Local Perturbation Bounds of the Discrete Matrix Inequality Linear Quadratic Regulator Problem for Differential-algebraic Systems

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***Abstract*— The theory of optimal control is concerned with operating a dynamical system at minimum cost. The situation where the system behavior is depicted by a set of linear differential equations and the cost is described by a quadratic function is considered to be the linear quadratic (LQ) problem. The discussed control problem can be implicitly accomplished by the solutions Q, Y of a system of linear matrix inequalities (LMIs). This paper is devoted to conditioning of the discrete-time LMI based linear quadratic regulator problem for differential-algebraic (DAE) systems. To compute the local perturbation limits of the matrix inequalities we introduce an appropriate right hand side, which is slightly perturbed. Based on the perturbation analysis we obtain tight linear perturbation bounds for the LMIs' solutions to the linear quadratic regulator problem. The results are illustrated by numerical examples.**

Keywords — discrete differential-algebraic systems, local perturbation bounds, LQR problem, condition numbers, linear matrix inequalities synthesis

I. Introduction

Recently LMIs have appeared as a powerful tool with applications across the major domains of systems and control. According to [1,2, 16] LMIs are straight byproduct of Lyapunov based criteria The LQR problem is a perfect representation in our considerations since it can be reduced to a convex problem that involve LMIs.

A very attractive feature of LMIs consists of the fact that many problems in systems and control can be easily reduced to LMI problems, which can be computed efficiently and numerically reliably Also LMI based design is a real technique with many theoretical and practical applications, thanks to the availability of efficient convex optimization algorithms [3] and software [4] in addition to the MATLAB package Yalmip and SeDuMi solver [5].

Singular systems or differential-algebraic systems present a fundamental mathematical framework for the modeling, simulation and control of complex dynamical systems existing in many areas of electrical and mechanical engineering. For the analysis of structural properties of descriptor systems like controllability, observability, stability, minimality, model predictive control, linear quadratic optimal regulator, optimal state regulation, state feedback and observer design have already been considered in [6, 7, 8, 9, 10, 11, 17]. Numerically stable algorithms for the analysis of singular systems have been proposed in [12, 13].

In this note we propose an approach to perform full local sensitivity analysis of the LMI based LQR problem for differential-algebraic systems via incorporating an appropriate right hand part in the considered matrix inequalities. As far as we know this is the first study on sensitivity of discrete DAE systems. After the investigated problem is solved the achieved results can be applied in several directions. To begin with it is possible to asses the errors in the calculated solution of the LQR problem, which are based on rounding errors and structured disturbances in the studied data. Second it is possible to investigate the robust stability and performance of the closed loop system with uncertain elements in the plant and in the controller.

Throughout the paper, we adopt the following notation: $R^{m\times n}$ - the set of real $m\times n$ matrices; $R^n = R^{n\times 1}$; I_n - the identity $n\times n$ matrix; e_n - the unity $n\times 1$ vector; N^T - the transpose of N; $N^\dagger$ - the pseudo inverse of N; $\| N\|_2 = \sigma_{\max}(N)$ - the spectral norm of N, where $\sigma_{\max}(N)$ is the maximum singular value of N; $vec(N)\in R^{mn}$ - the column-like vector appearance of $N\in R^{m\times n}$; $\Pi_{m,n}\in R^{mn\times mn}$ - the vec-permutation matrix, such that $vec(N^T)=\Pi_{m,n}vec(N)$; $N\otimes Q$ - denotes the Kroneker product of the matrices M and Q. The sign ":=" represents "equal by definition".

The rest of the note is structured in as shown below. The problem set up and objective is briefly studied in Section 2. In Section 3 local perturbation bounds of the discrete LMI based LQR problem for DAE systems are derived. Section 4 reveals a numerical example where the effectiveness of the linear bounds is given before we end up in Section 5 with our ending considerations.

II. Problem Set and Objective

Linear discrete DAE systems are described by the set of equations:

$$\begin{aligned} Ex(k+1) &= Ax(k)+Bu(k), \\ y(k) &= Cx(k),\ k=0,1,\ldots,L, \end{aligned} \tag{1}$$

here $x(k)\in R^n$, $u(k)\in R^m$ and $y(k)\in R^r$ are the system DAE state, input and output, and A, B, C and E are constant matrices of compatible size.

