Analytic Approximation of Nonlinearities with Memory

Kamen Perev[a)]

Systems and Control Department,
Technical University of Sofia
1000 Sofia, Bulgaria

[a)] *Electronic mail: kperev@tu-sofia.bg*

Abstract. This paper considers the problem of analytic approximation of nonlinearities with memory. The mathematical models for nonlinearities with memory used in the paper are differential-based, rate dependent and two-valued, involving the input signal velocity in its description. The nonlinear characteristics, considered in the paper, are the shifted unpolarized relay with hysteresis, relay with hysteresis and dead zone, and relay with hysteresis and saturation. Explicit formulas for their mathematical models are presented, containing the ideal relay as basic element in their description. The analytic approximation for nonlinearities with memory reduces to the problem of rational function approximation of the ideal relay switching behavior. The discontinuous jumps are presented by hyperbolic tangent functions, where the exponential terms are approximated by using the chained fraction method. The error of approximation is reduced by introducing a parameter in the hyperbolic tangent presentation and the corresponding errors of approximation are discussed.

INTRODUCTION

The existence of nonlinearities is important element in modeling classical control systems. In most of the cases, the nonlinear characteristics are linearized and the resultant dynamical system is addressed as a linear one. The wide scope of linear analysis and design techniques enable the practicing engineer to solve a large variety of control system problems. One of the difficulties of using linearization is the range of system operation, i.e. the operating signals should be small and the system behavior is reliable only in small neighborhood around the equilibrium position. Another problem with linearization is the requirement for differentiability of the nonlinear characteristics under consideration. The condition for differentiability of nonlinear characteristics is additional limitation over the system description. Very often the nature of the nonlinear element does not allow to use linearization techniques. Such nonlinear elements that rule out using linearization procedures are known as essential nonlinearities. The essential nonlinearities include all relay characteristics describing abrupt change in system behavior. Some of the most severe limitations for using linearization techniques are produced by the relay with hysteresis characteristics. The presence of hysteresis loops is a special feature in a large variety of nonlinear control systems. Hysteresis is a phenomenon that is observed in many physical processes like mechanical systems, ferromagnetic materials and electromagnetic devices. The substance of hysteresis is the existence of multiple state equilibria associated with system dynamics. Important property of the hysteresis loop is the introduction of dynamical component in the nonlinear element characterization. The dynamics of the hysteresis nonlinearity is illustrated by the memory effect in the output signal. This memory effect is described by the property, that the response to particular changes is a function of preceding responses. The hysteresis nonlinearity can lead to performance degradation mainly in positioning accuracy of system performance. The difficulty in modeling hysteresis loops results from the existence of multivalued behavior. For different input values, two output values of the hysteresis characteristic are possible and which one of these two values will occur depends on the history of the input.

There exist different approaches in the control literature for modeling the hysteresis loops [17]. These approaches can be divided into two main groups: i) operator-based or static models, which use operators to describe the physical phenomenon and ii) differential-based or dynamic models, which use differential equations to model the hysteresis characteristic [7], [12], [13]. The first group of hysteresis models includes the Preisach model [7], [3], the Krasnoselskii – Pokrovskii model [8], [19], the Prandtl – Ishlinskii model [11], [10], the Maxwell – Slip model [7], etc. A specific feature of the operator-based models is the rate-independent effect of the hysteresis loop. This effect means that the branches of the hysteresis loop are determined only by the past extremum values of the input, while the speed of the input variations has no influence on branching. The differential-based models use differential equations to describe the hysteresis phenomenon [18]. A specific feature of differential-based models is to underline the rate-dependent effect of the hysteresis loops and to indicate that branching depends on the input rate of change. Thus, the output present value depends not only on the input present value but also on its velocity. Main representatives of the differential-based models are the Bouc-Wen model [4], the Duhem model [5], the Jiles-Atherton model, the Chua model, the Hodgon model and others [14], [12], [7]. Other applications of hysteresis loops find in control system theory. A special type of hysteresis characteristic representation is the relay with hysteresis nonlinear element, which can be considered as representative of a delayed relay operator with thresholds [1]. In electrical engineering, the relay with hysteresis characteristic model is used to estimate the energy losses in electrical machines for the purpose of electrical device design [3]. Another very effective application of the relay with hysteresis characteristic is in automatic tuning of PID regulators, where a relay with hysteresis is inserted in the closed loop and is used for adjustment of the PID parameters [9]. Relay with hysteresis characteristics find also their application in many algorithms and logic schemes, where switching with phase delay is taking part. In many analysis and design methods for nonlinear control systems, a main requirement is to have differentiable nonlinear trajectories. The differentiability requirement is especially demanded for all methods related to Lyapunov stability theory. However, the differentiability condition is not satisfied when the system exhibits jump behavior, which is the case for relay with hysteresis characteristics. This situation is resolved when the hysteresis loop is approximated by using smooth functions.

