{R}^n, \end{aligned} \quad (35)$$

if and only if

H12:

$$\begin{aligned} &P_{N_{i-1}^*}(I_{i-1}(x_0^{i-1}(\tau_{i-1})) - M_{i-1}x_0^{i-1}(\tau_{i-1})) = 0 \\ &P_{N_{i-1}^*}(I_{i-1}'(x_0^{i-1}(\tau_{i-1}))x_1^{i-1}(\tau_{i-1}) \\ &\quad - M_{i-1}x_1^{i-1}(\tau_{i-1})) = 0 \\ &P_{N_{i-1}^*}(I_{i-1}'(x_0(\tau_{i-1}))x_k^{i-1}(\tau_{i-1}) \\ &\quad + \beta_{i-1k} - M_{i-1}x_k^{i-1}(\tau_{i-1})) = 0, \ k = 2,3,... \end{aligned}$$

In (35) the coefficients of the same powers of ε are equalized, then for c_k^i is obtained

$$\begin{aligned} c_0^i =& P_{N_{i-1}}\eta_i + N_{i-1}^+(I_{i-1}(x_0^{i-1}(\tau_{i-1})) \\ &- M_{i-1}x_0^{i-1}(\tau_{i-1})) - x_0^i(\tau_i) \\ c_1^i =& N_{i-1}^+(I_{i-1}'(x_0^{i-1}(\tau_{i-1})) \\ &- M_{i-1})x_1^{i-1}(\tau_{i-1}) - x_1^i(\tau_{i-1}) \\ c_k^i =& N_{i-1}^+((I_{i-1}'(x_0^{i-1}(\tau_{i-1})) \\ &- M_{i-1})x_k^{i-1}(\tau_{i-1}) + \beta_{i-1k}) - x_k^i(\tau_i), \ k = 2,3,... \end{aligned}$$

Then the boundary functions take the form

$$\begin{aligned} \Pi_0^i(\nu_i) =& X(\nu_i)[P_{N_{i-1}}\eta_i + N_{i-1}^+(I_{i-1}(x_0^{i-1}(\tau_{i-1})) \\ &- M_{i-1}x_0^{i-1}(\tau_{i-1})) - x_0^i(\tau_{i-1})] \\ \Pi_1^i(\nu_i) =& X(\nu_i)[N_{i-1}^+(I_{i-1}'(x_0^{i-1}(\tau_{i-1}))x_1^{i-1}(\tau_{i-1}) \\ &- M_{i-1}x_1^{i-1}(\tau_{i-1})) - x_1^i(\tau_{i-1})] \\ &+ \int_0^{\nu_i} X(\nu_i)X^{-1}(s)f_1^i(s)ds \\ \Pi_k^i(\nu_i) =& X(\nu_i)[N_{i-1}^+((I_{i-1}'(x_0^{i-1}(\tau_{i-1})) \\ &- M_{i-1})x_k^{i-1}(\tau_{i-1}) + \beta_{i-1k}) - x_k^i(\tau_{i-1})] \\ &+ \int_0^{\nu_i} X(\nu_i)X^{-1}(s)f_k^i(s)ds, \ k = 2,3,... \end{aligned} \quad (36)$$

The bound of the remainder term of the asymptotic series is done analogously to the case when the impulse conditions have the form (3).

If instead of the generalized impulse conditions (3) the function $x(t,\varepsilon)$ satisfies

$$M_i x(\tau_i - 0) + N_i x(\tau_i + 0) = \varepsilon I_i(x), \quad i = \overline{1,p} \quad (37)$$

and the condition H11 is fulfilled, then analogously to the last case is obtained