Definition 1. (System equivalence) [6]**.** Two systems (E,A,B,C) and $(\hat{E},\hat{A},\hat{B},\hat{C})$ are said to be (system) equivalent, denoted by $(E,A,B,C)\approx(\hat{E},\hat{A},\hat{B},\hat{C})$, if there exist nonsingular transformation matrices $L,R\in R^{n\times n}$ such that the equations

$$\hat{E}=LER, \hat{A}=LAR, \hat{B}=LB, \hat{C}=CR$$

hold true.

Definition 2. (Weierstrass normal form - WNF) [6]. For any regular system there exist two non-singular matrices $L, R \in R^{n\times n}$ such that by

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = T^{-1}x, x_1 \in R^r, x_2 \in R^{n-r}$$

the following decomposed representation can be obtained

$$\begin{aligned} x_1(k+1) &= \hat{A}_r x_1(k) + \hat{B}_1 u(k) \\ Nx_2(k) &= x_2(k) + \hat{B}_2 u(k) \end{aligned} \qquad (2)$$

Definition 3. (Index of nilpotence) [6]. The index of nilpotence ν, i.e. $\nu := \min\{q \mid N^q = 0\}$ is said to be index of a linear descriptor system. Systems with $\nu \ge 2$ are called high index singular systems.

In notation (2), the first relation is a forward recurrent equation which state is obtained only by initial state $x_1(0)$ and $u(k) = 0,1,...,L$. The second expression is a backward recurrence which state is only computed by final state $x_2(L)$ and $u(k) = 0,1,...,L$.

For the system, described in WNF, the state evolution can be described according to [6]:

$$\begin{aligned} x_1(k) &= \hat{A}_r x_1(0) + \sum_{i=0}^{k-1} \hat{A}_r^{\,k-i-1} \hat{B}_1 u(i) \\ x_2(k) &= N^{L-k} x_2(L) - \sum_{i=0}^{L-k-1} N^i \hat{B}_2 u(k+i) \end{aligned} \qquad (3)$$

Equation (3) for state $x_2(k)$ proposes that index one descriptor systems $\nu = 1$ and $N = 0$ will have no infinite poles. In such situation the system (1) is called causal and index one.

Further we study the linear discrete DAE system (1), and assume no direct relation between the input and the output signal. Throughout the paper we admit that the DAE system (1) is an index one system.

The equivalent system

$$(\hat{E}, \hat{A}, \hat{B}, \hat{C}) = \left(\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \hat{A}_r & 0 \\ 0 & I_{n-r} \end{bmatrix}, \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \end{bmatrix}, \begin{bmatrix} \hat{C}_1, \hat{C}_2 \end{bmatrix} \right),$$

is given in WNF where $\hat{A}_r \in R^{r\times r}$ is a stable matrix. The equivalent system is as follows

$$\begin{aligned} x_1(k+1) &= \hat{A}_r x_1(k) + \hat{B}_1 u(k) \\ y(k) &= \hat{C}_1 x_1(k) \end{aligned} . \qquad (4)$$

The system (4) in WNF exists after applying the expression (3b) for state $x_2(k)$.

In line ar quadratic regulator task for a given initial state $x(0)$ we have to to find a control law, which minimizes the cost function $\sum_{k=0}^{\infty} \left\{ x_1^T(k) Q_{1p} x_1(k) + u^T(k) R_{1p} u(k) \right\}$. In addition to it has to be obtained a quadratic Lyapunov function $V_1(x_1) = x_1^T P_1 x_1,\ P_1 > 0,$ such that

$V_1[x_1(k+1)] - V_1[x_1(k)] < -x_1^T(k)\left[Q_{1p} + K_1^T R_{1p} K_1\right] x_1(k)$.

We consider the linear quadratic regulator problem, it is necessary to ensure closed-loop stability and desired performance thus we design a state-feedback control *u(k)=K₁x₁(k)*.