This paper considers the problem of smooth function approximation of nonlinearities with memory and more specifically, nonlinear characteristics with hysteresis loops. The paper appears as an extension of previous result on the rational function approximation of relay with hysteresis [15]. Several different nonlinear characteristics with hysteresis loops are examined: shifted unpolarized relay characteristic with hysteresis, relay with hysteresis and dead zone, relay with hysteresis and saturation. Explicit expressions for these characteristics are developed, where the ideal relay plays important role. Using the approximation formula for the ideal relay, all three characteristics are described in terms of analytic functions.

RATIONAL FUNCTION APPROXIMATION OF THE IDEAL RELAY CHARACTERISTICS

The rational function approximation of the ideal relay characteristic plays central role in the approximation of hysteresis loops. The ideal relay characteristic can be presented by the following expression:

$$N(x) = sgn(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases} \tag{1}$$

This characteristic is nondifferentiable at the point $x = 0$, where the relay behavior changes with jump. Therefore, the relay characteristic at this point is not only nondifferentiable, but it is also discontinuous. The discontinuous jump of any nonlinear characteristic imposes heavy limitations for its analytic approximation. One possible approximation method is by using gate functions [16]. A common gate function is the hyperbolic tangent function:

$$N(x) \approx \tanh(Ax), \tag{2}$$

where A is a parameter. A gate function approximation of the ideal relay for different values of the parameter A is shown in figure 1. where for $A = 1$ (the dashed line), the approximation curve deviates to a considerable extent from the ideal relay characteristic. For $A = 2$ (the dotted line) and $A = 10$ (the dashed-dotted line), the approximation curve gets closer to the relay characteristic and the approximation error reduces considerably. Finally for $A = 100$ (the solid line), the difference between the true and approximated characteristics is insignificant. From figure 1 is clearly seen that by increasing the value of the parameter A, the accuracy of approximation is also increased.

We assume as criterion for accuracy of approximation the mean square approximation error:

$$\delta_{mse} = \left\{ \frac{1}{b-a} \int_a^b [N(x) - f(x)]^2 \, dx \right\}^{1/2}, \tag{3}$$

where $N(x)$ is given by (1) and $f(x) = \tanh(Ax)$ is the approximation function. Using (3), we obtain for $A = 1$, $\delta_{mse} = 0.3562$, for $A = 2$, $\delta_{mse} = 0.2502$, for $A = 10$, $\delta_{mse} = 0.1061$ and for $A = 100$, $\delta_{mse} = 0.01392$. Therefore, the experimental data confirms the theoretical inference that as higher is the value of the parameter value A, as smaller is the value of the approximation error δ_{mse}.