$$\begin{aligned}\Pi_0^i(\nu_i) =& X(\nu_i)[P_{N_{i-1}}\eta_i - N_{i-1}^+ M_{i-1} x_0^{i-1}(\tau_{i-1}) \\ &- x_0^i(\tau_{i-1})]\\ \Pi_1^i(\nu_i) =& X(\nu_i)[N_{i-1}^+(I_{i-1}(x_0^{i-1}(\tau_{i-1})) \\ &- M_{i-1}x_1^{i-1}(\tau_{i-1})) - x_1^i(\tau_{i-1})] \\ &+ \int_0^{\nu_i} X(\nu_i)X^{-1}(s)f_1^i(s)ds,\\ \Pi_2^i(\nu_i) =& X(\nu_i)[N_{i-1}^+(I_{i-1}'(x_0^{i-1}(\tau_{i-1}))x_1^{i-1}(\tau_{i-1}) \\ &- M_{i-1}x_2^{i-1}(\tau_{i-1})) - x_2^i(\tau_{i-1})] \\ &+ \int_0^{\nu_i} X(\nu_i)X^{-1}(s)f_2^i(s)ds,\\ \Pi_k^i(\nu_i) =& X(\nu_i)[N_{i-1}^+(I_{i-1}'(x_0^{i-1}(\tau_{i-1}))x_{k-1}^{i-1}(\tau_{i-1}) \\ &+ \beta_{i-1k-1} - M_{i-1}x_k^{i-1}(\tau_{i-1})) - x_k^i(\tau_{i-1})] \\ &+ \int_0^{\nu_i} X(\nu_i)X^{-1}(s)f_k^i(s)ds, \; k = 3,4,...\end{aligned}$$

if and only if

H13:

$$\begin{aligned}&P_{N_{i-1}^*} M_{i-1}x_0^{i-1}(\tau_{i-1}) = 0\\ &P_{N_{i-1}^*}(I_{i-1}(x_0^{i-1}(\tau_{i-1})) - M_{i-1}x_1^{i-1}(\tau_{i-1})) = 0\\ &P_{N_{i-1}^*}(I_{i-1}'(x_0^{i-1}(\tau_{i-1}))x_1^{i-1}(\tau_{i-1}) - M_{i-1}x_2(\tau_{i-1})) = 0\\ &P_{N_{i-1}^*}(I_{i-1}'(x_0^{i-1}(\tau_{i-1}))x_{k-1}^{i-1}(\tau_{i-1}) + \beta_{i-1k}\\ &\quad - M_{i-1}x_k^{i-1}(\tau_{i-1})) = 0, \; k = 3,4,...\end{aligned}$$

Theorem 3 *Let the conditions H1-H3, H5, H7 and H11 are fulfilled. Initial value problem with impulse effects (1), (2), (34) has asymptotic solution of the form (7). The coefficients of the regular series have the representation (8) for each interval $(\tau_{i-1}, \tau_i]$, $i = \overline{1,p+1}$. The boundary functions depend on an arbitrary n-dimensional constant vector and have the representation (14) when $t \in (\tau_0, \tau_1]$ and (36) when $t \in (\tau_{i-1}, \tau_i]$, $i = \overline{2,p+1}$, if and only if v, $I_i(x)$, $i = \overline{1,p}$ and $\varphi(t)$ satisfy H6 and H12.*

An inequalities

$$\|\Pi_k^i(\nu_i)\| \le \sigma \exp(-\kappa\nu_i), \; i = \overline{1,p+1}, \; k = 0,1,...$$

are fulfilled for the boundary functions where σ and κ are positive constants.

Remark 7: Let the impulse conditions have the form $\Delta x|_{t=\tau_i} = \varepsilon I_i(x)$. Obviously, when $\varepsilon \to 0$ the function $x_0(t)$ satisfies the impulse conditions $\Delta x|_{t=\tau_i} = 0$. Then $\lim_{\varepsilon\to 0} x(t,\varepsilon) = x_0(t)$ when $t \in [a,b]$. It is clear, that under the generalized impulse condition (37) for the asymptotic solution of the problem (1), (2), (37) is real the next limit

$$\lim_{\varepsilon\to 0} x(t,\varepsilon) = x_0(t), \quad t \in (a,b]\backslash\{\tau_1, \cdots, \tau_p\}.$$