We apply a linear matrix inequality method to solve the LQR problem, as considered in [1].

$$x_1^T\left[(\hat{A}_r + \hat{B}_1 K_1)^T P_1 + P_1(\hat{A}_r + \hat{B}_1 K_1)\right]x_1 < -x_1^T\left[Q_{1p} + K_1^T R_{1p} K_1\right]x, P_1 > 0. \quad (5)$$

Apply the Schur technique [14] then the above expression is transformed to:

$$\begin{bmatrix} -P_1 & (\hat{A}_r + \hat{B}_1 K_1)^T & K_1^T & I \\ (\hat{A}_r + \hat{B}_1 K_1) & -P_1^{-1} & 0 & 0 \\ K_1 & 0 & -R_{1p}^{-1} & 0 \\ I & 0 & 0 & -Q_{1p}^{-1} \end{bmatrix} < 0. \quad (6)$$

We pre- and post- multiply expression (6) by $diag[P_1^{-1}, I, I, I]$ and invoke new variables $Q_1 = P_1^{-1}, Q_1 > 0$ and $Y_1 = K_1 P_1^{-1}$ to obtain the following system of LMIs:

$$\begin{bmatrix} -Q_1 & (\hat{A}_r Q_1 + \hat{B}_1 Y_1)^T & Y_1^T & Q_1 \\ (\hat{A}_r Q_1 + \hat{B}_1 Y_1) & -Q_1 & 0 & 0 \\ Y_1 & 0 & -R_{1p}^{-1} & 0 \\ Q_1 & 0 & 0 & -Q_{1p}^{-1} \end{bmatrix} < 0, \quad (7)$$

$$Q_1 > 0$$

with respect to the variables Q_1 and Y_1.

This paper aims at obtaining local perturbation bounds of the linear matrix inequality system (7) needed to solve the LQR task. Throughout the paper we adopt the following notation:

$R_{1P}^{-1} = R_{i1p}, Q_{1P}^{-1} = Q_{i1P}, \Delta R_{1P}^{-1} = \Delta R_{i1P}, \Delta Q_{1P}^{-1} = \Delta Q_{i1P}$

Further we assume that the matrices $\hat{A}_r, \hat{B}_1, R_{i1p}, Q_{i1p}$ are subject to perturbations $\Delta\hat{A}_r, \Delta\hat{B}_1, \Delta R_{i1p}, \Delta Q_{i1p}$ and accept that they do not alter the sign of the linear matrix inequality system (7). In the perturbation investigation of the discrete LMI based LQR problem for DAE systems it is necessary to determine local perturbation limits of the LMIs (7) as dependence of the perturbations in the data $\hat{A}_r, \hat{B}_1, R_{i1p}, Q_{i1p}$.

III. Local Perturbation Bounds Determination

In this section we do sensitivity analysis of the linear matrix inequality (7) for the discrete DAE system (4) in WNF

$$\begin{bmatrix} -(Q_1+\Delta Q_1) & \hat{A}_r\hat{B}_1Q_1Y_1{}^T & (Y_1+\Delta Y_1)^T & (Q_1+\Delta Q_1) \\ \hat{A}_r\hat{B}_1Q_1Y_1 & -(Q_1+\Delta Q_1) & 0 & 0 \\ (Y_1+\Delta Y_1) & 0 & -R_{1p}{}^{-1} & 0 \\ (Q_1+\Delta Q_1) & 0 & 0 & -Q_{1p}{}^{-1} \end{bmatrix} < 0, \quad (8)$$

where

$$\hat{A}_r\hat{B}_1Q_1Y_1{}^T = (Q_1+\Delta Q_1)(\hat{A}_r+\Delta\hat{A}_r)^T + (Y_1+\Delta Y_1)^T(\hat{B}_1+\Delta\hat{B}_1)^T$$

$$\hat{A}_r\hat{B}_1Q_1Y_1 = (\hat{A}_r+\Delta\hat{A}_r)(Q_1+\Delta Q_1) + (\hat{B}_1+\Delta\hat{B}_1)(Y_1+\Delta Y_1).$$

Further we investigate the impact of the perturbations $\Delta\hat{A}_r, \Delta\hat{B}_1, \Delta R_{i1p}, \Delta Q_{i1p}$ on the perturbed linear matrix inequality solutions $Q_1*+\Delta Q_1$ and $Y_1*+\Delta Y_1$, where Q_1*, Y_1* and $\Delta Q, \Delta Y$ are the nominal solution of the expression (8) and the perturbations. The most significant part of the method we propose is connected with performing perturbation analysis of the expression (7) in a closely way as for a proper matrix equation after introducing an appropriate right hand part, which is a little bit perturbed. In this way for the linear matrix inequality (8) we obtain:

$$\begin{bmatrix} -(Q_1*+\Delta Q_1) & \hat{A}_r\hat{B}_1Q_1Y_1*^T & (Y_1*+\Delta Y_1)^T & (Q_1*+\Delta Q_1) \\ \hat{A}_r\hat{B}_1Q_1Y_1*^T & -(Q_1*+\Delta Q_1) & 0 & 0 \\ (Y_1*+\Delta Y_1) & 0 & -(R_{i1p}+\Delta R_{i1p}) & 0 \\ (Q_1*+\Delta Q_1) & 0 & 0 & -(Q_{i1p}+\Delta Q_{i1p}) \end{bmatrix} = \quad (9)$$

$$= N*+\Delta N_1 < 0$$

here

$$\hat{A}_r\hat{B}_1Q_1Y_1*^T = (Q_1*+\Delta Q_1)(\hat{A}_r+\Delta\hat{A}_r)^T + (Y_1*+\Delta Y_1)^T(\hat{B}_1+\Delta\hat{B}_1)^T,$$

$$\hat{A}_r\hat{B}_1Q_1Y_1* = (\hat{A}_r+\Delta\hat{A}_r)(Q_1*+\Delta Q_1) + (\hat{B}_1+\Delta\hat{B}_1)(Y_1*+\Delta Y_1)$$

and $N*$ is determined from the nominal LMI

$$\begin{bmatrix} -Q_1* & Q_1*\hat{A}_r{}^T+Y_1*^TB_1{}^T & Y_1*^T & Q_1* \\ \hat{A}_rQ_1*+B_1Y_1* & -Q_1* & 0 & 0 \\ Y_1* & 0 & -R_{i1p} & 0 \\ Q_1* & 0 & 0 & -Q_{i1p} \end{bmatrix} = N* < 0 \ . \quad (10)$$

The matrix ΔN_1 is introduced to represent the impact of the data and closed-loop performance perturbations, the rounding errors and the sensitivity of the interior point method that is used to solve the considered LMIs.

With the expression (10) the perturbed relation (9) can be represented as

$$\Delta_{Q_1} + \Omega_{Q_1} = \Delta N_1, \quad (11)$$

where

$$\Delta_{Q_1} = \begin{bmatrix} -\Delta Q_1 & \Delta Q_1\hat{A}_r{}^T & 0 & \Delta Q_1 \\ \hat{A}_r\Delta Q_1 & -\Delta Q_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \Delta Q_1 & 0 & 0 & 0 \end{bmatrix},$$

$$\Omega_{Q_1} = \begin{bmatrix} 0 & Q_1*\Delta\hat{A}_r{}^T+\Delta Y_1{}^T\hat{B}_1{}^T+Y_1*^T\Delta\hat{B}_1{}^T & \Delta Y_1{}^T & 0 \\ \Delta\hat{A}_rQ_1*+\hat{B}_1\Delta Y_1+\Delta\hat{B}_1Y_1* & 0 & 0 & 0 \\ \Delta Y_1 & 0 & -\Delta R_{i1p} & 0 \\ 0 & 0 & 0 & -\Delta Q_{i1p} \end{bmatrix}.$$

The nonlinear terms are annihilated since we perform linear sensitivity analysis. In this way we obtain the vectorized type of the expression (11)

$$vec(\Delta_{Q_1}) + vec(\Omega_{Q_1}) = vec(\Delta N_1), \quad (12)$$

here

$$vec(\Delta_{Q_1}) = [-I, \hat{A}_r\otimes I, 0, I, I\otimes\hat{A}_r, -I, 0,0,0,0,0,0, I, 0,0,0]^T vec(\Delta Q_1)$$

$$:= S_1\Delta q_1,$$

$$vec(\Omega_{Q_1}) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ (I\otimes Q_1*)\Pi_{n^2} & (\hat{B}_1\otimes I)\Pi_{n\times m} & (I\otimes Y_1*^T)\Pi_{m^2} & 0 & 0 \\ 0 & \Pi_{m\times m} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ (Q_1*\otimes I) & (I\otimes\hat{B}_1) & (Y_1*\otimes I) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -I & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -I \end{bmatrix}$$

$$\times \begin{bmatrix} vec(\Delta\hat{A}_r) \\ vec(\Delta Y_1) \\ vec(\Delta\hat{B}_1) \\ vec(\Delta R_{i1p}) \\ vec(\Delta Q_{i1p}) \end{bmatrix} = [S1_{t1}, S1_{t2}, S1_{t3}, S1_{t4}, S1_{t5}]\Delta_{AYBRQ} = S1_t\,\Delta_{AYBRQ}.$$