The next problem is to approximate the hyperbolic tangent function in terms of rational functions of the argument. The computation of the hyperbolic tangent function, available in the computer libraries in FORTRAN or C languages, are presented in [2]. We accept the partition of the positive real line into two intervals for evaluation of the hyperbolic

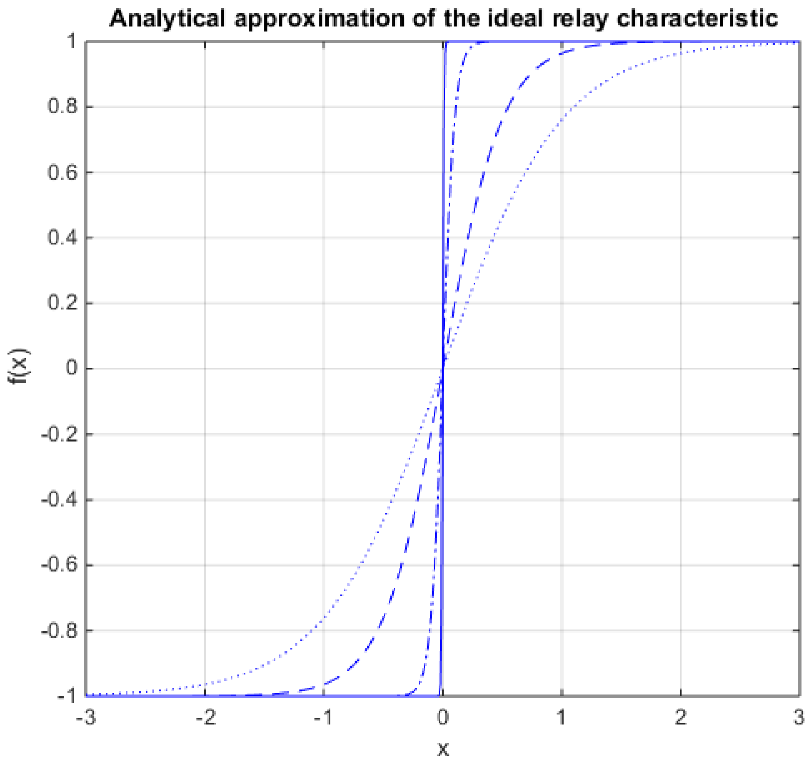

FIGURE 1. Approximation for the ideal relay $f(x) = \tanh(Ax)$

tangent function. For $x < x_{large} = 19.06155$, the the hyperbolic tangent function is presented by the expression [2]:

$$\tanh(x) = 1 - \frac{2}{\exp(2x)+1} \tag{4}$$

and for $x \geq x_{large}$, the value of the hyperbolic tangent function is considered as one. The value $x_{large} = 19.06155$ is obtained from the formula [2]:

$$\frac{2}{\exp(2x)} < B^{-(t+1)} \tag{5}$$

where B is the machine base and t is the number of significant digits in the machine representation, i.e. $B = 2$ and for double precision $t = 53$. The reason behind (5) is that for large x, the value of $\exp(2x) + 1 \gg 1$ and the second term in the right hand side of (4) is less than the machine representation error. The solution of (5), when the inequality is replaced by equality, gives the computed value for x_{large}.

The next step in the approximation procedure is to replace the exponential term in (4) by a rational function of the argument x. Due to the wide scope of application, the exponential function $\exp(x)$ is one of the most often approximated functions and there exist different approximation procedures. We apply the method of chained fractions for its rational function representation. The chained fraction expansion of the exponential function $\exp(x)$ can be presented as follows [6]:

$$\exp(x) = \left[\, 0;\ \tfrac{1}{1},\ \tfrac{-2x}{2+x},\ \tfrac{x^2}{6},\ \tfrac{x^2}{10},\ \tfrac{x^2}{14},\ \tfrac{x^2}{18},\ \tfrac{x^2}{22},\ \tfrac{x^2}{26},\ \tfrac{x^2}{30},\ \dots\ \tfrac{x^2}{4n+2},\ \dots \right]. \tag{6}$$

The above chained fraction can be rewritten in terms of two variables ratios as follows:

$$\exp(x) = \left[\, a_0,\ \tfrac{b_1}{a_1},\ \tfrac{b_2}{a_2},\ \tfrac{b_3}{a_3},\ \tfrac{b_4}{a_4},\ \tfrac{b_5}{a_5},\ \tfrac{b_6}{a_6},\ \dots \right]. \tag{7}$$