2. Example

Let in the problem (1)–(3)

$$A = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}, \; A_1(t) = \begin{bmatrix} 4t-1 & 1-5t \\ 0 & 0 \end{bmatrix}$$

$$\varphi(t) = \begin{bmatrix} 1-t \\ t \end{bmatrix}, \; D = [1 \;\; 2], \; v = [1]$$

$$\Rightarrow D^+ = \frac{1}{5}\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \; P_D = \frac{1}{5}\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}, \; P_{D^*} = 0.$$

$$N_1 = [1 \;\; 3], \; M_1 = [-1 \;\; 2], \; h_1 = [1], \tau_1 = \frac{1}{4}$$

$$\Rightarrow N_1^+ = \frac{1}{10}\begin{bmatrix} 1 \\ 3 \end{bmatrix} \; P_{N_1} = \frac{1}{10}\begin{bmatrix} 9 & -3 \\ -3 & 1 \end{bmatrix}, \; P_{N_1^*} = 0.$$

$$N_2 = [2 \;\; 3], \; M_2 = [1 \;\; 4], \; h_2 = [2], \; \tau_2 = \frac{1}{2}$$

$$\Rightarrow N_2^+ = \frac{1}{13}\begin{bmatrix} 2 \\ 3 \end{bmatrix}, \; P_{N_2} = \frac{1}{13}\begin{bmatrix} 9 & -6 \\ -6 & 4 \end{bmatrix}, \; P_{N_2^*} = 0,$$

$$X(\nu_i) = \begin{bmatrix} 1-2\nu_i & 4\nu_i \\ -\nu_i & 1+2\nu_i \end{bmatrix} e^{-\nu_i},$$

$$X^{-1}(\nu_i) = \begin{bmatrix} 1+2\nu_i & -4\nu_i \\ \nu_i & 1-2\nu_i \end{bmatrix} e^{\nu_i}.$$

The asymptotic expansion is sought in the form

$$x_i(t,\varepsilon) = x_0^i(t) + \Pi_0^i x(\nu_i) + \varepsilon[x_1^i(t) + \Pi_1^i x(\nu_i)] + O(\varepsilon^2)$$
$$i = 1,2,3.$$

If $i = 1$, then $t \in (0, \frac{1}{4}]$, $\nu_1 = t/\varepsilon$,
if $i = 2$, $t \in (\frac{1}{4}, \frac{1}{2}]$, $\nu_2 = (t - 1/4)/\varepsilon$,
and if $i = 3$, $t \in (\frac{1}{2}, 1]$, $\nu_3 = (t - 1/2)/\varepsilon$.

From (8) for the coefficients of the regular series is obtained

$$x_0^i(t) = -A^{-1}\varphi(t) = \begin{bmatrix} 5t-1 \\ 4t-1 \end{bmatrix} x_1^i(t)$$
$$= A^{-1}[(x_0^i)'(t) - A_1(t)x_0^i] = \begin{bmatrix} -11 \\ -7 \end{bmatrix}.$$

From (14) is found

$$\Pi_0^1(\nu_1) = \frac{1}{5}\begin{bmatrix} 2B_1 + 6 + (16 - 8B_1)\nu_1 \\ 7 - B_1 + (8 - 4B_1)\nu_1 \end{bmatrix} e^{-\nu_1},$$

$$\Pi_1^1(\nu_1) = \frac{1}{5}\begin{bmatrix} (-\frac{4}{3}B_1+\frac{8}{3})\nu_1^3+(5B_1-5)\nu_1^2+(31-3B_1)\nu_1+55 \\ (-\frac{2}{3}B_1+\frac{4}{3})\nu_1^3+(\frac{3}{2}B_1-\frac{1}{2})\nu_1^2+15\nu_1+35 \end{bmatrix} e^{-\nu_1}.$$

From (18) is found

$$\Pi_0^2(\nu_2) = \frac{1}{10}\begin{bmatrix} 3B_2-\frac{5}{4}+(\frac{35}{2}-10B_2)\nu_2 \\ -B_2+\frac{15}{4}+(\frac{35}{4}-5B_2)\nu_2 \end{bmatrix} e^{-\nu_2},$$

$$\Pi_1^2(\nu_2) = \begin{bmatrix} (\frac{35}{48}-\frac{5}{12}B_2)\nu_2^3+(\frac{3}{8}B_2-\frac{5}{32})\nu_2^2+(\frac{1}{4}B_2-\frac{15}{16})\nu_2+90\nu_2+113 \\ (\frac{35}{96}-\frac{5}{24}B_2)\nu_2^3+(-\frac{1}{8}B_2+\frac{15}{32})\nu_2^2+45\nu_2+79 \end{bmatrix} e^{-\nu_2},$$