The mathematical manipulations that we carried out above allow us to obtain the following relation

$$S_1 \Delta q_1 + S1_{t1} vec(\Delta \hat{A}_r) + S1_{t2} vec(\Delta Y_1) + S1_{t3} vec(\Delta \hat{B}_1) + + S1_{t4} vec(\Delta R_{i1p}) + S1_{t5} vec(\Delta Q_{i1p}) = vec(\Delta N_1) \quad (13)$$

Like this the relative first order perturbation bound for the solution $Q_1 *$ of the LMI (7) can be obtained

$$\frac{\|\Delta q_1\|_2}{\|vec(Q_1*)\|_2} \le \frac{1}{\|vec(Q_1*)\|_2}\left(R_1 \frac{\|vec(\Delta \hat{A}_r)\|_2}{\|vec(\hat{A}_r)\|_2} + R_2 \frac{\|vec(\Delta Y_1)\|_2}{\|vec(Y_1*)\|_2}\right) + \frac{1}{\|vec(Q_1*)\|_2}\left(R_3 \frac{\|vec(\Delta \hat{B}_1)\|_2}{\|vec(\hat{B}_1)\|_2} + R_4 \frac{\|vec(\Delta R_{i1p})\|_2}{\|vec(R_{i1p})\|_2}\right) + \frac{1}{\|vec(Q_1*)\|_2}\left(R_5 \frac{\|vec(\Delta Q_{i1p})\|_2}{\|vec(Q_{i1p})\|_2} + N_1 \frac{\|vec(\Delta N_1)\|_2}{\|vec(N*)\|_2}\right) \quad (14)$$

where

$$\frac{R_1}{\|vec(Q_1*)\|_2} = \frac{\|S_1^{\dagger}\|_2 \|S1_{t1}\|_2 \|vec(\hat{A}_r)\|_2}{\|vec(Q_1*)\|_2},$$

$$\frac{R_2}{\|vec(Q_1*)\|_2} = \frac{\|S_1^{\dagger}\|_2 \|S1_{t2}\|_2 \|vec(Y_1*)\|_2}{\|vec(Q_1*)\|_2},$$

$$\frac{R_3}{\|vec(Q_1*)\|_2} = \frac{\|S_1^{\dagger}\|_2 \|S1_{t3}\|_2 \|vec(\hat{B}_1)\|_2}{\|vec(Q_1*)\|_2},$$

$$\frac{N_1}{\|vec(Q_1^*)\|_2} = \frac{\|S_1^{\dagger}\|_2 \|vec(N*)\|_2}{\|vec(Q_1^*)\|_2}$$

$$\frac{R_4}{\|vec(Q_1*)\|_2} = \frac{\|S_1^{\dagger}\|_2 \|S1_{t4}\|_2 \|vec(R_{i1p})\|_2}{\|vec(Q_1*)\|_2},$$

$$\frac{R_5}{\|vec(Q_1*)\|_2} = \frac{\|S_1^{\dagger}\|_2 \|S1_{t5}\|_2 \|vec(Q_{i1p})\|_2}{\|vec(Q_1*)\|_2}.$$

can be denoted as the individual relative conditioning of the LMI (7) with respect to the perturbations $\Delta\hat{A}_r, \Delta\hat{B}_1, \Delta R_{i1p}, \Delta Q_{i1p}$ and ΔY_1.

Applying a similar procedure approach the relative perturbation limits for the solution Y_1* of the LMI (7) can be determined using the following equality

$$\Delta_{Y1} + \Omega_{Y1} = \Delta N_2, \quad (15)$$

here

$$\Delta_{Y_1} = \begin{bmatrix} \mathbf{0} & \Delta Y_1^T \hat{B}_1^T & \Delta Y_1^T & \mathbf{0} \\ \hat{B}_1 \Delta Y_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \Delta Y_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix},$$

$$\Omega_{Y_1} = \begin{bmatrix} -\Delta Q_1 & \Delta Q_1 \hat{A}_r^T + Q_1^* \Delta\hat{A}_r^T + Y_1^{*T} \Delta\hat{B}_1^T & 0 & \Delta Q_1 \\ \hat{A}_r \Delta Q_1 + \Delta\hat{A}_r Q_1^* + \Delta\hat{B}_1 Y_1^* & -\Delta Q_1 & 0 & 0 \\ 0 & 0 & -\Delta R_{i1p} & 0 \\ \Delta Q_1 & 0 & 0 & -\Delta Q_{i1p} \end{bmatrix}.$$