Then $\exp(x)$ can be evaluated as a ratio of two indexed polynomials, with indices corresponding to their degree, and satisfying the recurrence relations [6]:

$$P_k(x) = a_k P_{k-1}(x) + b_k P_{k-2}(x), \tag{8}$$

$$Q_k(x) = a_k Q_{k-1}(x) + b_k Q_{k-2}(x), \qquad k = 1,2,3,\ldots \tag{9}$$

where the initial polynomials are defined as follows:

$$P_{-1}(x) = 1, \quad Q_{-1}(x) = 0, \quad P_0(x) = a_0, \quad Q_0(x) = 1. \tag{10}$$

The n^{th} order approximation of the exponential function is given by the expression $\exp(x) \approx \frac{P_n(x)}{Q_n(x)}$. The first few polynomials are obtained as follows: $P_1(x) = 1$, $Q_1(x) = 1$; $P_2(x) = 2+x$, $Q_2(x) = 2-x$; $P_3(x) = x^2+6x+12$, $Q_3(x) = x^2-6x+12$ ..., $P_9(x) = 518918400+259459200x+60540480x^2+8648640x^3+831600x^4+211440x^5+2520x^6+272x^7+x^8$, $Q_9(x) = 518918400-259459200x+60540480x^2-8648640x^3+831600x^4-211440x^5+2520x^6-272x^7+x^8$, etc. The chained fraction approximation of the exponential function for the value $x = 1$ and $n = 9$ is given by the expression $e \approx \frac{P_9(x)}{Q_9(x)} = 2.718281828459045$, that is evaluated with an error $\varepsilon = O(10^{-16})$. The corresponding $\tanh(Ax)$ for $|x| < x_{large}$ is approximated by the expression: $\tanh(Ax) \approx 1 - \frac{2}{\frac{P_9(2Ax)}{Q_9(2Ax)}+1}$.

RATIONAL FUNCTION APPROXIMATION FOR NONLINEARITIES WITH MEMORY

The nonlinearities with memory characteristics play important role for modeling nonlinear phenomena, where the system behavior changes with jump depending on the signal sign of velocity before the jump occurs. One of the main relay characteristic with memory is the characteristic of shifted unpolarized relay with positive hysteresis, shown in fig. 2.

The mathematical expression describing this nonlinear element is given as follows:

$$N(x) = \begin{cases} c, & x > b, \\ 0, & x < a, \\ c, & a < x \le b, \;\; \dot{x} < 0 \\ 0, & a \le x < b, \;\; \dot{x} > 0 \end{cases} \tag{11}$$

Using the ideal relay model, we can obtain the following expression for the shifted unpolarized relay with positive hysteresis model:

$$N(x) = \frac{c}{2}\left\{1 + 0.5sgn(x-b)\left[1+sgn(\dot{x})\right] + 0.5sgn(x-a)\left[1-sgn(\dot{x})\right]\right\} \tag{12}$$

We make the following notations:

$$N_1(x) = 0.5sgn(x-b)\left[1+sgn(\dot{x})\right] \tag{13}$$

$$N_2(x) = 0.5sgn(x-a)\left[1-sgn(\dot{x})\right] \tag{14}$$

Then we have $N(x) = \frac{c}{2}\left[1+N_1(x)+N_2(x)\right]$. When $\dot{x} > 0$, $N_2(x) = 0$. Then, if $x > b$, $N_1(x) = 1$ and $N(x) = c$. If $x < b$, $N_1(x) = -1$ and $N(x) = 0$. When $\dot{x} < 0$, $N_1(x) = 0$. Then, if $x > a$, $N_2(x) = 1$ and $N(x) = c$. If $x < a$, $N_2(x) = -1$ and $N(x) = 0$. From the above derivations follows, that no matter of the velocity sign, if $x > b$, $N(x) = c$ and if $x < a$, $N(x) = 0$. The two-valued region of the hysteresis loop is in the interval $a < x < b$, where if the argument velocity is positive, $N(x) = 0$ and if the argument velocity is negative, $N(x) = c$. The analytic approximation of expression (11) is developed by replacing the $sgn(x)$ function in terms of the function $\tanh(Ax)$. For example, the approximations are implemented along the line $sgn(x-b) \approx \tanh[A(x-b)]$, $sgn(x-a) \approx \tanh[A(x-a)]$ and $sgn(\dot{x}) \approx \tanh(A\dot{x})$. Then for values of the argument $x < x_{large}$, the hyperbolic tangent function is substituted by $\tanh(Ax) \approx 1 - \frac{2}{P_n(2Ax)/Q_n(2Ax)+1}$.