$$\Pi_0^3(\nu_3) = \frac{1}{26}\begin{bmatrix} (-28B_3-82)\nu_3-53+6B_3) \\ (-14B_3-41)\nu_3-4B_3-47 \end{bmatrix} e^{-\nu_3},$$

$$\Pi_1^3(\nu_3) = \begin{bmatrix} \frac{1}{26}[(\frac{7}{3}B_3+\frac{41}{6})\nu_3^3+(\frac{-31}{2}B_3-\frac{111}{4})\nu_3^2+(\frac{35}{2}+12B_3)\nu_3]+30\nu_3+17 \\ \frac{1}{26}[(\frac{7}{6}+\frac{41}{12}B_3)\nu_3^3+(-6B_3-\frac{35}{4})\nu_3^2]+15\nu_3+16 \end{bmatrix} e^{-\nu_3}.$$

Then when $t \in (0, \frac{1}{4}]$

$$x_1(t,\varepsilon) = \begin{bmatrix} 5t-1+\frac{1}{5}(2B_1+6+(16-8B_1)\nu_1)e^{-\nu_1} \\ 4t-1+\frac{1}{5}(7-B_1+(8-4B_1)\nu_1)e^{-\nu_1}, \end{bmatrix}$$
$$+\varepsilon\begin{bmatrix} -11+\frac{1}{5}[(\frac{8}{3}-\frac{4}{3}B_1)\nu_1^3+(5B_1-5)\nu_1^2+(31-3B_1)\nu_1+55]e^{-\nu_1} \\ -7+\frac{1}{5}[(\frac{4}{3}-\frac{2}{3}B_1)\nu_1^3+(\frac{3}{2}B_1-\frac{1}{2})\nu_1^2+15\nu_1+35]e^{-\nu_1} \end{bmatrix} + O(\varepsilon^2)$$

where $\nu_1 = t/\varepsilon$, $B_1 = 2\xi_1 - \xi_2$-parameter,

$t \in (\frac{1}{4}, \frac{1}{2}]$

$$x_2(t,\varepsilon) = \begin{bmatrix} 5t-1+\frac{1}{10}[3B_2-\frac{5}{4}+(\frac{35}{2}-10B_2)\nu_2]e^{-\nu_2} \\ 4t-1+\frac{1}{10}[-B_2+\frac{15}{4}+(\frac{35}{4}-5B_2)\nu_2]e^{-\nu_2} \end{bmatrix}$$
$$+\varepsilon\begin{bmatrix} -11+[\frac{1}{10}[(\frac{35}{48}-\frac{5}{12}B_2)\nu_2^3+(\frac{3}{8}B_2-\frac{5}{32})\nu_2^2+\frac{1}{4}B_2-\frac{15}{16})\nu_2]+90\nu_2+113]e^{-\nu_2} \\ -7+[\frac{1}{10}](\frac{35}{96}-\frac{5}{24}B_2)\nu_2^3+(-\frac{1}{8}B_2+\frac{15}{32})\nu_2^2]+45\nu_2+79]e^{-\nu_2} \end{bmatrix} + O(\varepsilon^2),$$

$\nu_2 = (t-1/4)/\varepsilon$, $B_2 = 3\eta_{21} - \eta_{22}$-parameter,

$t \in (\frac{1}{2}, 1]$

$$x_3(t,\varepsilon) = \begin{bmatrix} 5t-1+[\frac{1}{26}(-28B_3-82)\nu_3-53+6B_3)]e^{-\nu_3} \\ 4t-1+\frac{1}{26}[(-14B_3-41)\nu_3-4B_2-47]e^{-\nu_3} \end{bmatrix}$$
$$+\varepsilon\begin{bmatrix} -11+[\frac{1}{26}[(\frac{7}{3}B_3+\frac{41}{6})\nu_3^3+(\frac{-31}{2}B_3-\frac{111}{4})\nu_3^2+(\frac{35}{2}+12B_3)\nu_3]+30\nu_3+17]e^{-\nu_3} \\ -7+[\frac{1}{26}[(\frac{7}{6}+\frac{41}{12}B_3)\nu_3^3+(-6B_3-\frac{35}{4})\nu_3^2]+15\nu_3+16]e^{-\nu_3} \end{bmatrix} + O(\varepsilon^2)$$

where $\nu_3 = (t-1/2)/\varepsilon$, $B_3 = 3\eta_{31} - 2\eta_{32}$-parameter.