The nonlinear elements are neglected due to the reason that we perform linear sensitivity analysis. Thus the vectorized form of the relation (15) is presented below

$$vec(\Delta_{Y_1}) + vec(\Omega_{Y_1}) = vec(\Delta N_2), \quad (16)$$

where

$$vec(\Delta_{Y_1}) = [0, (\hat{B}_1 \otimes I)\Pi_{nxm}, \Pi_{mxm}, 0, (I \otimes \hat{B}_1), 0,0,0, I, 0,0,0,0,0,0,0]^T vec(\Delta Y_1) = V_1 \Delta y_1,$$

$$vec(\Omega_{Y_1}) = \begin{bmatrix} 0 & -I & 0 & 0 & 0 \\ (I \otimes Q_1^*)\Pi_{n^2} & (\hat{A}_r \otimes I) & (I \otimes Y_1^{*T})\Pi_{m^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ (Q_1^* \otimes I) & (I \otimes \hat{A}_r) & (Y_1^* \otimes I) & 0 & 0 \\ 0 & -I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -I & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -I \end{bmatrix} \times$$

$$\times \begin{bmatrix} vec(\Delta\hat{A}_r) \\ vec(\Delta Q_1) \\ vec(\Delta\hat{B}_1) \\ vec(\Delta R_{i1p}) \\ vec(\Delta Q_{i1p}) \end{bmatrix} = [V1_{t1}, V1_{t2}, V1_{t3}, V1_{t4}, V1_{t5}] \Delta_{AYBRQ} = V1_t \Delta_{AYBRQ}.$$

Thus we obtain the relation below

$$V1\Delta y_1+V1_{t1}vec(\Delta\hat{A}_r)+V1_{t2}vec(\Delta Q_1)+V1_{t3}vec(\Delta\hat{B}_1)$$
$$+V1_{t4}vec(\Delta R_{i1p})+V1_{t5}vec(\Delta Q_{i1p})=vec(\Delta N_2). \quad (17)$$

After the mathematical transformations we derive the relative perturbation limit for the solution Y_1* of the linear matrix inequality (7)

$$\frac{\|\Delta y_1\|_2}{\|vec(Y_1*)\|_2}\le\frac{1}{\|vec(Y_1*)\|_2}\left(M_1\frac{\|vec(\Delta\hat{A}_r)\|_2}{\|vec(\hat{A}_r)\|_2}+M_2\frac{\|vec(\Delta Q)\|_2}{\|vec(Q*)\|_2}\right)$$
$$+\frac{1}{\|vec(Y_1*)\|_2}\left(M_3\frac{\|vec(\Delta\hat{B}_1)\|_2}{\|vec(\hat{B}_1)\|_2}+M_4\frac{\|vec(\Delta R_{i1p})\|_2}{\|vec(R_{i1p})\|_2}\right) \quad (18)$$
$$+\frac{1}{\|vec(Y_1*)\|_2}\left(M_5\frac{\|vec(\Delta Q_{i1p})\|_2}{\|vec(Q_{i1p})\|_2}+N_2\frac{\|vec(\Delta N_2)\|_2}{\|vec(N*)\|_2}\right)$$

where

$$\frac{M_1}{\|vec(Y_1*)\|_2}=\frac{\|V_1^{\dagger}\|_2\|V1_{t1}\|_2\|vec(\hat{A}_r)\|_2}{\|vec(Y_1*)\|_2},$$
$$\frac{M_2}{\|vec(Y_1*)\|_2}=\frac{\|V_1^{\dagger}\|_2\|V1_{t2}\|_2\|vec(Q_1*)\|_2}{\|vec(Y_1*)\|_2},$$
$$\frac{M_3}{\|vec(Y_1*)\|_2}=\frac{\|V_1^{\dagger}\|_2\|V1_{t3}\|_2\|vec(\hat{B}_1)\|_2}{\|vec(Y_1*)\|_2},$$
$$\frac{N_2}{\|vec(Y_1*)\|_2}=\frac{\|V_1^{\dagger}\|_2\|vec(N*)\|_2}{\|vec(Y_1*)\|_2},$$
$$\frac{M_4}{\|vec(Y_1*)\|_2}=\frac{\|V_1^{\dagger}\|_2\|V1_{t4}\|_2\|vec(\Delta R_{i1p})\|_2}{\|vec(Y_1*)\|_2},$$
$$\frac{M_5}{\|vec(Y_1*)\|_2}=\frac{\|V_1^{\dagger}\|_2\|V1_{t5}\|_2\|vec(\Delta Q_{i1p})\|_2}{\|vec(Y_1*)\|_2}.$$

are the individual relative conditioning of the LMI (7) towards the perturbations $\Delta\hat{A}_r, \Delta\hat{B}_1, \Delta R_{i1p}, \Delta Q_{i1p}$ and ΔQ_1.