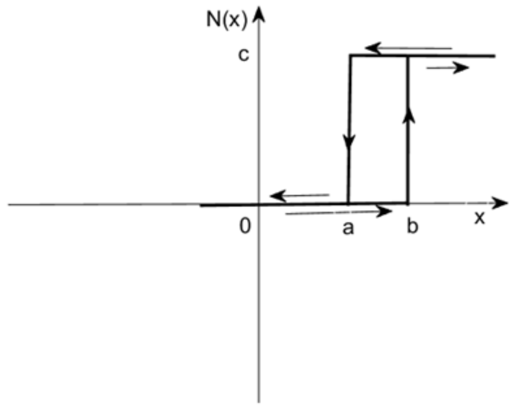

FIGURE 2. Shifted unpolarized relay with hysteresis

In this sense we have $sgn(x-b) \approx 1 - \frac{2}{P_n(2A(x-b))/Q_n(2A(x-b))+1}$, similarly $sgn(x-a) \approx 1 - \frac{2}{P_n(2A(x-a))/Q_n(2A(x-a))+1}$ and finally $sgn(\dot{x}) \approx 1 - \frac{2}{P_n(2A(\dot{x}))/Q_n(2A(\dot{x}))+1}$.

Another member of the relay with memory set is the characteristic of polarized relay with hysteresis and dead zone, shown in fig. 3.

The mathematical model for this characteristic is given by the expression:

$$N(x) = \begin{cases} c, & x > b, \\ -c & x < -b \\ 0, & |x| < a, \\ c, & a < x \leq b, \quad \dot{x} < 0 \\ 0, & a \leq x < b, \quad \dot{x} > 0 \\ -c & -b \leq x < -a, \quad \dot{x} > 0 \\ 0, & -b < x \leq -a, \quad \dot{x} < 0 \end{cases} \tag{15}$$

Using the sign function expression, we can obtain the following model for the polarized relay with hysteresis and dead zone characteristic:

$$\begin{aligned} N(x) \;=\; & \tfrac{c}{2}\left\{1+0.5sgn(x-b)\left[1+sgn(\dot{x})\right]+0.5sgn(x-a)\left[1-sgn(\dot{x})\right]\right\} - \\ & \tfrac{c}{2}\left\{1+0.5sgn(-x-a)\left[1+sgn(\dot{x})\right]+0.5sgn(-x-b)\left[1-sgn(\dot{x})\right]\right\} \end{aligned} \tag{16}$$

We make the following notations:

$$N_1(x) = 0.5sgn(x-b)\left[1+sgn(\dot{x})\right] \tag{17}$$

$$N_2(x) = 0.5sgn(x-a)\left[1-sgn(\dot{x})\right] \tag{18}$$

$$N_3(x) = 0.5sgn(-x-a)\left[1+sgn(\dot{x})\right] \tag{19}$$

$$N_4(x) = 0.5sgn(-x-b)\left[1-sgn(\dot{x})\right] \tag{20}$$

and the model of the nonlinear element can be represented as:

$$N(x) = \frac{c}{2}\left[N_1(x)+N_2(x)-N_3(x)-N_4(x)\right] \tag{21}$$

Further, we assume that $\dot{x} > 0$, then $N_2(x) = N_4(x) = 0$. If $x > b$, then $N_1(x) = 1$, $N_3(x) = -1$ and $N(x) = c$. If $a < x < b$, then $N_1(x) = -1$, $N_3(x) = -1$ and $N(x) = 0$. If $|x| < a$ then $N_1(x) = -1N_3(x) = -1$ and $N(x) = 0$. If