3. Almost nonlinear singularly perturbed system

A singularly perturbed system of the form

$$\varepsilon\frac{dx}{dt} = Ax + \varepsilon f(t,x,\varepsilon) + \varphi(t), \tag{38}$$
$$t \in [a,b],\ t \neq \tau_i,\ i = \overline{1,p},\ \ll \varepsilon < 1,$$
$$a \equiv \tau_0 < \tau_1 < \cdots < \tau_p < \tau_{p+1} \equiv b,$$

is considered. The matrix A satisfies H1, vector-function $\varphi(t)$-H3 and instead of H2 is fulfilled the following condition

H14: The function $f(t,x,\varepsilon)$ is nonlinear vector function, which is partially continuous with respect to the first argument with breaks of first kind in the points τ_i, $i=\overline{1,p}$, continuously differentiable in the neighbourhood of the solution of the degenerate system (4) with respect to the second argument and continuous with respect to the third argument when $\varepsilon \in (0,\varepsilon_0]$.

It is sought n-dimensional vector function $x(t,\varepsilon)$, which is partially continuous with respect to t and continuous with respect to ε, $x(\cdot,\varepsilon) \in C((a,b]\backslash(\tau_1,...\tau_p)$, $x(t,\cdot) \in C(0,\varepsilon_0]$ and which satisfies the system (38), the generalized initial condition (2) and the generalized impulse conditions (3) where the matrices M_i, N_i and the vectors h_i satisfy the condition H4.

3.1. Asymptotic expansion

The formally asymptotic expansion of the solution of the problem (38), (2), (3) is sought in the form (7). The series (7) is substituted in the system (38):

$$\varepsilon\frac{d}{dt}\sum_{k=0}^{\infty}\varepsilon^k(x_k^i(t)+\Pi_k^i(\nu_i)) = A\sum_{k=0}^{\infty}\varepsilon^k(x_k^i(t)+\Pi_k^i(\nu_i))$$
$$+\varepsilon f(t,\sum_{k=0}^{\infty}\varepsilon^k(x_k^i(t)++\Pi_k^i(\nu_i),\varepsilon)+\varphi(t). \quad (39)$$

The function $f(t,\sum_{k=0}^{\infty}\varepsilon^k(x_k^i(t)+\Pi_k^i(\nu_i)),\varepsilon)$ is represented as follows

$$f\Big(t,\sum_{k=0}^{\infty}\varepsilon^k(x_k^i(t)+\Pi_k^i(\nu_i)),\varepsilon\Big) = f\Big(t,\sum_{k=0}^{\infty}\varepsilon^k x_k^i(t),\varepsilon\Big)$$
$$+f\Big(\varepsilon\nu_i+\tau_{i-1},\sum_{k=0}^{\infty}\varepsilon^k(x_k^i(\varepsilon\nu_i+\tau_{i-1})+\Pi_k^i(\nu_i)),\varepsilon\Big)$$
$$-f\Big(\varepsilon\nu_i+\tau_{i-1},\sum_{k=0}^{\infty}\varepsilon^k x_k^i(\varepsilon\nu_i+\tau_{i-1}),\varepsilon\Big)$$
$$=\overline{f}(t)+\Pi^i(\nu_i) \quad (40)$$

where $\nu_i = (t-\tau_{i-1})/\varepsilon$, $i=\overline{1,p+1}$. The function $\overline{f}(t) = f(t,\sum_{k=0}^{\infty}\varepsilon^k x_k^i(t),\varepsilon)$ is expanded in the Taylor series in the neighbourhood of the points $(t,x_0^i(t),0)$

$$\overline{f}(t) = f(t,x_0^i(t),0)+\varepsilon[f_x'(t,x_0^i(t),0)x_1^i(t)+g_1^i(t)]$$
$$+\varepsilon^2[f_x'(t,x_0^i(t),0)x_2^i(t)+g_2^i(t)]+\cdots$$
$$+\varepsilon^k[f_x'(t,x_0^i(t),0)x_k^i(t)+g_k^i(t)]+\cdots \quad (41)$$

where the functions $g_k^i(t)$, $k=1,2,\cdots$ are given by x_ν^i, $\nu=\overline{0,k-1}$.