IV. Illustrative Example [10]

Pay attention to the discrete index one DAE system (1) given in WNF, i.e.

$$\hat{E}=\begin{bmatrix}1&0&\vdots&0&0\\0&1&\vdots&0&0\\\cdots&\cdots&\cdots&\cdots&\cdots\\0&0&\vdots&0&0\\0&0&\vdots&0&0\end{bmatrix},\ \hat{A}=\begin{bmatrix}0.5&0&\vdots&0&0\\0&0.7&\vdots&0&0\\\cdots&\cdots&\cdots&\cdots&\cdots\\0&0&\vdots&0&0\\0&0&\vdots&0&0\end{bmatrix},$$
$$\hat{B}=\begin{bmatrix}0\\1\\\cdots\\1\\-1\end{bmatrix},$$

$$Q_{1p}=\begin{bmatrix}1&0\\0&2\end{bmatrix},\ R_{1p}=1.$$

In the note we determine local bounds, that is why perturbations in the system matrices are taken in a way as to get rid of the nonlinear terms in the mathematical transformations, delivered above, i.e.:

$$\Delta R_{ip}=R_{ip}\times10^{-i},\ \Delta Q_{ip}=Q_{ip}\times10^{-i},$$
$$\Delta\hat{A}_r=A_r\times10^{-i},\ \Delta\hat{B}_1=\hat{B}_1\times10^{-i},$$
$$\Delta N_1=N_1*\times10^{-i},\ \Delta N_2=N_2*\times10^{-i},$$
$$\Delta Q_1=Q_1*\times10^{-i},\ \Delta Y_1=Y_1*\times10^{-i} \text{ for } i=8,7...4.$$

The perturbed solutions $Q_1*+\Delta Q_1$ and $Y_1*+\Delta Y_1$ are made using the approach presented in [15] and applying the code [4]. With the help of the worked out method we derive the relative local perturbation bounds (14) and (18), respectively, for the solutions Q_1* and Y_1* of the LMIs (7).

For the considered amount of perturbations we state the local bounds and present the delivered results in the table given below

Table 1

i	$\frac{\|\Delta q_1\|_2}{\|vec(Q_1^*)\|_2}$	**Bound (14)**	$\frac{\|\Delta y_1\|_2}{\|vec(Y_1^*)\|_2}$	**Bound (18)**
8	6.25 *10^-8	1.01 *10^-7	7.93 *10^-8	0.98 *10^-7
7	6.25 *10^-7	1.01 *10^-6	7.93 *10^-7	0.98 *10^-6
6	6.25 *10^-6	1.01 *10^-5	7.93 *10^-6	0.98 *10^-5
5	6.25 *10^-5	1.01 *10^-4	7.93 *10^-5	0.98 *10^-4
4	6.25 *10^-4	1.01 *10^-3	7.93 *10^-4	0.98 *10^-3

To carry out the perturbation analysis the discrete linear matrix inequality based LQR problem for DAE systems we apply the derived solution methodology, which gives opportunity to end up with the perturbation bounds (14) and (18). The obtained local limits are narrow and similar to the real relative perturbation bounds $\frac{\| \Delta q_1 \|_2}{\| vec(Q_1{*}) \|_2}$ and $\frac{\| \Delta y_1 \|_2}{\| vec(Y_1{*}) \|_2}$. Based on the presented trial results we can come to a conclusion that the studied method is appropriate for delivering the local perturbation bounds of the discrete linear matrix inequality based LQR problem for DAE systems.

V. CONCLUSION

This note is devoted to conditioning of the local perturbation bounds of the discrete LMI based LQR problem for DAE systems. At the same moment we reveal how the estimates of the individual conditioning of the considered LMIs can be obtained. We achieve tight local perturbation bounds for the matrix inequalities representing the problem solution. Using condition numbers we are able to estimate the degree of uncertainty in the solution in presence of errors (measurement, round off, modeling) in the data. Afterwards an illustrative example is demonstrated aiming at to depict the applicability and performance of the obtained results.

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