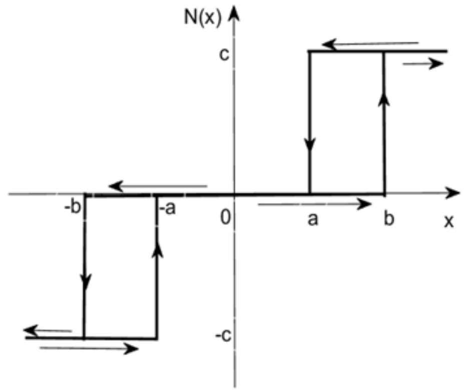

FIGURE 3. Relay with hysteresis and dead zone

$-b < x < -a$, then $N_(x) = -1$, $N_3(x) = 1$ and $N(x) = -c$. If $x < -b$, then $N_1(x) = -1$, $N_3(x) = 1$ and $N(x) = -c$. Next we assume that $\dot{x} < 0$, then $N_1(x) = N_3(x) = 0$. If $x > b$, then $N_2(x) = 1$, $N_4(x) = -1$ and $N(x) = c$. If $a < x < b$, then $N_2(x) = 1$, $N_4(x) = -1$ and $N(x) = c$. If $|x| < a$, then $N_2(x) = -1$, $N_4(x) = -1$ and $N(x) = 0$. If $-b < x < -a$, then $N_2(x) = -1$, $N_4(x) = -1$ and $N(x) = 0$. Finally, if $x < -b$, then $N_2(x) = -1$, $N_4(x) = 1$ and $N(x) = -c$.

Similarly to the previous case, the analytic approximation of expression (16) is obtained by replacing the sign function in terms of the hyperbolic tangent function, see (2). Another nonlinear element with memory is the relay with hysteresis and saturation characteristic, shown in fig. 4.

The mathematical expression for this characteristic is given as follows:

$$N(x) = \begin{cases} c & x > b, \\ -c & x < -b \\ x-a & -b+2a \le x \le b, \quad \dot{x} > 0 \\ x+a, & -b \le x \le b-2a, \quad \dot{x} < 0 \\ c, & b-2a < x \le b, \quad \dot{x} < 0 \\ -c & -b \le x < -b+2a, \quad \dot{x} > 0 \end{cases} \tag{22}$$

The model for the relay with hysteresis and saturation can be written in the form:

$$\begin{aligned} N(x) \;=\; & \tfrac{c}{4}\left[1+sgn(\dot{x})\right]\left\{\tfrac{x-a}{c}\left[sgn(x-2a+b)-sgn(x-b)\right]+sgn(x-b)-sgn(-x+2a-b)\right\}+ \\ & \tfrac{c}{4}\left[1-sgn(\dot{x})\right]\left\{\tfrac{x+a}{c}\left[sgn(x+b)-sgn(x+2a-b)\right]+sgn(x+2a-b)-sgn(-x-b)\right\} \end{aligned} \tag{23}$$

where for simplicity of the expressions, we have assumed that $b = 3a$ and $c = 2a$. We make the following notations:

$$N_1(x) = \frac{c}{4}\left[1+sgn(\dot{x})\right]\left\{\frac{x-a}{c}\left[sgn(x-2a+b)-sgn(x-b)\right]+sgn(x-b)-sgn(-x+2a-b)\right\} \tag{24}$$

$$N_2(x) = \frac{c}{4}\left[1-sgn(\dot{x})\right]\left\{\frac{x+a}{c}\left[sgn(x+b)-sgn(x+2a-b)\right]+sgn(x+2a-b)-sgn(-x-b)\right\} \tag{25}$$