The functions $\Pi^i f(\nu_i)$, $i=\overline{1,p+1}$ are also expanded in the Taylor series, but in the neighbourhood of the points $(\tau_{i-1},x_0^i(\tau_{i-1})+\Pi_0^i(\nu_i),0)$, $i=\overline{1,p+1}$

$$\Pi^i f(\nu_i)=f\Big(\varepsilon\nu_i+\tau_{i-1},\sum_{k=0}^{\infty}\varepsilon^k(x_k^i(\varepsilon\nu_i+\tau_{i-1})+\Pi_k^i(\nu_i)),\varepsilon\Big)$$
$$-f\Big(\varepsilon\nu_i+\tau_{i-1},\sum_{k=0}^{\infty}\varepsilon^k(x_k^i(\varepsilon\nu_i+\tau_{i-1}),\varepsilon\Big)$$
$$=f(\tau_{i-1},x_0^i(\tau_{i-1})+\Pi_0^i(\nu_i),0)-f(\tau_{i-1},x_0^i(\tau_{i-1}),0)$$
$$+\varepsilon[f_\varepsilon'(\tau_{i-1},x_0^i(\tau_{i-1})+\Pi_0^i(\nu_i),0)-f_\varepsilon'(\tau_{i-1},x_0^i(\tau_{i-1}),0)]$$
$$+\varepsilon\nu_i[f_t'(\tau_{i-1},x_0^i(\tau_{i-1})+\Pi_0^i(\nu_i),0)-f_t'(\tau_{i-1},x_0^i(\tau_{i-1}),0)]$$
$$+f_t'(\tau_{i-1},x_0^i(\tau_{i-1}+\Pi_0^i(\nu_i)),0)[\sum_{k=0}^{\infty}\varepsilon^k(x_k^i(\varepsilon\nu_i+\tau_{i-1})$$
$$+\Pi_k^i(\nu_i))-x_0^i(\tau_{i-1})-\Pi_0^i(\nu_i)]-f_x'(\tau_{i-1},x_0^i(\tau_{i-1}),))$$
$$\times[\sum_{k=0}^{\infty}\varepsilon^k(x_k^i(\varepsilon\nu_i+\tau_{i-1})-x_0^i(\tau_{i-1})]+\cdots.$$

The functions $x_k^i(\varepsilon\nu_i+\tau_{i-1})$ are expanded in the Taylor series in the neighbourhood of the points τ_{i-1}, $i=\overline{1,p+1}$

$$x_k^i(\varepsilon\nu_i+\tau_{i-1}) = x_k^i(\tau_{i-1})+(x_k^i)'(\tau_{i-1})\varepsilon\nu_i$$
$$+\frac{(x_k^i)''(\tau_{i-1})}{2!}\varepsilon^2\nu_i^2+\cdots,\quad k=0,1,2,...$$

The last equalities are substituted in $\Pi^i f(\nu_i)$, the coefficients before equal powers of ε are grouped and for the last function is obtained:

$$\Pi^i f(\nu_i) = f(\tau_{i-1},x_0^i(\tau_{i-1})+\Pi_0^i(\nu_i),0)$$
$$-f(\tau_{i-1},x_0^i(\tau_{i-1}),0)+\sum_{k=1}^{\infty}\varepsilon^k[f_x'(\tau_{i-1},x_0^i(\tau_{i-1})$$
$$+\Pi_0^i(\nu_i),0)\Pi_k^i(\nu_i)+\Pi^i g_k(\nu_i)] \quad (42)$$

where the functions $\Pi^i g_k(\nu_i)$ are expressed by $\Pi_\nu^i(\nu_i)$, $\nu=\overline{0,k-1}$.

In (39) variables are separated by t and ν_i, $f(t,x,\varepsilon)$ from (40) is substituted with corresponding expansion (41) and (42), the coefficients before equal powers of ε are equalized and for the elements of regular series is obtained

$$x_k^i(t) = \begin{cases} -A^{-1}\varphi_i(t), & k=0 \\ A^{-1}[(x_0^i)'(t)-f(t,x_0^i(t),0)], & k=1 \\ A^{-1}[(x_{k-1}^i)'(t)-f'(t,x_0^i(t),0)x_{k-1}^i(t)-g_{k-1}^i(t)], & k=2,3... \end{cases} \quad (43)$$

Systems of the form (9) are obtained for the coefficients of the singular series, but in this case the functions $f_k^i(\nu_i)$ have the form:

$$f_k^i(\nu_i) = \begin{cases} 0, & k=0 \\ f(\tau_{i-1}, x_0^i(\tau_{i-1}) + \Pi_0^i(\nu_i), 0) - f(\tau_{i-1}, x_0^i(\tau_{i-1}), 0), & k=1 \\ f'_x(\tau_{i-1}, x_0^i(\tau_{i-1}) + \Pi_0^i(\nu_i), 0)\Pi_{k-1}^i(\nu_i) + \Pi^i g_{k-1}(\nu_i), & k=2,3,... \end{cases} \quad (44)$$

Therefore analogously to point 1.2 the next theorem is real.

Theorem 4 *Let the conditions H1, H3-H5, H7 and H14 are fulfilled. Initial value problem with impulse effects (38),(2), (3) has asymptotic solution of the form (7). The coefficients of the regular series have the form (43) for each interval* $(\tau_{i-1}, \tau_i]$, $i = \overline{1, p+1}$. *The boundary functions depend on an arbitrary* $n-$*dimensional constant vector and have the representation (14) when* $t \in [\tau_0, \tau_1]$ *and (18) when* $t \in (\tau_{i-1}, \tau_i]$, $i = \overline{2, p+1}$, *according to functions (44), if and only if* v, h_i, $i = \overline{1,p}$ $\varphi(t)$ *are satisfied H6 and H8. For the boundary functions are real the next inequalities*

$$\|\Pi_k^i(\nu_i)\| \le \sigma \exp(-\kappa\nu_i), \ i = \overline{1, p+1}, \ k = 0, 1, ...$$

where σ *and* κ *are positive constants.*

The proof of this theorem is similar to the proof of Theorem1. The bound of the remainder term of the asymptotic series is done analogously to the way in point 1.3. In this case the remarks 1-7 are held too.

References

[1] Bainov, D.D. and Covachev, V. (1994) *Impulsive differential equations with small parameter*, World Scientific Publishing Co. Ptc. Ltd., Singapore.

[2] Bainov, D.D. and Hristova, S.G. (1986) Asymptotics of the solution of initial value problem for a singularly perturbed linear system with impulses. II Justification of the algorithm, *Sci.Proc.Plovdiv Univ.Math.* **24**.

[3] Flato, L. and Levinson, N. (1963) Periodic solution of singularly perturbed systems, *J. Rational Mech. Anal.* **4**.

[4] Halanaj, A. and Veskler, O. (1971) *Theory of impulsive systems* Moscow, Mir (in Russian).

[5] Karandjulov, L.I. (1999) Generalized Cauchy problem for linear pulse differential systems, In *Mathematics and Education in Mathematics*, Proceedings of twenty eighth Spring Conference of the Union of Bulgarian Mathematicians, Montana, April 5 - 8, 1999, pp.120–127.

[6] Lancaster, L. (1982) *Matrix theory*, Nauka, Moscow (in Russian).

[7] Lakshmikatham, V., Bainov, D.D. and Simeonov, P.S. (1989) Theory of Impulsive Differential equations, *World Scientific, Singapore-New Jersey-London- Hong-Kong.*

[8] Nached, M.Z. (1976) *Generalize Inverse and Applications*, Ed. by Nached M.Z., New York, San Francisco, London - Acad.Press.

[9] Penrose, R. (1956) On best approximate solution on linear matrix equations, In *Proc. Cambridge Philos. Soc.* **52**, pp.17–19.

[10] Penrose, R. (1955) A generalize inverse for matrices, In *Proc. Cambridge Philos. Soc.* **51**, pp.406–413.

[11] Samoilenko, A.M. and Perestyik, N.A. (1987) *Differential equations with impulse actions*, Kiev (in Russian).

[12] Vasil'eva, A.B. and Butuzov, V.F. (1973) *Asymptotic expansions of solution of singularly perturbed equations*, Nauka, Moscow (in Russian).

[13] Vasil'eva, A.B. and Butuzov, V.F. (1978) *Singularly perturbed equations in critical case*, Moscow State University (in Russian).