$$N_3(x) = \frac{x-a}{c}\left[sgn(x-2a+b)-sgn(x-b)\right] \tag{26}$$

$$N_4(x) = \frac{x+a}{c}\left[sgn(x+b)-sgn(x+2a-b)\right] \tag{27}$$

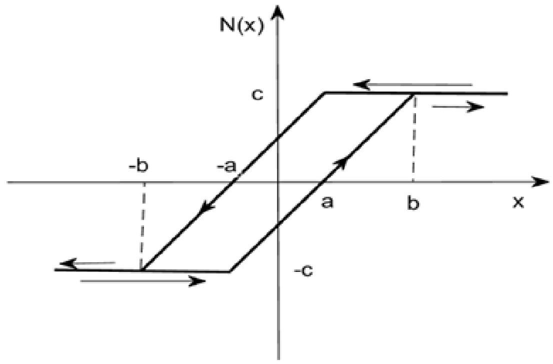

FIGURE 4. Relay with hysteresis and saturation

Then, using (26) and (27), expressions (24) (25) can be represented as:

$$N_1(x) = \frac{c}{4}\left[1 + sgn(\dot{x})\right]\left[N_3(x) + sgn(x-b) - sgn(-x+2a-b)\right] \tag{28}$$

$$N_2(x) = \frac{c}{4}\left[1 - sgn(\dot{x})\right]\left[N_4(x) + sgn(x+2a-b) - sgn(-x-b)\right] \tag{29}$$

$$N(x) = N_1(x) + N_2(x) \tag{30}$$

Then, we have:

$$N_5(x) = N_3(x) + sgn(x-b) - sgn(-x+2a-b) \tag{31}$$

$$N_6(x) = N_4(x) + sgn(x+2a-b) - sgn(-x-b) \tag{32}$$

First we assume that $\dot{x} > 0$ and therefore, $N_2(x) = 0$. If $x > b$, then $N_3(x) = 0$, $N_5(x) = 2$ and $N(x) = c$. If $a < x < b$, then $N_3(x) = 2\frac{x-a}{c}$, $N_5(x) = N_3(x)$ and $N(x) = (x-a)$. If $|x| < a$, then $N_3(x) = 2\frac{x-a}{c} = -2\frac{a-x}{c}$, $N_5(x) = N_3(x)$ and $N(x) = (x-a)$. If $-b < x < -a$, then $N_3(x) = 0$, $N_5(x) = -2$ and $N(x) = -c$. If $x < -b$, then $N_3(x) = 0$, $N_5(x) = -2$ and $N(x) = -c$. Next we assume that $\dot{x} < 0$ and therefore $N_1(x) = 0$. If $x > b$, then $N_4(x) = 0$, $N_6(x) = 2$ and $N(x) = c$. If $a < x < b$, then $N_4(x) = 0$, $N_6(x) = 2$ and $N(x) = c$. If $|x| < a$, then $N_4(x) = 2\frac{x+a}{c}$, $N_6(x) = N_4(x)$ and $N(x) = (x+a)$. If $-b < x < -a$, then $N_4(x) = 2\frac{x+a}{c}$, $N_6(x) = N_4(x)$ and $N(x) = (x+a)$. Finally, if $x < -b$, then $N_4(x) = 0$, $N_6(x) = -2$ and $N(x) = -c$.

The analytic approximation of expression (23) is obtained by replacing the $sgn(x)$ function in terms of the function $\tanh(Ax)$ and then for $|x| < x_{large}$ we have $\tanh(Ax) \approx 1 - \frac{2}{\frac{P_n(2Ax)}{Q_n(2Ax)}+1}$.

CONCLUSION

The paper considers the problem of analytic approximation of nonlinearities with memory. Three different nonlinear characteristics are presented: shifted unpolarized relay with hysteresis, relay with hysteresis and dead zone and relay with hysteresis and saturation. Explicit mathematical formulas for these characteristics are derived, which includes the sign function as a basic element for describing the jump behavior of these characteristics. A specific feature of nonlinearities with memory is the existence of hysteresis loops and therefore, two valued intervals in the mathematical model. The existence of two valued intervals introduces the velocity of the input argument into the nonlinear element description. The focus of our derivations is to approximate the discontinuous jump in the nonlinear characteristic. A rational function approximation of the hyperbolic tangent function as a gate function for the jump in the hysteresis loop is used. In this case, the chained fractions method is implemented to approximate the exponential function in the hyperbolic tangent expression. Finally, some information about the error of approximation is also discussed.

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